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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten.*

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq (25)" is acceptable, but not "the preceding Eq " Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

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BOOK REVIEW SECTION

Strömungs- und Temperaturgrenzschichten. By Alfred Walz. G. Braun, Publishers, Karlsruhe, 1966. xvi + 260 pp. D.M. 56.–.

This book is not a textbook which introduces students into the field of boundary layer theory, but is written to help the designing engineer to compute shear and heat transfer in a simple, approximate manner. For this reason, emphasis is placed on integral methods. The author states explicitly that the flood of present-day publications makes it nearly impossible to present all available methods; that is understandable. However, it is regrettable that Lighthill's method (Proc. Roy. Soc. London **A202**, 359 (1950)) is omitted. This method allows one to compute heat-transfer if one has a reasonable estimate of the shear at the wall; it is older and gives very good results.

Laminar and turbulent layers are treated. The difficulty in predicting the transition region is discussed. Ample space is given to the discussion of the underlying physical facts. The author is known for his ability to simplify computations and this ability shows up in numerous nomograms and diagrams which help the designer. Comparisons of computed results with experiments and some exact solutions are presented.

The reviewer feels that these approximate procedures are attractive as long as they are simple and guarantee reasonably good results. However, on some occasions the author suggests the use of electronic computers. If that is needed, then one should really abandon integral methods and take finite difference methods for solving the governing partial differential equations directly. There are very rapid methods available today.

Practical engineers will appreciate the last part of the book. It gives step-by-step computing guides for three procedures, described earlier, for computing boundary layers.

I. Flügge-Lotz (*Stanford*)

Random number generators. By Birger Jansson. Almqvist & Wiksell, Stockholm, 1966. 205 pp. Sw. Kr. 42.

This work contains a survey, partly historical, of the state of the art of generating "random" numbers as of 1966. A good portion of the monograph is original. Six classes of random number generation methods are discussed: dice-like methods, tables of random numbers, physical devices, arithmetic processing of a previously generated random number source, arithmetic procedures, and digits of irrational numbers. There is a great deal of careful consideration given to the statistical questions arising from tests of various procedures based on special distributions. To determine serial correlation coefficients of sequences generated by the much-used congruential generators the author devotes a chapter to some quite original work involving the so-called Dedekind sum and its reciprocity law.

The work is a valuable source of information on all kinds of generators and their behavior. It contains many interesting suggestions for further work and should be in the library of every respectable computing center.

D. H. Lehmer (*Berkeley*)

Statistical continuum theories. By Mark J. Beran. Interscience Publishers, New York, 1968. xv + 424 pp. $17.50.

This book is concerned with the statistical formulation of classical field theories. In particular, problems of flow through porous media, turbulence, elastic and electrical properties of heterogeneous media, and electromagnetic fields with statistical properties are dealt with. The opening chapter introduces the subject with a familiar model, the harmonic oscillator with a) constant coefficients, b) coefficients known functions of time, c) coefficients having known statistical properties, and d) coefficients that are functions of the dependent variable. In each case, the system is subject to a random forcing function. The techniques used are the prototypes of some of those used in the later chapters. In addition, an introduction to a one-dimensional continuum problem is given through the consideration of a statistically homogenous medium with a random dielectric constant.

Chapter two presents a review of basic ideas in probability theory and the theory of functionals. The necessity and motivation for this have been shown in the first chapter. Chapter three consists of an outline of general methods of formulating and solving statistical continuum problems.

The remainder of the book is concerned with special topics and the special assumptions needed to get solutions, since the equations governing the characteristic functionals are too difficult to solve. Approximate solutions are sought by considering statistical moment equations. Unfortunately, the resulting infinite hierarchy of statistical moment equations is also intractable, so assumptions must be made to reduce the set to a finite one.

The book is clear and concise. Numerous references bring the material up to about 1966. The author states that it is his hope that the book will be suitable for second- or third-year graduate students in engineering and physics as well as for research workers in fields using statistical formulations. That hope is admirably realized.

EDWARD SAIBEL (*Pittsburgh, Pa.*)

Conformal mapping on Riemann surfaces. By H. Cohn. McGraw-Hill Book Co., New York, 1967. xiv + 325 pp. $12.95.

The theory of Riemann surfaces represents one of the most important chapters of the theory of analytic functions of one complex variable. It has various applications, and many chapters of this theory have been studied from different points of view; sometimes different names have been used for the same notions. This book represents a very valuable survey of this theory. It will be welcomed by anyone interested in this field and should not be missing in any graduate mathematical library. It is directed to senior undergraduate or beginning graduate students with a minimal preparation. The book consists of 14 chapters.

