UNSTEADY, SELF-SIMILAR, TWO-DIMENSIONAL SIMPLE WAVE FLOWS*

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1. Introduction. The two-dimensional, unsteady, isentropic motion of a polytropic gas, that is a gas whose pressure $p(x, y, t)$ and density $\rho(x, y, t)$ are related by

$$p = K\rho^{\gamma}, \qquad (\gamma > 1)$$

where γ is the adiabatic index of the gas and K is a constant, is governed by the equations of continuity and momentum which may be put in the form

$$c_t + uc_x + vc_y + \frac{\gamma - 1}{2}(cu_x + cv_y) = 0, \tag{1.1}$$

$$u_t + uu_x + vu_y + \frac{2}{\gamma - 1}cc_x = 0, \tag{1.2}$$

$$v_t + uv_x + vv_y + \frac{2}{\gamma - 1}cc_y = 0. \tag{1.3}$$

In Eqs. (1.1)—(1.3) the subscripts denote partial derivatives, $u(x, y, t)$ and $v(x, y, t)$ are the x and y components of the velocity respectively, and $c(x, y, t)$ is the local speed of sound which is given by

$$c^2 = dp/d\rho = K\gamma\rho^{\gamma-1}.$$

Recently Mackie [1] has investigated directly solutions of the system (1.1)–(1.3) in which the independent variables x, y, and t occur only in the combination x/t and y/t, so-called self-similar solutions; whereas Pogodin, Suchkov, and Ianenko [2] have first considered from a quite general standpoint solutions in which $c = c(u, v)$, which they have termed "traveling wave" solutions, and then specialized to the self-similar case. Furthermore, Suchkov [3], Ermolin and Sidorov [4], and Levine [5], [6] have studied certain physical problems which have only velocities but no characteristic length parameter in their formulation and therefore possess self-similar solutions which may be written as

$$u(x, y, t) = c_0U(X, Y), v(x, y, t) = c_0V(X, Y), c(x, y, t) = c_0F(X, Y), \tag{1.4}$$

where

$$X = x/c_0t, \qquad Y = y/c_0t. \tag{1.5}$$

Here c_0 is a constant with the dimensions of a velocity so that U, V, F, X, and Y have the convenience of being dimensionless.

A consideration of the mapping from the U, V-plane (hodograph plane) to the X, Y-plane (which in self-similar flow problems may be thought of as a "snapshot"

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of the physical x, y-plane at some unit time) shows that there are three types of solutions—uniform (U = const., V = const.), simple wave ($U = U(F)$, $V = V(F)$) and mixed wave ($F = F(U, V)$). In this paper we shall study the class of self-similar, simple wave solutions of (1.1)–(1.3) which are of interest because of the occurrence in physical problems of regions in the X, Y-plane where $U = U(F)$, $V = V(F)$ (see, for example, [4]). In addition, this class of solutions possesses certain properties analogous to those exhibited by simple waves in two-dimensional, steady flow.

2. Properties of simple waves. In the self-similar variables (1.4) and (1.5) the system (1.1)–(1.3) takes the form

$$(U - X)F_X + (V - Y)F_Y + \frac{\gamma - 1}{2}(FU_X + FV_Y) = 0, \tag{2.1}$$

$$(U - X)U_X + (V - Y)U_Y + \frac{2}{\gamma - 1}FF_X = 0, \tag{2.2}$$

$$(U - X)V_X + (V - Y)V_Y + \frac{2}{\gamma - 1}FF_Y = 0. \tag{2.3}$$

If the flow is assumed to be irrotational, then as a result of the dynamic similarity there will exist a dimensionless velocity potential Φ defined by

$$\phi = c_0^2 t\Phi(X, Y),$$

where ϕ is the customary velocity potential and thus satisfies $u = \phi_x$, $v = \phi_y$. From (2.1)–(2.3) it follows that the equation for Φ is

$$\{(\Phi_X - X)^2 - F^2\}\Phi_{XX} + 2(\Phi_X - X)(\Phi_Y - Y)\Phi_{XY} + \{(\Phi_Y - Y)^2 - F^2\}\Phi_{YY} = 0. \tag{2.4}$$

The function F^2 is given by the unsteady form of Bernoulli's theorem, namely

$$\Phi - X\Phi_X - Y\Phi_Y + \tfrac{1}{2}\Phi_X^2 + \tfrac{1}{2}\Phi_Y^2 + \frac{F^2}{\gamma - 1} = \frac{F_0^2}{\gamma - 1},$$

where F_0 is a constant.

