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ON WEIGHTED AVERAGES AT A JUMP DISCONTINUITY*

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Muldoon [1], following some work of Lorch and Szego [2], has investigated what amount to integral functionals which select the weighted average $\frac{1}{3}f(0^+) + \frac{2}{3}f(0^-)$ of a function at a jump discontinuity, as opposed to the usual $\frac{1}{2}(f0^+) + \frac{1}{2}f(0^-)$ of the Dirichlet kernel of Fourier series. We indicate here a simple modification of Muldoon's results to get other weighted averages. We discuss a conjecture about the relation of the various weights to the analytic continuation of the kernel functions.

The results obtained by Muldoon are essentially the following. Under certain rather natural conditions on $f(x)$ one can obtain results of the form

$$\lim_{\nu\to\infty} \int_{-\infty}^{\infty} f(t)\nu Ai[\nu(t - x)]\, dt = \tfrac{1}{3}f(x^+) + \tfrac{2}{3}f(x^-) \tag{1}$$

with corresponding results on the appropriate half intervals. These results use the fact that the Airy function $Ai(z)$ is that solution of

$$y'' - zy = 0 \tag{2}$$

which is strictly decaying for positive z and purely oscillatory for negative z.

Our initial generalization is then to look at the integral

$$F(x) = \lim_{\nu\to\infty} \int_a^b f(t)h_\nu(t - x)\, dt \tag{3}$$

where the kernel $h_\nu(z)$ is the appropriate solution of

$$(d^2y/dz^2) + \nu^{m-2}(-1)^m z^{m-2} y = 0. \tag{4}$$

(It turns out to be convenient to write the exponent as $m - 2$.) It is well known (e.g., see Watson [3, p. 97]) that (4) is related to Bessel's equation, and that the solutions to (4) may be written as combinations of

$$y_k = \sqrt{z}\, B_{1/m}\left(\frac{2\nu}{m} z^{m/2}\right) \tag{5}$$

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where $B_{1/m}(\tau)$ is any of the appropriate Bessel functions. It is necessary to make a distinction between odd and even m. We temporarily consider the case where m is odd. Then the appropriate solution turns out to be

$$h_\nu(z) = A(\nu)\sqrt{z}\,K_{1/m}\left(\frac{2\nu}{m} z^{m/2}\right) \tag{6}$$

where the normalizing factor is conveniently chosen as

$$A(\nu) = \nu^{3/m}\{\Gamma(1/m)\Gamma(2/m)[1 + \cos\pi/m](m^{-1+3/m})\}^{-1} \tag{7}$$

and $K_{1/m}$ is the well-known modified Bessel function. It is easily seen that, for $m = 3$, Eq. (6) reduces to

$$h_\nu(z) = \nu\frac{2\sin\pi/3}{1 + \cos\pi/3}\sqrt{z}\,K_{1/3}(\tfrac{2}{3}\nu z^{3/2}) = \nu^{2/3}Ai(\nu^{2/3}z)$$

which is essentially Muldoon's kernel.

Our normalization factor (7) is chosen as a result of the following formulas taken from Magnus, Oberhettinger and Soni [4]:

$$\int_0^\infty t^{\mu-1}K_\alpha(at)\,dt = 2^{\mu-2}a^{-\mu}\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\alpha)\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\alpha), \quad \mathrm{Re}\,(\mu \pm \alpha) > 0, \tag{8}$$

$$\int_0^\infty t^{\mu-1}J_\alpha(at)\,dt = 2^{\mu-1}a^{-\mu}\frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\alpha)}{\Gamma(1 + \frac{1}{2}\alpha - \frac{1}{2}\mu)}, \quad -\mathrm{Re}\,\alpha < \mathrm{Re}\,\mu < \tfrac{3}{2}, \tag{9}$$

and

$$2K_\alpha(te^{l\pi i}) = \Gamma(\alpha)\Gamma(1-\alpha)[e^{-\pi i\alpha(2l-1)/2}J_{-\alpha}(te^{\pi i/2}) - e^{\pi i\alpha(2l-1)/2}J_\alpha(te^{\pi i/2})]. \tag{10}$$

Eq. (10) is derived from formulas on p. 66 and p. 69 of [4] and is needed because of the multivalued character of the Bessel functions. We use these formulas to derive

THEOREM 1. *For ν real and positive and $h_\nu(z)$ defined by* (6) *and* (7) *we have, for m odd and* ≥ 3,

$$\int_0^\infty h_\nu(z)\,dz = \frac{1}{2(1 + \cos\pi/m)}, \tag{11}$$

$$\int_{-\infty}^0 h_\nu(z)\,dz = \frac{(1 + 2\cos\pi/m)}{2(1 + \cos\pi/m)} \tag{12}$$

Proof. Eq. (11) follows from the definitions (6) and (7) and formula (8), under the elementary change of variables $t = z^{m/2}$. Since z is real and positive we take the real positive square root. We obtain $\alpha = 1/m$, $a = 2\nu/m$ and $\mu = 3/m$. The restriction on (8), Re $(\mu \pm \alpha) > 0$, then holds for all $m > 0$.

