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THE DIFFRACTION OF A PLANE COMPRESSIONAL ELASTIC WAVE BY A RIGID CIRCULAR DISC*

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Abstract. The paper deals with the low-frequency diffraction of a plane compressional elastic wave incident obliquely on a rigid circular disc embedded in an infinite elastic medium. The motion of the disc, both rotational and translational, has been discussed in detail. By letting the mass of the disc go to infinity one obtains the results for diffraction by a fixed disc. Far-field amplitude of the scattered field has also been obtained. This can be used to calculate the scattering cross-section of the disc. It is found that for long wavelengths the scattering coefficient varies as the fourth power of the wave number if the disc is movable, whereas it is independent of the wave number if the disc is fixed.

1. Introduction. The diffraction of electromagnetic and sound waves by finite obstacles has been the subject of extensive studies in the past. In particular, the diffraction by a circular disc has received considerable attention. Recently Jones [1] reviewed the different methods used for the low- and high-frequency diffraction by a circular disc and gave a method particularly suitable for high-frequency diffraction. In this paper he takes a typical axisymmetric problem of diffraction of plane sound waves by a sound-soft circular disc. Jones [2] also discusses the general nonaxisymmetric problem of diffraction of electromagnetic waves by a circular disc at high frequencies using the method of [1]. The low-frequency diffraction of obliquely incident plane electromagnetic waves was earlier discussed by Lure'e [3], who reduces the dual integral equations governing the nth Fourier component of the diffracted wave into a Fredholm integral equation of the second kind. The kernel of this equation is a Magnus type kernel, which he expresses in terms of Bessel and Struve functions that are expanded in the powers of the small wave number.

In comparison to the vast literature on electromagnetic and sound wave diffraction, the problem of diffraction of elastic waves by finite obstacles has not been discussed at length. The reason, of course, is the inherent coupling of governing equations. Recently Mal et al. [4] have considered the axisymmetric problem of diffraction of compressional waves by an immovable rigid circular disc. By expressing the displacements in terms of two scalar functions and then decomposing these into symmetric and antisymmetric components about the plane of the disc they were able to uncouple the dual integral

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equations governing the vertical and horizontal components of the scattered field. By following a method given in [5] they reduced the dual equations into a single integral equation, which they then proceeded to solve for long wave lengths. Mal [6] uses this same approach to discuss the diffraction of axisymmetric elastic waves by a circular crack in an infinite elastic medium. This last problem has also been discussed by Robertson [7] for an incident plane compressional wave. However, the general nonaxisymmetric problem of diffraction of elastic waves by an arbitrary obstacle is a considerably more difficult problem and does not seem to have received much attention. Mal [8][1] has recently considered a problem of nonaxisymmetric diffraction of a shear wave by a circular crack. The method used is similar to the one used here. In a recent paper Barratt and Collins [9] give the formulae for calculating the scattering cross-section of an arbitrary obstacle (both in two and three dimensions) for an incident plane wave in terms of the amplitudes of the far-field scattered displacement components. This is a generalization of an earlier result of Jones [10] for scalar waves.

The present paper deals with the solution of a nonaxisymmetric elastic diffraction problem which is amenable to analytical tools developed for the electromagnetic and sound wave diffraction. This is the problem of diffraction of a plane obliquely incident compressional wave by a rigid circular disc embedded in an infinite elastic medium. The disc is assumed to be free to move. It is the object of this paper to discuss the motion of the disc and derive expressions for the far-field displacement components. These can then be used to calculate the scattering cross-section by the formulae given in [9]. The method used is as follows.

The displacement components have been expressed in terms of three scalar functions ϕ, ψ and x, which are solutions of the scalar wave equation. By assuming harmonic time-dependence and by assuming suitable forms of the functions ϕ, ψ, and x, it has been possible to uncouple the dual integral equations governing the vertical and angular motion of the disc from those governing the lateral motion. This last is governed by simultaneous dual integral equations and it has not been possible to uncouple them. Then, following a method given by Noble [11], we have reduced the dual equations governing the vertical and angular motion into single Fredholm integral equations of the second kind with Magnus type kernels. For the lateral motion we get a pair of simultaneous equations for each Fourier component. Instead of expressing the kernels in terms of Bessel and Struve functions (as is done in [3]) we express them, following Noble [12], in terms of definite integrals that are suitable for expansion at low frequencies. The equations governing the vertical and angular motions can then be solved by straightforward iteration. The equations for vertical motion reduce to the ones discussed by us [13] when the wave is incident normally. On the other hand, the iterates for the simultaneous equations governing the lateral motion are to be solved from simultaneous integral equations that can be reduced to solving algebraic equations. This approach can be used for other problems of low-frequency diffraction by circular discs and cracks. But this is not suitable for high-frequency diffraction. In a later communication we wish to discuss this latter problem.

2. Equations. Consider cylindrical polar coordinates r, θ, z with origin at the center of the disc and z-axis normal to the plane of the disc. The displacement vector $\mathbf{u}$ satisfies the equation

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[1]The author is grateful to the referee for bringing this paper to his attention.

$$(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - \mu\nabla \wedge \nabla \wedge \mathbf{u} = \rho \frac{\partial^2}{\partial t^2}\mathbf{u}, \tag{1}$$

where λ, μ are Lamé constants and ρ is the density of the elastic medium. The solution to (1) will be assumed of the form $\mathbf{u}(\mathbf{r})e^{-i\omega t}$. The components u_r, u_θ and u_z of $\mathbf{u}(\mathbf{r})$ can be expressed in terms of scalar functions ϕ, ψ and x as

$$\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial x}{\partial\theta}, \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r}\frac{\partial\psi}{\partial\theta\,\partial z} - \frac{\partial x}{\partial r}, \\ u_z &= \frac{\partial\phi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} + k_2^2\psi. \end{aligned} \tag{2}$$

Here ϕ, ψ and x satisfy the equations

$$(\nabla^2 + k_1^2)\phi = 0, \qquad k_1^2 = \omega^2/C_1^2, \tag{3}$$

$$(\nabla^2 + k_2^2)\psi = 0, \qquad k_2^2 = \omega^2/C_2^2, \tag{4}$$

C_1, C_2 being the two wave speeds.