The first part of the book is a review of some results in complex analysis (Cauchy integral, topological considerations, Riemann sphere, analytic continuation). In the second part, Riemann manifolds (elliptic functions, mainfolds over the z-sphere, triangulated and abstract manifolds) are discussed. In the third part, the existence theorems (topological considerations, harmonic differentials, physical intuitions) are presented. In part IV, real existence proofs (conformal mapping, boundary behavior, alternating procedure) are given. In part V, algebraic applications (Riemann's existence theorem, Riemann-Roch and Abel's theorems) are indicated. Finally, minimal principles and infinite manifolds are discussed in the appendices.

As stressed in the Introduction, the author tried to present the material in such a way that it can be covered by a one-semester course. This is probably the reason why the author does not treat the automorphic functions, the lemma of Schwarz-Pick, and the invariant (hyperbolic) metric.

The reviewer hopes that in not too distant future a second book by the author will appear in which he will make a survey, in the same expert manner as in the present volume, of the topics not discussed in the present book.

STEFAN BERGMAN (*Stanford, Calif.*)

Set theory and the continuum hypothesis. By Paul J. Cohen, W. A. Benjamin, Inc., New York, 1966. 154 pp. $8.00.

The author's own description in the preface cannot be improved on: "The notes that follow are based on a course given at Harvard University, Spring 1965. The main objective was to give the proof of the independence of the continuum hypothesis. To keep the course as self-contained as possible we included background material in logic and axiomatic set theory as well as an account of Gödel's proof of the consistency of the continuum hypothesis. Our review of logic is of necessity rather sketchy although we have tried to cover some of the fundamental concepts such as formal systems, undecidable statements and recursive functions. Actually, with the exception of the Löwenheim-Skolem theorem, none of the results of the first chapter are used in the later work and the reader who has had an introductory course in logic may omit this chapter. Its primary purpose is to accustom mathematicians who are not specialists in logic to the strictly precise point of view which is necessary when dealing with questions in the foundations of mathematics. Also, it is intended to clarify certain common confusions such as that of the concept of an undecidable statement in a particular axiom system with the concept of an unsolvable problem, which concerns methods of computation."

The book will undoubtedly appeal to a wide audience, indeed to two large classes of readers. Those who are already specialists in logic or axiomatic set theory will find the last chapter the most readable and complete version of Cohen's independence theorem (and of a few of the many independence results which have been proved since, using 'Cohen's method') yet published and, as Cohen hopes, the non-specialist should find the whole book a self-contained route to these theorems. Cohen's style, attempting to emphasise the intuitive ideas rather than give all the formal details (i.e. to write about logic and set theory in the same way as one would about any other branch of mathematics), is refreshing, and has been too rarely used by authors in this field. Too many textbooks of logic have been written by authors who appear to believe that because one is discussing formalisms one must be formalistic, a point of view which might suggest that Dr. Spock should have written in baby language. However the reader should be warned that what is obvious to Cohen may take him some time to think out; this is not easy bedtime reading and will probably be more successful as a supplement to an explanatory lecture course than as a text for self study, although there are enough hints for the bright student who is prepared to work through for himself all the things he is advised to.

As Cohen says, the first chapter is not essential to the main theorems, but most of what is in it is desirable background material for the non-logician. It is a delightfully concise presentation of the most important results in mathematical logic and can be thoroughly recommended to any non-logician who wants to read these in only 50 pages. Naturally the pace is smart; to get to the Gödel completeness theorem on p. 16 must be a record. In one or two places the presentation is so concise that it might lead the reader into error. For example, the subtleties of Gödel's second incompleteness theorem, concerned with what needs to be formally provable about the proof predicate used in the formula expressing consistency, are not mentioned. And the statement at the foot of p. 45, 'The requirement that the axioms be given recursively is essential, otherwise we could take for Σ the set of all true statements of Z', is incorrect, for this particular Σ, Consis Σ cannot even be expressed in Σ, never mind proved. Indeed, as Rosser pointed out many years ago, Gödel's second theorem does apply to many systems whose theorems are not recursively enumerable. As far as I know there is no natural example known of any consistent system in which a formula naturally expressing its consistency is provable.

The treatment becomes a little (but not much—it is still a good deal faster than the Gödel monograph) less condensed in the next two chapters in which Zermelo-Fraenkel (ZF) and Gödel-Bernays (GB) set theory are developed and Gödel's results on the consistency of the continuum hypothesis and axiom of choice are proved. For the specialist it is nice to have this version of the Gödel results in ZF, and following the original paper rather than the later monograph. Cohen also sketches out the latter line of argument, discusses the relation between ZF and GB and then proves the existence of a minimal standard model.