Equation (2.4) was derived by Mackie [1]. It is valid throughout the entire region of the X, Y-plane in which a physical problem is posed. Note that (2.4) has a certain resemblance to the equation for the potential ϕ in two-dimensional, steady flow

$$(\phi_x^2 - c^2)\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + (\phi_y^2 - c^2)\phi_{yy} = 0. \tag{2.5}$$

Returning now to Eqs. (2.1)–(2.3), we see that these equations take a particularly simple form in the special case $U = U(F)$, $V = V(F)$, since the derivatives of U and V with respect to X and Y may now be replaced by derivatives with respect to F only. If this is done, we have

$$(U - X)F_X + (V - Y)F_Y + \frac{\gamma - 1}{2}(FU_FF_X + FV_FF_Y) = 0, \tag{2.6}$$

$$(U - X)U_FF_X + (V - Y)U_FF_Y + \frac{2}{\gamma - 1}FF_X = 0, \tag{2.7}$$

$$(U - X)V_FF_X + (V - Y)V_FF_Y + \frac{2}{\gamma - 1}FF_Y = 0. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$U_F F_Y = V_F F_X \ , \tag{2.9}$$

that is, *a self-similar, simple wave flow must be irrotational.* Using (2.9) in either (2.7) or (2.8) yields

$$U_F X + V_F Y - \left[\frac{2}{\gamma - 1} F + UU_F + VV_F\right] = 0. \tag{2.10}$$

Furthermore, if (2.7) and (2.8) are multiplied by U_F and V_F respectively and added, and if the resulting terms containing F are eliminated by means of (2.6), then we have

$$U_F^2 + V_F^2 = \left(\frac{2}{\gamma - 1}\right)^2. \tag{2.11}$$

Equations (2.10) and (2.11) are the basic equations governing self-similar, simple waves and will be used for further investigation. (It is interesting to note that two somewhat more general equations of which (2.10) and (2.11) are special cases may be derived from (1.1)–(1.3) by directly seeking solutions of the form $u = u(c)$, $v = v(c)$. This was apparently first done by Ianenko [7]. From these equations it follows as above that a flow in which u and v are functions of c only must be irrotational so that, in particular, the two-dimensional, isentropic, steady, simple wave motion of a gas is irrotational. However, since we are concerned only with self-similar solutions, the derivation given here and the resulting equations are sufficient.)

In a simple wave region we have the following result.

THEOREM 1. *If Eqs.* (2.10) *and* (2.11) *hold in some domain* $\mathfrak{D}$ *in the X, Y-plane, then equation* (2.4) *is in general hyperbolic in* $\mathfrak{D}$. *More specifically, Eq.* (2.4) *cannot be elliptic anywhere in* $\mathfrak{D}$ *and cannot be parabolic except possibly on a curve in* $\mathfrak{D}$.

Proof. Let

$$L = (U - X)^2 + (V - Y)^2 - F^2. \tag{2.12}$$

Since Eqs. (2.11) and (2.12) are invariant under rotation, we may always perform a suitable rotation to obtain a coordinate system in which both V_F and U_F are locally nonzero. Considering this done, Eq. (2.10) may then be solved for the quantity $V - Y$ so that

$$V - Y = -(U - X)\frac{U_F}{V_F} - \frac{2}{\gamma - 1}\frac{1}{V_F}. \tag{2.13}$$

Substituting (2.13) into (2.12) and squaring yields

$$L = (U - X)^2\left[1 + \frac{U_F^2}{V_F^2}\right] + \frac{4}{\gamma - 1}\frac{U_F}{V_F^2}(U - X)F + F^2\left[\left(\frac{2}{\gamma - 1}\right)^2\frac{1}{V_F^2} - 1\right].$$

If (2.11) is used in the above expression for L, we obtain

$$L = \frac{1}{V_F^2}\left[\frac{2}{\gamma - 1}(U - X) + U_F F\right]^2, \tag{2.14}$$

and similarly

$$L = \frac{1}{U_F^2}\left[\frac{2}{\gamma - 1}(V - Y) + V_F F\right]^2.$$

Clearly $L \geq 0$ throughout $\mathfrak{D}$. However, $\Phi_X = U$ and $\Phi_Y = V$ so that this implies that equation (2.4) cannot be elliptic anywhere in $\mathfrak{D}$.

Suppose $L = 0$ in any region contained in $\mathfrak{D}$. Then differentiation of the above two expressions for L with respect to X and Y leads to a contradiction.

We have not dealt with the possibility that L may be zero along a curve. After we have established Theorem 2 we shall show that this situation does indeed occur, but that in any simple wave region there is only one such curve.