The incident plane compressional wave is given by

$$\phi^{(i)}e^{-i\omega t} = \phi_0 \exp\,[ik_1(z\cos\alpha + x\sin\alpha) - i\omega t], \tag{5}*$$

where α is the direction of propagation of the wave measured from the positive direction of the z-axis.

The reflected field will be given in terms of $\phi^{(r)}$, $\psi^{(r)}$ and $x^{(r)}$. The motion of the disc will be governed by the equations

$$-m\omega^2 U_z = \int_0^1\int_0^{2\pi} [(\sigma_{zz})_+ - (\sigma_{zz})_-]r\,d\theta\,dr, \tag{6}$$

$$-m\omega^2 U_x = \int_0^1\int_0^{2\pi} [(\sigma_{xz})_+ - (\sigma_{xz})_-]r\,d\theta\,dr, \tag{7}$$

$$-I\omega^2\Omega = \int_0^1\int_0^{2\pi} [(\sigma_{zz})_+ - (\sigma_{zz})_-]r^2\cos\theta\,d\theta\,dr, \tag{8}$$

where $\mathbf{U}e^{-i\omega t}$ and $\Omega e^{-i\omega t}$ are respectively the displacement, and rotation about the y-axis, of the disc, m is the mass and I is the moment of inertia of the disc about a diameter. The boundary conditions are, at $z = 0$,

$$u_z^{(i)} + u_z^{(r)} = U_z + r\Omega\cos\theta, \qquad 0 \le r < 1, \tag{9}$$

$$u_r^{(i)} + u_r^{(r)} = U_x\cos\theta, \qquad 0 \le r < 1, \tag{10}$$

$$u_\theta^{(i)} + u_\theta^{(r)} = -U_x\sin\theta, \qquad 0 \le r < 1. \tag{11}$$

Besides, the stresses and the displacement must be continuous across the plane $z = 0$ for $r > 1$.

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*i within brackets stands for the incident field. Otherwise $i = \sqrt{-1}$.

Using the well-known relation

$$\exp\,[ik_1 r \sin\alpha \cos\theta] = \sum_0^\infty \epsilon_n i^n J_n(k_1 r \sin\alpha) \cos n\theta, \qquad \epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0, \end{cases} \tag{12}$$

the incident displacement field on the disc can be written as

$$u_z^{(i)} = u_0 \sum_0^\infty \epsilon_n i^n J_n(k_1 r \sin\alpha) \cos n\theta, \qquad 0 \le r < 1, \tag{13}$$

$$u_r^{(i)} = v_0 \left[iJ_1(k_1 r \sin\alpha) + \sum_1^\infty \{ i^{n+1} J_{n+1}(k_1 r \sin\alpha) + i^{n-1} J_{n-1}(k_1 r \sin\alpha) \} \cos n\theta \right], \qquad 0 \le r < 1, \tag{14}$$

$$u_\theta^{(i)} = -v_0 \sum_{n=1}^\infty \{ i^{n-1} J_{n-1}(k_1 r \sin\alpha) - i^{n+1} J_{n+1}(k_1 r \sin\alpha) \} \sin n\theta, \qquad 0 \le r < 1, \tag{15}$$

where $u_0 = ik_1\phi_0 \cos\alpha$, $v_0 = ik_1\phi_0 \sin\alpha$.

The functions $\phi^{(r)}$, $\psi^{(r)}$ and $x^{(r)}$ can be chosen in the following manner:

$$\phi^{(r)} = \sum_0^\infty \epsilon_n \phi_n^{(r)} \cos n\theta, \tag{16}$$

$$\psi^{(r)} = \sum_0^\infty \epsilon_n \psi_n^{(r)} \cos n\theta, \tag{17}$$

$$x^{(r)} = \sum_1^\infty \epsilon_n x_n^{(r)} \sin n\theta, \tag{18}$$

with

$$\phi_n^{(r)} = \frac{1}{\omega^2} \int_0^\infty \left[\frac{k^2 A_n}{\nu_1} \mp kP_n \right] J_n(kr) \exp\,[-\nu_1 |z|]\, dk, \tag{19}$$

$$\psi_n^{(r)} = \frac{1}{\omega^2} \int_0^\infty \left[\mp C_n + \frac{k}{\nu_2} Q_n \right] J_n(kr) \exp\,[-\nu_2 |z|]\, dk, \tag{20}$$

$$x_n^{(r)} = \frac{1}{\omega^2} \int_0^\infty \frac{k_2^2}{\nu_2} B_n J_n(kr) \exp\,[-\nu_2 |z|]\, dk. \tag{21}$$

Here

$$\begin{aligned} \nu_1 &= (k^2 - k_1^2)^{1/2}, & k \ge k_1 \\ &= -i(k_1^2 - k^2)^{1/2}, & 0 \le k < k_1 , \end{aligned}$$

and

$$\begin{aligned} \nu_2 &= (k^2 - k_2^2)^{1/2}, & k \ge k_2 \\ &= -i(k_2^2 - k^2)^{1/2}, & 0 \le k < k_2 . \end{aligned}$$