The last chapter is the raison d'être of the book. After discussing the intuitive motivation of the notion of forcing he goes on, assuming axiom SM (that there is a standard model for ZF), to define forcing in much the same way as in his original paper (*Proc. Nat. Acad. Sci.* **50** (1963) 1143–1148, **51** (1964) 105–110). Apart from incorporating various simplifications suggested by others, it differs from this in using the original Gödel method (via predicative definitions) of building up the constructible sets instead of that via the function F used in the Gödel monograph. He then shows the existence of a complete sequence of forcing conditions and hence defines a model N which he proves to be a model for ZF, GCH (Generalised Continuum Hypothesis), AC (Axiom of Choice) and $V \neq L$ (there exist non-constructable sets). He goes on to indicate generalisations of the forcing concept which he uses to produce a model for ZF in which CH fails. He briefly mentions Easton's results that one can achieve $2^{\lambda'_\alpha} = \lambda'_{g(\alpha)}$ for all regular λ'_α and all monotonic functions g such that $\lambda'_{g(\alpha)}$ is not cofinal with any cardinal $\leq\lambda'_\alpha$. Next he produces a model in which AC fails, the continuum not being well-orderable, and sketches very briefly theorems of Levy and Halpern and Feferman strengthening this in various ways (e.g. in the above model every set can be ordered). The next section deals with the construction of models in which cardinalities are changed; e.g., one in which there are only countably many constructible real numbers, one (due to Levy and Feferman; the proof is only sketched) in which the continuum is the union of a countable number of countable sets. He points out how one could eliminate the appeal to SM in all this by purely syntactical arguments expressible entirely in elementary number theory. He ends with Sierpinski's proof that GCH implies AC and a section speculating on how one should regard statements like CH now known to be independent. He feels that eventually CH may come to be regarded as obviously false.

Of course the incredible fruitfulness of Cohen's method has ensured that the results mentioned in this last chapter are very far from the best now known. At the time of writing the most up-to-date survey of these post-Cohen independence results is to be found in 'A survey of recent results in set theory' by A. R. D. Mathias (shortly to be published in the Proceedings of the UCLA Set Theory Institute), which states over 200 such theorems. But the present book deserves a lasting place as an introductory textbook by the originator of the method.

J. C. SHEPHERDSON (*Berkeley*)

Lectures on quantum field theory. By P.A.M. Dirac. Academic Press, New York, 1966. viii + 151 pp. $7.50.

In this collection of lectures given at the Belfer Graduate School of Science, Prof. Dirac expounds his present view of quantum field theory. The emphasis is on foundations; S-matrix theory, the apparatus of Feynman graphs, etc., of interest to the practical field theorist, are not touched on. The student's memory is first refreshed on basic quantum mechanical principles and quantum mechanical systems of a finite number of degrees of freedom. Then he is brought through an elementary exposition of the quantum theory of fields at about the level of Wentzel's old book. These basic ideas—many of them originally due to Dirac himself—are presented more or less in their original form and notation. It is necessary to say that most of this material, which has gained much in mathematical concision, clarity, and generality over the intervening years and is now common knowledge, suffers from this unnecessarily old-fashioned presentation. We cite for example the exposition of dual vector spaces, the Clifford algebra of the γ-matrices, and the theory of anti-particles *via* the old hole theory.

The theme of the book is Prof. Dirac's oft-repeated dictum, "The Heisenberg Picture is a good picture, and the Schrödinger Picture is a bad picture." This means that the Schrödinger Picture is unusable because physical states, certain elements of Hilbert Space, are not in the domain of the type of operators representing the time evolution of the system which arise in quantum field theory. This is illustrated by constructing a model Hamiltonian H and state Ω_0 such that $H\Omega_0$ exists but $H^2\Omega_0$ does not, because of the infinite (constant) zero point energy, thus invalidating the existence of states like exp $(-iHt)\Omega_0$. By "using the Heisenberg Picture", he means restricting oneself entirely to an operator algebra, in particular the Heisenberg equations of motion. This will necessitate various new interpretive rules to extract physical information about particles, which hopefully will replace the more familiar probabilistic ones in terms of states in Hilbert Space. This program is illustrated in the final chapters by calculating to second order in e the electron anomalous magnetic moment and Lamb shift by integrating the Heisenberg motion equations. But the ultra-violet divergences (which are of course more serious than the infinite zero point energy, exorcizable by a mere normal ordering) remain in both Pictures, and somewhat mar his thesis that the Heisenberg Picture is "good."