THEOREM 2. *Along each characteristic of one family of characteristics of Eq.* (2.4) *F, U, and V are constant, and furthermore, each of these is a straight line.*

Proof. From (2.12) it is easily seen that the characteristics of Eq. (2.4) are given by

$$\frac{dY}{dX} = \frac{(U - X)(V - Y) \pm F\sqrt{L}}{(U - X)^2 - F^2}$$

For the appropriate family the elimination of $V - Y$ from the above by means of (2.13) and the taking into account of Eq. (2.14) yield

$$dY/dX = -U_F/V_F .$$

However, the irrotationality condition (2.9) implies

$$dY/dX = -U_F/V_F = -F_X/F_Y \tag{2.15}$$

Thus along each characteristic of one family of characteristics we have from (2.15) that

$$F_X \, dX + F_Y \, dY = 0,$$

and F is constant along each characteristic.

If F is constant along some curve, then U, V, U_F, and V_F are also constant along this curve since they are functions of F only. In particular, the ratio U_F/V_F is constant, so that (2.15) implies the last part of the theorem.

Equation (2.15) and Theorem 2 allow us to explain fully the last part of Theorem 1. Any characteristic in the family consisting of straight lines may be suitably rotated so that its slope is zero, and hence $U_F' = 0$ on this rotated characteristic by (2.15), where U' denotes the velocity component in the new coordinate system. The invariance of L and (2.14) then imply that at the point $X' = U'L$ is zero. However, since U' is constant along each characteristic there will only be one such point on each characteristic. If these points are now joined, the continuity of the quantities in (2.12) implies that a curve will result along which $L = 0$; that is, there is in general one curve along which Eq. (2.4) is parabolic. It is interesting to note that for the special simple wave considered in [1] $U \equiv 0$ so that on the line $X = 0$ Eq. (2.4) is parabolic.

Let us comment briefly on Theorems 1 and 2. Firstly, these theorems have their obvious analogies in two-dimensional, supersonic flow where it is well known that Eq. (2.5) is hyperbolic and that in a simple wave region one family of characteristics in the x, y-plane consists of straight lines. Secondly, it is readily seen from Eq. (2.10) that the curves along which F, U, and V are constant are straight lines in the X, Y-plane, since this equation is linear in X and Y, and this has, in fact, been noted in [4]. However, it is certainly not obvious that these lines are characteristics—a result which is apparently not established in [4].

Up until now we have not related solutions of Eqs. (2.10) and (2.11) to actual physical motions. In order to do this we first introduce the terminology Riemann wave. By the term Riemann wave we mean the solution to the following problem. Let polytropic gas with adiabatic index γ occupy the region $y \geq x \tan \theta$, $-\infty < x < \infty$ at time $t = 0$. Furthermore, suppose at this instant it is flowing uniformly with constant x and y velocity components u_1 and v_1 respectively and with acoustic speed c_1. At time $t = 0$ the gas is allowed to expand into the vacuum $y < x \tan \theta$. Thus a Riemann wave is simply the one-dimensional, unsteady flow which results when a semi-infinite column of gas behind a piston is allowed to expand into a vacuum by the instantaneous removal of the piston when viewed by an observer moving with constant velocity in the plane.

By a fairly straightforward analysis of a Riemann wave (for details see [6]) the following theorem relating simple waves and Riemann waves may be established.

THEOREM 3. *A linear simple wave, that is, a solution of* (2.10) *and* (2.11) *of the form*

$$U = \alpha_1 F + \beta_1, \qquad V = \alpha_2 F + \beta_2,$$

where α_i, β_i $(i = 1, 2)$ *are constants, is a Riemann wave. Conversely, any Riemann wave is a linear simple wave.*

From equation (2.15) we obtain

COROLLARY. *Suppose that in a simple wave region the lines along which F is constant are parallel. Then this simple wave is a Riemann wave.*

In two-dimensional, steady, supersonic flow the assumption that the characteristics in a simple wave are straight leads one to the physical problem of flow round a bend, that is, Prandtl-Meyer flow. The situation in unsteady, self-similar flow is, however, somewhat more complicated. Although here also one family of characteristics is straight in a simple wave region, the exact physical nature of a nonlinear, simple wave is not easy to interpret and cannot, in general, be identified globally as the solution of a definite physical problem. (Consider, for example, the solution

$$U = \frac{2}{\gamma - 1} \sin F, \qquad V = \frac{2}{\gamma - 1} \cos F, \qquad F = X \cos F - X \sin F$$

of Eqs. (2.10) and (2.11), where F is suitably restricted.) Nevertheless, nonlinear simple waves do occur in the solution of physical problems (see [4]) as part of the total solution.

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