The boundary conditions (9)–(11) and Eqs. (13)–(15) then lead to the following equations:

$$C_n = -A_n , \qquad n \ge 0, \tag{22}$$

$$\frac{1}{\omega^2}\int_0^\infty k\left[\frac{k^2}{\nu_2}Q_0+\nu_1P_0\right]J_0(kr)\,dk = U_z - u_0J_0(k_1r\sin\alpha), \qquad 0\le r<1, \tag{23}$$

$$\frac{1}{\omega^2}\int_0^\infty k\left[\frac{k^2}{\nu_2}Q_1+\nu_1P_1\right]J_1(kr)\,dk = +\tfrac{1}{2}r\Omega - iu_0J_1(k_1r\sin\alpha), \qquad 0\le r<1, \tag{24}$$

$$\frac{1}{\omega^2}\int_0^\infty k\left[\frac{k^2}{\nu_2}Q_n+\nu_1P_n\right]J_n(kr)\,dk = -i^nu_0J_n(k_1r\sin\alpha), \qquad 0\le r<1, \quad n>1, \tag{25}$$

$$\int_0^\infty k[(2k^2-k_2^2)P_n+2k^2Q_n]J_n(kr)\,dk = 0, \qquad r>1, \quad n\ge 0, \tag{26}$$

$$Q_n = -P_n, \qquad n\ge 0, \tag{27}$$

$$\frac{1}{\omega^2}\int_0^\infty k\left[\frac{k^2}{\nu_1}-\nu_2\right]A_0J_1(kr)\,dk = iv_0J_1(k_1r\sin\alpha), \qquad 0\le r<1, \tag{28}$$

$$\frac{1}{\omega^2}\int_0^\infty \left[\left(\frac{k^2}{\nu_1}-\nu_2\right)A_1\frac{dJ_1(kr)}{dr}+\frac{k_2^2}{\nu_2}B_1\frac{1}{r}J_1(kr)\right]dk$$
$$= \tfrac{1}{2}U_z + \tfrac{1}{2}v_0J_2(k_1r\sin\alpha) - \tfrac{1}{2}v_0J_0(k_1r\sin\alpha), \qquad 0\le r<1, \tag{29}$$

$$\frac{1}{\omega^2}\int_0^\infty \left[\left(\frac{k^2}{\nu_1}-\nu_2\right)A_1\frac{1}{r}J_1(kr)+\frac{k_2^2}{\nu_2}B_1\frac{dJ_1(kr)}{dr}\right]dk$$
$$= \tfrac{1}{2}U_z - \tfrac{1}{2}v_0J_2(k_1r\sin\alpha) - \tfrac{1}{2}v_0J_0(k_1r\sin\alpha), \qquad 0\le r<1, \tag{30}$$

$$\frac{1}{\omega^2}\int_0^\infty \left[\left(\frac{k^2}{\nu_1}-\nu_2\right)A_n\frac{dJ_n(kr)}{dr}+\frac{k_2^2}{\nu_2}B_n\frac{n}{r}J_n(kr)\right]dk$$
$$= -\tfrac{1}{2}i^{n+1}v_0J_{n+1}(k_1r\sin\alpha) - \tfrac{1}{2}i^{n-1}v_0J_{n-1}(k_1r\sin\alpha), \qquad 0\le r<1, \quad n>1, \tag{31}$$

$$\frac{1}{\omega^2}\int_0^\infty \left[\left(\frac{k^2}{\nu_1}-\nu_2\right)A_n\frac{n}{r}J_n(kr)+\frac{k_2^2}{\nu_2}B_n\frac{dJ_n(kr)}{dr}\right]dk$$
$$= \tfrac{1}{2}i^{n+1}v_0J_{n+1}(k_1r\sin\alpha) - \tfrac{1}{2}i^{n-1}v_0J_{n-1}(k_1r\sin\alpha), \qquad 0\le r<1, \quad n>1, \tag{32}$$

$$\int_0^\infty A_0kJ_1(kr)\,dk = 0, \qquad r>1, \tag{33}$$

$$\int_0^\infty \left[A_n\frac{dJ_n(kr)}{dr}+B_n\frac{n}{r}J_n(kr)\right]dk = 0, \qquad r>1, \quad n\ge 1, \tag{34}$$

$$\int_0^\infty \left[A_n\frac{n}{r}J_n(kr)+B_n\frac{dJ_n(kr)}{dr}\right]dk = 0, \qquad r>1, \quad n\ge 1. \tag{35}$$

Using (27) in Eqs. (23)–(26) one obtains the following dual integral equations to solve for P_0, P_1, and P_n $(n>1)$:

$$\int_0^\infty k\left[\frac{k^2}{\nu_2}-\nu_1\right]P_0J_0(kr)\,dk = -\omega^2U_z + \omega^2u_0J_0(k_1r\sin\alpha), \qquad 0\le r<1, \tag{36a}$$

$$\int_0^\infty kP_0J_0(kr)\,dk = 0, \qquad r>1, \tag{36b}$$

$$\int_0^\infty k\left[\frac{k^2}{\nu_2}-\nu_1\right]P_1J_1(kr)\,dk = -\tfrac{1}{2}\omega^2r\Omega + iu_0J_1(k_1r\sin\alpha), \qquad 0\le r<1, \tag{37a}$$

$$\int_0^\infty kP_1J_1(kr)\,dk = 0, \qquad r > 1, \tag{37b}$$

$$\left.\begin{aligned} \int_0^\infty k\left[\frac{k^2}{\nu_2} - \nu_1\right]P_nJ_n(kr)\,dk &= i^n u_0\omega^2 J_n(k_1 r\sin\alpha), \qquad 0 \le r < 1, \\ \int_0^\infty kP_nJ_n(kr)\,dk &= 0, \qquad r > 1, \end{aligned}\right\} \quad (n > 1). \tag{38a, 38b}$$