Eq. (12) follows from (6) and (7) with formulas (9) and (10). Some care must be taken with the root of z. We take $\arg z = \pi$. This makes the variable (at) in (9) real if we take $l = m + 1/2$ in (10). For the use of (9) we have $t = (e^{-\pi i}z)^{m/2}$, and again $\alpha = \pm 1/m$, $a = 2\nu/m$ and $\mu = 3/m$. Thus the restrictions on (9) are satisfied for $m > 2$, i.e.: all positive integer exponents in (4). Straightforward calculations give (12). Q.E.D.

The normalization factor (7) is now obvious. We note that (11) and (12) reduce to Muldoon's results for $m = 3$, as they should.

The results of Theorem 1 are for m odd. The case of m even requires a slightly different kernel. The integrals of both a modified Bessel function K_ν and an ordinary Bessel function J_ν converge when integrated over the positive real axis. However, examination of (10) shows that the integral of K_ν diverges over the negative real axis. This leads us to the conclusion that the kernel must be essentially oscillatory (i.e., a Bessel function) on both sides of the origin. This is what one might expect, since for $m = 2$ Eq. (4) shows that the desired integrals are essentially the Fourier integrals. Thus, for m even, we take for our kernel

$$h_\nu(z) = B(\nu)\sqrt{z}\; Y_{1/m}\left(\frac{2\nu}{m} z^{m/2}\right), \tag{13}$$

where it turns out that $B(\nu)$ is conveniently taken to be

$$B(\nu) = \frac{m^{-3/m}\nu^{3/m}\pi}{2(1+a)\cos(\pi/m)\Gamma(1-1/m)\Gamma(2/m)} \quad \text{and} \quad a = \frac{\Gamma(1/m)}{\Gamma(-1/m)}. \tag{14}$$

We then find

THEOREM 2. *For ν real and positive and $h_\nu(z)$ defined by* (13) *and* (14) *we have*

$$\int_0^\infty h_\nu(z)\,dz = \frac{1}{2(1+a)} \tag{15}$$

$$\int_{-\infty}^0 h_\nu(z)\,dz = \frac{1+2a}{2(1+a)}, \tag{16}$$

where $m = 4, 6, \ldots$

Proof. Eqs. (15) and (16) are straightforward consequences of the formulas of [4, p. 91 and p. 68]. By direct computation the answer is independent of whether $\arg z$ is π or $-\pi$ in (16), where we use

$$\int_0^\infty t^{\mu-1}Y_\alpha(at)\,dt = -2^{\mu-1}\pi^{-1}a^{-\mu}\cos[\pi/2(\mu-\alpha)] \times \Gamma(\tfrac{1}{2}\alpha - \tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}\alpha + \tfrac{1}{2}\mu) \tag{17}$$

$$\operatorname{Re}(\mu \pm \alpha) > 0, \quad \operatorname{Re}\mu < 3/2$$

$$Y_\alpha(te^{il\pi}) = e^{-il\pi\alpha}Y_\alpha(t) + 2i\sin(l\pi\alpha)\cot\pi\alpha J_\alpha(t),$$

with exactly the same substitutions as in Theorem 1. Q.E.D.

We now take these kernels and show that one may obtain the obvious generalizations of Eq. (1). There are a number of such results possible for different conditions on $f(x)$. Some general formulas were obtained by Lebesgue [5], with some rather restrictive conditions on $f(x)$. Somewhat less restrictive results were obtained by Muldoon for the Airy function kernel. We give just one of the more useful modifications of Muldoon's results for the above kernels.

For our proof we will need not only the results of Theorem 1 but what amounts to a localization property, namely that as $\nu \to \infty$ all of the value of the integrals (11) and (12) comes from the neighborhood of the origin. Thus we need

LEMMA 1. *For ν real and positive and $h_\nu(z)$ given by* (6) *and* (7), *for any fixed* $\eta > 0$,

$$\lim_{\nu\to\infty}\int_0^\eta h_\nu(z)\,dz = \frac{1}{2(1+\cos\pi/m)}$$

for m odd and $\geqq 3$.

Proof. It is sufficient to show that

$$\lim_{\nu \to \infty} \int_{\eta}^{\infty} h_\nu(z)\, dz = 0.$$

But this equation is a specific case of a result (Eq. (25)) proven as part of Theorem 3.