Prof. Dirac's philosophy, in the face of this continuing crisis in quantum field theory, is summed up in the following quotation, taken from the section called "Cut offs": "Quantum electrodynamics cannot be a complete theory in itself. We find that we cannot make quantum electrodynamics consistently relativistically invariant for high energy processes. Well, that does not really matter." His viewpoint then is that it is all right to abandon relativistic invariance—that is, to prefer one special inertial frame—at very high energies, because the unknown interaction with other particles, which sets in at these energies, presumably will restore it again.

His conviction that field theory must make a choice between yielding finite, sensible numbers ("be a logical theory" as he puts it) and Lorentz invariance (to repeat: prefer no inertial frame over another) is clearly expressed. This pessimistic prejudice is very widely shared today, and goes so deep as effectively to preclude further thinking on this vital question. In this reviewer's opinion, it is extremely unfortunate for physics that this view, that a physical cut-off and Lorentz invariance are incompatible, so pernicious because it poses false and desperate alternatives, is allowed to go unexamined and unchallenged. It has in fact been demonstrated to be false within the last five years, but these ideas have not yet found their way into currency nor been subjected to experimental check.

R. L. INGRAHAM (*University Park, N. M.*)

Principles of structural stability. By Hans Ziegler. Blaisdell Publishing Company, Waltham, Mass., 1968. vii + 150 pp. $5.50.

This short book is an outgrowth of two articles by Professor Ziegler on stability. The articles were published in 1953 and 1956 and are referenced frequently in the literature. This book is rather well organized, and the treatment is superior to the presentations of the articles.

The basic notion here is to classify and discuss stability problems for a wide class of structures. The system classifications are slightly modified from those of classical mechanics but, roughly, are gyroscopic conservative, nongyroscopic conservative, dissipative, circulatory and instationary. The techniques of solution considered are the imperfection method, the equilibrium method, the energy method and the dynamic method.

A number of theorems are presented which help to clarify the roles of the various methods. A variety of physical problems are used to illustrate the concepts under discussion. The classical column problems and rotating shaft-disc combinations are presented in detail. Problems for plates, rotating shafts subjected to compression, shimmy of trailers, torsion of shafts and flutter are used for illustrative purposes.

This reviewer finds the book to be sophisticated and deceptively easy to read, and feels that this book is better suited for one who actively works in structural stability or as a supplementary graduate text rather than as a senior or first year graduate course text as the author suggests.

PAUL R. PASLAY (*Providence, R. I.*)

Integrals and operators. By J. E. Segal and R. E. Kunze. McGraw-Hill Publishing Company, New York, 1968. xi + 308 pp. $10.50.

Most books on modern integration theory share a wearisome dreariness, due to the fact that they deal with the most complex of the basic tools of present-day mathematics, requiring a large number of abstract concepts to be handled with care (compare to general topology or linear algebra!), and that the book usually stops short of any significant application of the theory which might stimulate the reader's interest. The most attractive feature of the present volume is that its main objective is to avoid these defects and, as soon as the basic notions of integration theory have been secured, to lead the reader at once into a fairly detailed exposition of two of its most important applications, integration on a locally compact group and the spectral theory of operators. Indeed, this reviewer wholly shares the authors' belief that spectral theory is *the* most important application of integration, because it brings to light the inner necessity and unavoidability of the concepts of Stieltjes measure and Stieltjes-Lebesgue integral, even when dealing with the most classical problems of the theory of differential equations with the smoothest possible coefficients; and it now appears as a historical oddity that integration theory was developed before spectral theory and for completely different purposes.

The book is about evenly divided, the first half being devoted to "abstract" integration, which is then specialized to locally compact spaces, making the transition to the applications of functional analysis. Very little is required of the reader as background beyond the rudiments of algebra and fundamental topology: the exposition includes such topics as uniform spaces or Hilbert space (developed from scratch), and complete proofs of fundamental results such as the Stone-Weierstrass or the Hahn-Banach theorem are given in the text.

Although all applications given in the text have to do with integration on locally compact spaces, and the integral always appears as a linear functional, the authors have not dared violate the American taboo against the Bourbaki approach to integration as a part of duality theory, probably with an eye to applications to probability theory; they stick therefore to the usual σ-rings of sets, somewhat mitigated by using the Daniell definition of the integral, with a novel way of treating limiting processes by the introduction of the topology of *sequential* pointwise convergence. Some attention is given to the pathology of "large" measure spaces (where the total space does not belong to the σ-ring), which are of very slight importance in the applications treated in the book, and also to a "relatively expendable" result (in the authors' own words), Lebesgue's differentiability theorem. On the other hand, it is a pity that such useful notions as bounded measures or measures with compact support have been relegated to exercises.