Also, using the relations

$$\frac{dJ_n(kr)}{dr} = \frac{k}{2}(J_{n-1} - J_{n+1}), \qquad \frac{n}{r}J_n(kr) = \frac{k}{2}(J_{n-1} + J_{n+1}) \tag{39}$$

in Eqs. (29)–(31) and (33)–(35), one gets the following system of dual integral equations to solve for A_1, B_1, and A_n, B_n $(n > 1)$:

$$\int_0^\infty k\left[\left(\frac{k^2}{\nu_1} - \nu_2\right)A_1 + \frac{k_2^2}{\nu_2}B_1\right]J_0(kr)\,dk = U_x\omega^2 - v_0\omega^2 J_0(k_1 r\sin\alpha), \qquad 0 \le r < 1, \tag{40a}$$

$$\int_0^\infty k\left[\left(\frac{k^2}{\nu_1} - \nu_2\right)A_1 - \frac{k_2^2}{\nu_2}B_1\right]J_2(kr)\,dk = -v_0\omega^2 J_2(k_1 r\sin\alpha), \qquad 0 \le r < 1, \tag{40b}$$

$$\int_0^\infty k[A_1 + B_1]J_0(kr)\,dk = 0, \qquad r > 1, \tag{40c}$$

$$\int_0^\infty k[A_1 - B_1]J_2(kr)\,dk = 0, \qquad r > 1, \tag{40d}$$

$$\int_0^\infty \left[\left(\frac{k^2}{\nu_1} - \nu_2\right)A_n + \frac{k_2^2}{\nu_2}B_n\right]kJ_{n-1}(kr)\,dk = -i^{n-1}\omega^2 v_0 J_{n-1}(k_1 r\sin\alpha), \quad 0 \le r < 1, \tag{41a}$$

$$\int_0^\infty \left[\left(\frac{k^2}{\nu_1} - \nu_2\right)A_n - \frac{k_2^2}{\nu_2}B_n\right]kJ_{n+1}(kr)\,dk = i^{n+1}\omega^2 v_0 J_{n+1}(k_1 r\sin\alpha), \qquad 0 \le r < 1, \tag{41b}$$

$$\int_0^\infty k(A_n + B_n)J_{n-1}(kr)\,dk = 0, \qquad r > 1, \tag{41c}$$

$$\int_0^\infty k(A_n - B_n)J_{n+1}(kr)\,dk = 0, \qquad r > 1. \tag{41d}$$

Eqs. (28) and (33) determine A_0.

3. Solution (vertical and angular oscillation). The solutions P_n $(n \ge 0)$ satisfying Eqs. (36)–(38) can be obtained as (see Noble [11, p. 358,])

$$P_n(k) = \left(\frac{2k}{\pi}\right)^{1/2}\int_0^1 \xi^{1/2}\lambda_n(\xi)J_{n-1/2}(k\xi)\,d\xi, \tag{42}$$

$$\lambda_n(x) + \frac{1}{\pi}\int_0^1 \lambda_n(\xi)T_n(x,\xi)\,d\xi = x^{-n}H_n(x), \qquad 0 \le x < 1, \tag{43}$$

$$T_n(x,\xi) = \pi\int_0^\infty (x\xi)^{1/2}\omega t H(t)J_{n-1/2}(xt)J_{n-1/2}(\xi t)\,dt, \tag{44}$$

$$1 + \omega H(t) = \frac{2k}{k_1^2 + k_2^2}\left(\frac{k^2}{\nu_2} - \nu_1\right), \tag{45}$$

$$H_n(x) = \frac{d}{dx}\int_0^x \frac{f_n(\rho)\rho^{n+1}}{(x^2 - \rho^2)^{1/2}}\,d\rho. \tag{46}$$

Here

$$f_0(\rho) = -\frac{2\omega^2}{k_1^2 + k_2^2}\,[U_z - u_0 J_0(k_1\rho \sin\alpha], \tag{47}$$

$$f_1(\rho) = -\frac{2\omega^2}{k_1^2 + k_2^2}\,[\tfrac{1}{2}\rho\Omega - iu_0 J_1(k_1\rho \sin\alpha)], \tag{48}$$

$$f_n(\rho) = \frac{2\omega^2}{k_1^2 + k_2^2}\, i^n u_0 J_n(k_1\rho \sin\alpha). \tag{49}$$

The kernel $T_n(x, \xi)$ in Eq. (43) can easily be expressed (see Noble [12, pp. 343–344]), for $x > \xi$, as

$$T_n(x, \xi) = \frac{2\pi i k_2^2}{1 + \gamma^2}\,(x\xi)^{1/2}\left[\int_0^1 \frac{t^4}{\sqrt{1 - t^2}}\,H_{n-1/2}^{(1)}(k_2 xt)J_{n-1/2}(k_2\xi t)\,dt \right.$$
$$\left. + \int_0^\gamma t^2\sqrt{\gamma^2 - t^2}\,H_{n-1/2}^{(1)}(k_2 xt)J_{n-1/2}(k_2\xi t)\,dt\right], \qquad \gamma = \frac{k_1}{k_2}. \tag{50}$$

Use of Eqs. (47)–(49) in (46) gives

$$H_0(x) = -\beta[U_z - u_0 \cos(k_1 x \sin\alpha)], \qquad \beta = \frac{2\omega^2}{k_1^2 + k_2^2}, \tag{51}$$

$$H_1(x) = -\beta[\Omega x^2 - iu_0 x \sin(k_1 x \sin\alpha)], \tag{52}$$

$$H_n(x) = i^n \beta u_0 \sqrt{\frac{\pi k_1 \sin\alpha}{2}}\, x^{n+1/2} J_{n-1/2}(k_1 x \sin\alpha). \tag{53}$$

Equation (43) will now be solved approximately for small values of k_1 and k_2. The zeroth-order approximations are

$$\lambda_0^{(0)}(x) = -\beta(U_z - u_0), \tag{54}$$

$$\lambda_1^{(0)}(x) = -\beta\Omega x. \tag{55}$$

It is easily seen that $\lambda_n(x) = O(k_1^n)$ for $n > 1$. Substitution of Eqs. (54)–(55) in (42) gives

$$P_0^{(0)}(k) = \frac{2}{\pi}\lambda_0^{(0)}\frac{\sin k}{k}, \tag{56}$$

$$P_1^{(0)}(k) = -\beta\Omega\left(\frac{2}{\pi k}\right)^{1/2} J_{3/2}(k). \tag{57}$$

These expressions are pertinent to calculating the elastostatic resistance to the displacement of the disc perpendicular to itself and the rotation about the y-axis. Thus if the disc is displaced by an amount U_0 parallel to the z-axis and rotated through an angle Ω_0 about the y-axis, the resistance on the disc will be

$$\mathbf{F}_{(\text{static})} = -U_0 F_0 \mathbf{i}_z, \qquad F_0 = 16\,\mu/(1 + \gamma^2),$$

and the resisting torque τ is given by

$$\tau = -\Omega_0 \tau_0 \mathbf{i}_j, \qquad \tau_0 = 16\,\mu/3(1 + \gamma^2).$$

To obtain higher-order corrections it will be necessary to expand $T_n(x, \xi)$ in powers of k_2. These expansions are

$$T_0(x, \xi) = k_2 T_0^{(1)} + k_2^2 T_0^{(2)} + \cdots, \tag{58}$$

$$T_1(x, \xi) = k_2^2 T_1^{(2)} + \cdots, \tag{59}$$

$$T_2(x, \xi) = k_2^2 T_2^{(2)} + \cdots, \tag{60}$$

with

$$T_0^{(n)} = +\frac{2i^n}{(1+\gamma^2)(n-1)!}[(x+\xi)^{n-1} + (|x-\xi|)^{n-1}] \cdot \int_0^{\pi/2} [\gamma^{n+2}\sin^n\theta\cos^2\theta + \sin^{n+2}\theta]\, d\theta, \qquad n \geq 1, \tag{61}$$

$$T_1^{(n)} = -\frac{2i^n}{(1+\gamma^2)(n-1)!}[(x+\xi)^{n-1} - (|x-\xi|)^{n-1}] \cdot \int_0^{\pi/2} [\gamma^{n+2}\sin^n\theta\cos^2\theta + \sin^{n+2}\theta]\, d\theta, \qquad n \geq 2, \tag{62}$$

$$T_2^{(2)} = \frac{\pi(\gamma^4+3)}{12(1+\gamma^2)}\frac{\xi^2}{x}, \qquad x > \xi. \tag{63}$$

It will be assumed that $\lambda_n(x)$ can be expanded in powers of k_2. Under this assumption, and using (58)–(63) in (43), it is found that

$$\lambda_0^{(0)}(x) = \lambda_0^{(0)}(x) + k_2\lambda_0^{(1)}(x) + k_2^2\lambda_0^{(2)}(x), \tag{64}$$

$$\lambda_1(x) = \lambda_1^{(0)}(x) + k_2\lambda_1^{(1)}(x) + k_2^2\lambda_1^{(2)}(x), \tag{65}$$

$$\lambda_2(x) = k_2^2\lambda_2^{(2)}(x) + \cdots, \tag{66}$$

and the $\lambda_n^{(i)}(x)$ are given by

$$\begin{aligned}
\lambda_0^{(1)}(x) &= -\lambda_0^{(0)} t_1, \qquad t_1 = \frac{4i}{3\pi}\frac{2+\gamma^3}{1+\gamma^2}, \\
\lambda_0^{(2)}(x) &= -\tfrac{1}{2}\beta u_0(\gamma\sin\alpha)^2 x^2 - \lambda_0^{(0)}\left[t_1^2 + \frac{3+\gamma^4}{8(1+\gamma^2)}(1+x^2)\right], \\
\lambda_1^{(1)}(x) &= i\beta u_0(\gamma\sin\alpha)x, \\
\lambda_1^{(2)}(x) &= \frac{\beta\Omega(3+\gamma^4)}{8(1+\gamma^2)}x(1-\tfrac{1}{3}x^2), \\
\lambda_2^{(2)}(x) &= -\tfrac{1}{3}\beta u_0(\gamma\sin\alpha)^2 x^2.
\end{aligned} \tag{67}$$

Now, it can easily be shown that the total thrust on the disc in the z-direction is given by

$$F_z = \frac{8\mu}{C_2^2}\int_0^1 \lambda_0(\xi)\, d\xi. \tag{68}$$

Using this and (64), (67) in Eq. (6) U_z is found to be

$$U_z = u_0 \frac{\mathfrak{F} - \frac{1}{6}k_2^2(\gamma \sin \alpha)^2}{\mathfrak{F} - m\omega^2(1 + \gamma^2)/16\mu}, \tag{69}$$

where

$$\mathfrak{F} = 1 - t_1 k_2 - k_2^2(t_1^2 + (3 + \gamma^4)/6(1 + \gamma^2)).$$

When $\alpha = 0$, this expression for U_z agrees with that obtained in Ref. [13], Eq. (26).