Coming to integration on locally compact groups, a noteworthy feature is that the existence and uniqueness of Haar measure are proved in a more general setting, that of a uniformly locally compact space on which acts a uniformly equicontinuous transformation group (a theorem due to the first author). But convolution is only given a very sketchy and scattered treatment.

The last three chapters are the best ones, in the reviewer's opinion. The authors are to be highly commended for having put in the forefront what they call "algebraic integration theory" (or, equivalently, the structure of commutative Hilbert algebras), and deducing from it the Hilbert spectral theorem in its most intuitive and striking form, the unique "model" for a self-adjoint bounded operator being multiplication in a L^2 space by a function belonging to L^∞; this does away with the clumsy "projection-valued spectral measure" of the traditional description by $\int \lambda \, dE(\lambda)$ (although the authors, out of respect for tradition, add a short paragraph on spectral theory considered from that point of view). The last chapter deals very sketchily with two very important topics, representation of locally compact groups and unbounded operators (why the latter should be inserted between two paragraphs on group representations is a mystery to the reviewer). An interesting feature is the inclusion (for the first time in a textbook) of the von Neumann uniqueness theorem in harmonic analysis, from which the authors elegantly deduce the Plancherel theorem (but the Pontrjagin duality is only included in an excercise).

J. Dieudonné (*Nice, France*)

Stability of parallel flows. By Robert Betchov and William O. Criminale, Jr. Academic Press, New York, London, 1967. xi + 330 pp. $16.00.

This is the first book dealing with this subject since Lin's classical monograph, *The theory of hydrodynamic stability* (Cambridge University Press 1955), though several excellent review or survey articles (W. H. Reid, page 249 of *Basic developments in fluid dynamics*, edited by M. Holt, Academic Press, New York (1965) and J. T. Stuart, Appl. Mech. Rev. **18**, 523 (1965)) or chapters in books (J. T. Stuart, Chapter 9 of *Laminar boundary layers*, edited by L. Rosenhead, Oxford Press (1963)) have appeared. The present text, which might well be subtitled "An Engineer Looks at the Stability of Parallel Flows with the Help of a High-Speed Computer", is complementary to the works just cited.

The primary emphasis is on solving the Orr-Sommerfeld equation, or slight modifications of it, for a variety of problems—the stability of boundary layers, shear layers, jets and wakes. The technique of solution is always by direct numerical methods, which in itself is not an easy task, and the authors provide an informative discussion of why it is difficult to solve the Orr-Sommerfeld equation numerically, and how these difficulties can be overcome. This reviewer would have been happy to have had this material considerably expanded.

The authors have made a serious effort to interpret and explain the physics of each problem, relate these ideas to their analysis, and finally to interpret their results. The first part of the book is written in a leisurely introductory manner and provides considerable insight into the physics of the Orr-Sommerfeld equation, the Rayleigh equation, and the critical layer and linear hydrodynamic stability theory in general. The second part of the text is more of a review nature; three-dimensional disturbances are considered, as well as the effects of compressibility, stratification, magnetic fields, flexible boundaries, etc. Also there is a short chapter on nonlinear effects which is presently such an important area of research. The work of D. Benney and C. C. Lin (Phys. Fluids **3**, 656–657 (1960), and D. Benney, Phys. Fluids **7**, 319–326 (1964)) is discussed; but the important work of J. T. Stuart (J. Fluid Mech. **9**, 353–370 (1960)) and J. Watson (J. Fluid Mech. **9**, 371–389 (1960)) and of W. Eckhaus (*Studies in nonlinear stability theory,* Springer, Berlin (1965)) is mentioned only briefly. Of course the present book was published before W. Reynolds and M. Potter (J. Fluid Mech. **27**, 465–492 (1967)) showed, following the ideas of Stuart and Watson, the existence of subcritical instabilities for plane Poiseuille flow.

While the applied mathematician can undoubtedly complain at several points about the level of rigor and/or the lack of completeness in the presentation, there is no doubt that the authors have provided a considerable amount of practical information about the physics of the Orr-Sommerfeld equation and its solution for a variety of problems.

It is interesting to note (an example of modern technology) that in addition to about 240 cited references, Mr. Lucker of the Princeton Library has provided an additional bibliography of approximately 460 references, representing a survey of the literature from 1954 through 1966.

R. C. DiPrima (*Troy, New York*)