Similarly, the torque on the disc is given by

$$\tau = \frac{8\mu}{C_2^2} \int_0^1 \xi \lambda_1(\xi)\, d\xi. \tag{70}$$

Eq. (70) together with (64) and (67) will give τ, which when substituted for the right-hand side of Eq. (8) gives Ω. Thus

$$\Omega = ik_2 u_0(\gamma \sin \alpha) \Big/ \left[1 - \frac{k_2^2(3 + \gamma^4)}{10(1 + \gamma^2)} - \frac{3I\omega^2(1 + \gamma^2)}{16\mu}\right]. \tag{71}$$

It has been shown by Barratt and Collins [9] that the scattering cross-section of the obstacle due to an incident plane compressional wave can be computed by knowing the first term in the far-field approximate expression for $\phi^{(r)}$. So here an expression for $\phi^{(r)}$ at large distance from the disc due to its vertical and angular oscillations about the y-axis will be obtained. Similar expressions for $\psi^{(r)}$ can also be obtained. Note that the contributions to $\phi^{(r)}$ due to these oscillations are given by the equation

$$\phi_1^{(r)} = -\frac{1}{\omega^2} \sum_{n=0}^{\infty} \epsilon_n \int_0^\infty kP_n J_n(kr) e^{-\gamma_1 z} \cos n\theta\, dk. \tag{72}$$

Using Eq. (42), this can be written as

$$\phi_1^{(r)} = -\frac{1}{\omega^2}\sqrt{\frac{2}{\pi}} \sum_0^\infty \epsilon_n \int_0^\infty k^{3/2} P_n J_n(kr) e^{-\gamma_1 z} \int_0^1 \xi^{1/2} \lambda_n(\xi) J_{n-1/2}(k\xi) \cos n\theta\, d\xi\, dk. \tag{73}$$

It can easily be shown that for large values of R ($R = (r^2 + z^2)^{1/2}$)

$$\phi_1^{(r)} \sim \frac{ik_1}{\omega^2 R} e^{ik_1 R} \cos \theta \sum_0^\infty \epsilon_n P_n(k_1 \sin \theta) \cos n\theta i^{-n}. \tag{74}$$

On substitution of P_n from (42) and using Eq. (67) it may be seen that

$${}_1u_R^P = \frac{\partial \phi_1^{(r)}}{\partial R} \sim u_0 g(\theta) \frac{e^{ik_1 R}}{R}, \tag{75}$$

where

$$g(\theta) = +(2\beta/\pi C_1^2) \cos \theta\, [(U_{z1} - 1)\mathfrak{F} + \tfrac{1}{6}k_2^2(\gamma \sin \alpha)^2 - \tfrac{2}{3}k_1 k_2(\gamma \sin \alpha) \sin \theta \cos \theta (1 - \Omega_1)],$$

$$U_{z1} = \frac{\mathfrak{F} - \frac{1}{6}k_2^2(\gamma \sin \alpha)^2}{\mathfrak{F} - m\omega^2(1 + \gamma^2)/16\mu},$$

$$\Omega_1 = 1 \Big/ \left[1 - \frac{k_2^2(3 + \gamma^4)}{10(1 + \gamma^2)} - \frac{3I\omega^2(1 + \gamma^2)}{16\mu}\right]. \tag{76}$$

For $\alpha = 0$ the expression for $g(\theta)$ agrees with the result derived in Ref. [13], Eq. 31.

4. Solution (lateral oscillation). The solution A_0 satisfying (28) and (33) is

$$A_0 = \left(\frac{2k}{\pi}\right)^{1/2} \int_0^1 \xi^{1/2}\theta_0(\xi)J_{1/2}(\xi k)\, d\xi, \tag{77}$$

$$\theta_0(x) + \frac{1}{\pi}\int_0^1 \theta_0(\xi)M_0(x, \xi)\, d\xi = x^{-1}G_{10}(x), \qquad 0 \leq x < 1, \tag{78}$$

where

$$M_0(x, \xi) = \pi \int_0^\infty \omega t L(t)(x\xi)^{1/2}J_{1/2}(xt)J_{1/2}(\xi t)\, dt,$$

$$1 + \omega L(t) = \frac{2t}{k_1^2 + k_2^2}\left(\frac{t^2}{\nu_1} - \nu_2\right),$$

$$G_{10}(x) = i v_0 \beta \frac{d}{dx}\int_0^x \frac{\rho^2 J_1(k_1\rho \sin\alpha)}{(x^2 - \rho^2)^{1/2}}\, d\rho.$$

In the same way as in the previous section one obtains an expansion for $\theta_0(x)$ in powers of k_2 as

$$\theta_0(x) = k_2\theta_0^{(1)} + O(k_2^3), \tag{79}$$

with

$$\theta_0^{(1)} = i v_0 \beta(\gamma \sin\alpha)x.$$

To solve for A_n, B_n $(n \geq 1)$ satisfying Eqs. (40), (41) let

$$A_n + B_n = p_n, \qquad A_n - B_n = q_n.$$

Then Eqs. (40)–(41) transform to

$$\int_0^\infty [1 + \omega N(k)]p_n J_{n-1}(kr)\, dk$$
$$= \alpha_0[U_x\delta_{1n} - i^{n-1}v_0 J_{n-1}(k_1 r \sin\alpha)] - f_n(r), \qquad 0 \leq r < 1, \tag{80}$$

$$\int_0^\infty [1 + \omega N(k)]q_n J_{n+1}(kr)\, dk = i^{n+1}\alpha_0 v_0 J_{n+1}(k_1 r \sin\alpha) - g_n(r), \qquad 0 \leq r < 1, \tag{81}$$

$$\int_0^\infty kN(k)p_n J_{n-1}(kr)\, dk = 0, \qquad r > 1, \tag{82}$$

$$\int_0^\infty kN(k)q_n J_{n+1}(kr)\, dk = 0, \qquad r > 1, \tag{83}$$

where

$$\delta_{1n} = 1, \qquad n = 1$$
$$= 0, \qquad n \neq 1$$

$$\alpha_0 = \frac{4\omega^2}{k_1^2 + 3k_2^2},$$

$$1 + \omega N(k) = \frac{2k}{k_1^2 + 3k_2^2}\left[\frac{k^2}{\nu_1} - \nu_2 + \frac{k_2^2}{\nu_2}\right],$$

$$f_n(r) = \alpha \int_0^\infty [1 + \omega K(k)] q_n J_{n-1}(kr)\, dk,$$

$$g_n(r) = \alpha \int_0^\infty [1 + \omega K(k)] p_n J_{n+1}(kr)\, dk,$$

$$1 + \omega K(k) = \frac{2k}{k_1^2 - k_2^2}\left[\frac{k^2}{\nu_1} - \nu_2 - \frac{k_2^2}{\nu_2}\right], \qquad \alpha = \frac{k_1^2 - k_2^2}{k_1^2 + 3k_2^2}.$$

The solutions p_n, q_n satisfying Eqs. (80)–(83) are

$$p_n = \left(\frac{2k}{\pi}\right)^{1/2} \int_0^1 \xi^{1/2}\theta_n(\xi) J_{n-3/2}(k\xi)\, d\xi, \tag{84}$$

$$q_n = \left(\frac{2k}{\pi}\right)^{1/2} \int_0^1 \xi^{1/2}\mu_n(\xi) J_{n+1/2}(k\xi)\, d\xi, \tag{85}$$

$$\theta_n(x) + \frac{1}{\pi}\int_0^1 \theta_n(\xi) M_n(x, \xi)\, d\xi = x^{-n+1} F_{1n}(x), \qquad 0 \le x < 1, \tag{86}$$

$$\mu_n(x) + \frac{1}{\pi}\int_0^1 \mu_n(\xi) M_{n+2}(x, \xi)\, d\xi = x^{-n-1} G_{1n}(x), \qquad 0 \le x < 1, \tag{87}$$

where

$$M_n(x, \xi) = \pi(x\xi)^{1/2} \int_0^\infty \omega t N(t) J_{n-3/2}(xt) J_{n-3/2}(\xi t)\, dt, \tag{88}$$

$$F_{1n}(x) = \frac{d}{dx}\int_0^x [\alpha_0 U_x \delta_{1n} - i^{n-1}\alpha_0 v_0 J_{n-1}(k_1\rho \sin\alpha) - f_n(\rho)] \frac{\rho^n}{(x^2 - \rho^2)^{1/2}}\, d\rho, \tag{89}$$

$$G_{1n}(x) = \frac{d}{dx}\int_0^x [i^{n+1}\alpha_0 v_0 J_{n+1}(k_1\rho \sin\alpha) - g_n(\rho)] \frac{\rho^{n+2}}{(x^2 - \rho^2)^{1/2}}\, d\rho. \tag{90}$$

In the same way as in the previous section it can be shown that

$$M_n(x, \xi) = \frac{2\pi i k_2^2}{3 + \gamma^2}(x\xi)^{1/2}\left[\int_0^\gamma \frac{t^4}{(\gamma^2 - t^2)^{1/2}} H_{n-3/2}^{(1)}(k_2 xt) J_{n-3/2}(k_2\xi t)\, dt \right.$$
$$\left. + \int_0^1 t^2 (1 - t^2)^{1/2} + \frac{1}{(1 - t^2)^{1/2}} H_{n-3/2}^{(1)}(k_2 xt) J_{n-3/2}(k_2\xi t)\, dt\right], \qquad x > \xi. \tag{91}$$

In the following it will be assumed that θ_n, μ_n can be expanded in series of powers of k_2 for small k_2. First let $n = 1$.

The zeroth-order approximation can easily be shown to be

$$\mu_1^{(0)} = 0, \qquad \theta_1^{(0)} = \alpha_0(U_x - v_0). \tag{92}$$

Eq. (92) can be used to calculate the resistance to the lateral displacement of a rigid circular disc embedded in an infinite elastic medium. Note that the force on the disc causing its lateral motion is given by

$$F_x e^{-i\omega t} = -\frac{16\mu}{C_2^2}\int_0^1 \theta_1(\xi) e^{-i\omega t}\, d\xi. \tag{93}$$

Thus the elastostatic resistance to the lateral displacement V_0 of the disc will be

$$F_{x(\text{static})} = -\frac{16\mu\alpha_0 V_0}{C_2^2}. \tag{94}$$

To obtain first-order corrections to θ_1, μ_1 it may be noted that

$$\int_0^x \frac{\rho f_1(\rho)}{(x^2 - \rho^2)^{1/2}}\, d\rho \simeq k_2\alpha \int_x^1 \frac{x}{\xi}\, \mu_1^{(1)}(\xi)\, d\xi, \tag{95}$$

$$\int_0^x \frac{\rho^3 g_1(\rho)}{(x^2 - \rho^2)^{1/2}}\, d\rho \simeq \tfrac{1}{2}k_2\alpha \int_0^x x^2\theta_1^{(1)}(\xi)\left(1 - \frac{3\xi^2}{x^2}\right) d\xi, \tag{96}$$

$$M_1^{(1)}(x, \xi) \simeq \frac{4ik_2}{3 + \gamma^2}\, m_1^{(1)}, \qquad m_1^{(1)} = \tfrac{2}{3}(\gamma^3 + 2), \tag{96}$$

$$\frac{d}{dx}\int_0^x \frac{\rho J_0(k_1\rho \sin\alpha)}{(x^2 - \rho^2)^{1/2}}\, d\rho \simeq 1 - \tfrac{1}{2}(k_1 \sin\alpha)^2 x^2, \tag{97}$$

$$\frac{d}{dx}\int_0^x \frac{\rho^3 J_2(k_1\rho \sin\alpha)}{(x^2 - \rho^2)^{1/2}}\, d\rho \simeq \tfrac{1}{3}(k_1 \sin\alpha)^2 x^4. \tag{98}$$

Thus Eqs. (86)–(87) will lead to the following equations for determination of the first-order corrections $\theta_1^{(1)}(x)$ and $\mu_1^{(1)}(x)$:

$$\theta_1^{(1)}(x) + \frac{4i\theta_1^{(0)}}{\pi(3 + \gamma^2)}\, m_1^{(1)} = -\alpha \frac{d}{dx}\int_x^1 \frac{x}{\xi}\, \mu_1^{(1)}(\xi)\, d\xi, \tag{99}$$

$$\mu_1^{(1)}(x) = -\frac{\alpha}{2x^2}\frac{d}{dx}\int_0^x x^2\theta_1^{(1)}(\xi)\left(1 - \frac{3\xi^2}{x^2}\right) d\xi. \tag{100}$$

It is easily seen that these equations have the solutions

$$\theta_1^{(1)}(x) = -\frac{4i\theta_1^{(0)}}{\pi(3 + \gamma^2)}\, m_1^{(1)}, \qquad \mu_1^{(1)}(x) = 0. \tag{101}$$

Second-order corrections can similarly be obtained to be

$$\theta_1^{(2)}(x) = A + Bx^2, \tag{102}$$

$$\mu_1^{(2)}(x) = [C + \tfrac{2}{3}\alpha B]x^2, \tag{103}$$

where

$$B = \frac{1}{1 - \alpha^2}\left[\frac{2\theta_1^{(0)} m_1^{(2)}}{\pi(3 + \gamma^2)} + \tfrac{1}{2}\alpha_0 v_0(\gamma \sin\alpha)^2 + \tfrac{3}{2}\alpha C\right],$$

$$C = -\tfrac{1}{3}\alpha_0 v_0(\gamma \sin\alpha)^2 - \frac{15\alpha}{16(\gamma^2 - 1)}\, \theta_1^{(0)},$$

$$A = -\tfrac{1}{3}\alpha^2 B - \frac{1}{\pi}\left(\frac{4im_1^{(1)}\theta_1^{(1)}}{3 + \gamma^2} - 2\,\frac{\theta_1^{(0)} m_1^{(2)}}{3 + \gamma^2}\right) - \tfrac{1}{2}\alpha C,$$

$$m_1^{(2)} = \frac{\pi}{16}(3\gamma^4 + 5).$$

Now using

$$\theta_1(x) = \theta_1^{(0)}(x) + k_2\theta_1^{(1)}(x) + k_2^2\theta_1^{(2)}(x), \tag{104}$$

in Eq. (93) and in turn in Eq. (7), one obtains

$$U_x = v_0\, \frac{\mathcal{G} - \tfrac{1}{6}k_2^2(\gamma \sin\alpha)^2}{\mathcal{G} - m\omega^2 C_2^2/16\mu\alpha_0} \tag{105}$$

where

$$\mathcal{G} = 1 - \frac{1}{\pi} M_1^{(1)} + k_2^2\left[\frac{8m_1^{(2)}}{\pi(3+\gamma^2)} + \frac{(M_1^{(1)})^2}{\pi^2 k_2^2}\right].$$

Eq. (105) can be used to calculate the resistance to forced lateral oscillation of amplitude v_0 of a rigid circular disc embedded in an infinite elastic medium to be

$$F_x e^{-i\omega t} = -\frac{16\mu\alpha_0 v_0}{C_2^2}\,\mathcal{G}e^{-i\omega t}. \tag{106}$$

For $n = 2$,

$$\theta_2 \simeq -i\alpha_0 k_1 v_0 \sin \alpha x + O(k_1^3), \qquad \mu_2 \simeq O(k_1^3). \tag{107}$$

As before, an approximate expression for the potential function $\phi^{(r)}$ due to the lateral oscillation can be obtained to be

$$\phi_2^{(r)} \sim \frac{k_1 \sin\theta}{\omega^2} \sum_0^\infty \epsilon_n i^{-n} A_n(k_1 \sin\theta) \cos n\theta \frac{e^{ik_1 R}}{R}. \tag{108}$$

The contribution of $\phi_2^{(r)}$ to the radial component of the displacement is then

$${}_2u_R^P = \frac{\partial\phi_2^{(r)}}{\partial R} \sim v_0 h(\theta)\frac{e^{ik_1 R}}{R}, \tag{109}$$

where

$$h(\theta) = \frac{\sin\theta}{C_1^2}\left[-k_1^2\left(\frac{2\beta}{3\pi} + \frac{2\alpha_0}{3\pi}\cos 2\theta\right)\sin\alpha\sin\theta \right.$$
$$\left. + \frac{2\alpha_0}{\pi}\{(U_{x1} - 1)[\mathcal{G} - \tfrac{1}{6}k_1^2(\gamma\sin\theta)^2] + \tfrac{1}{6}k_2^2(\gamma\sin\alpha)^2\}\cos\theta\right], \tag{110}$$

$U_{x1} = U_x/v_0$.

The scattering cross-section Σ_p can now be calculated from the equation

$$\Sigma_p = \frac{4\pi}{k_1}\mathcal{I}[g(\alpha) + h(\alpha)]. \tag{111}$$

It can be verified that Σ_p is proportional to k_2^4 .

In conclusion it may be noted that if the disc was kept fixed, the scattering cross-section would be independent of the wave-length in the limit of long wave-length.

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