For the interval $(-\infty, 0)$ the desired result is a bit harder to prove. We need

LEMMA 2. *For ν real and positive and $h_\nu(z)$ is given by* (6) *and* (7), *for any fixed* $\eta > 0$,

$$\lim_{\nu \to \infty} \int_{-\eta}^{0} h_\nu(z)\, dz = \frac{(1 + 2 \cos \pi/m)}{2(1 + \cos \pi/m)}$$

for m odd and $\geqq 3$.

Proof. Again it is sufficient to show that

$$\lim_{\nu \to \infty} \int_{-\infty}^{-\eta} h_\nu(z)\, dz = 0.$$

From (6), (7), and (10) we need to consider integrals of the form

$$\lim_{\nu \to \infty} C_1 \nu^{3/m} \int_{\eta}^{\infty} \sqrt{t}\, \mathcal{C}_{1/m}\left(\frac{2\nu}{m} t^{m/2}\right) dt$$

where C_1 is independent of ν and $\mathcal{C}$ is a Bessel function of real argument. Making a change of variables $\tau = t^{m/2}$, we find we must consider

$$\lim_{\nu \to \infty} \nu^{3/m} \int_{\eta}^{\infty} \tau^{3/m-1} \mathcal{C}_{1/m}\left(\frac{2\nu}{m} \tau\right) d\tau. \tag{18}$$

Now (18) is an improper integral whose integrand is not in $\mathcal{L}_1$. We interpret (18) as

$$\lim_{\nu \to \infty} \lim_{b \to \infty} \int_{\eta}^{b} \tau^{3/m-1} \mathcal{C}_{1/m}\left(\frac{2\nu}{m} \tau\right) d\tau \tag{18a}$$

For fixed η take ν so large that we may use the asymptotic representation of the Bessel function. Then (18a) may be written

$$\lim_{\nu \to \infty} \lim_{b \to \infty} C_2 \nu^{3/m-1/2} \int_{\eta}^{b} \tau^{3/m-3/2} \cos\left(\frac{2\nu}{m} \tau + \alpha\right) [1 + O(1/\nu\tau)]\, d\tau$$

$$= \lim_{\nu \to \infty} \lim_{b \to \infty} \left\{ C_2 \nu^{3/m-3/2} \left[+ \tau^{3/m-3/2}(1 + O(1/\nu\tau)) \sin(2\nu\tau/m + \alpha) \Big|_{\eta}^{b} \right.\right.$$

$$\left.\left. + (3/m - 3/2) \int_{\eta}^{b} \tau^{3/m-5/2} f(\tau) \sin\left(\frac{2\nu\tau}{m} + \alpha\right) d\tau \right]\right\}$$

where C_2 is independent of ν and $f(\tau)$ is obtained by differentiating the known asymptotic expressions for the Bessel functions and is easily shown to be $1 + O(1/\nu\tau)$.

Taking the limit $b \to \infty$ for the integrated term gives zero for the upper limit for $m > 2$. The term under the integral sign is in $\mathcal{L}_2$ and is thus no problem. Thus (18a) becomes

$$\lim_{\nu\to\infty}\left\{C_2\nu^{3/m-3/2}\left\{\eta^{3/m-3/2}(1+O(1/\nu\eta))\sin\left(\frac{2\nu}{m}\eta+\alpha\right)\right.\right.$$
$$\left.\left.+(3/m-3/2)\int_\eta^\infty \tau^{3/m-5/2}f(\tau)\sin\left(\frac{2\nu}{m}\tau+\alpha\right)d\tau\right\}\right\}.$$

But this limit is clearly zero. Q.E.D.

The following theorem is then typical of our principal results.

THEOREM 3. *Let $f(x)$ be defined on $(-b, \infty)$ where $b > 0$ and $x > -b$. Assume*

(i) *f is of bounded variation on $[-b, x]$*

(ii) *f is Lebesgue integrable on $[x, a]$, each $a > x$.*

(iii) $\lim_{a\to\infty}\int_a^\infty f(x+ct)\exp(-(2/m)t^{m/2})\,dt = 0$ *uniformly in* c, $0 \leqq c \leqq 1$

(iv) $\lim_{h\to 0+} h^{-1}\int_x^{x\pm h}[f(t)-f(x\pm)]\,dt = 0.$

Then, if m is odd and ≥ 3,

$$\lim_{\nu\to\infty}\int_{-b}^\infty f(t)h_\nu(t-x)\,dt = \alpha f(x^+)+\beta f(x^-) \tag{19}$$

where

$$\alpha = 1/2(1+\cos\pi/m) \tag{20}$$

$$\beta = (1+2\cos\pi/m)/(2(1+\cos\pi/m)) \tag{21}$$

Proof. Before our proof we remark on the hypotheses. The distinction in (i) and (ii) is due to the fact that the kernel is decaying for positve values of the argument and merely oscillatory for negative values. This fact also makes it difficult to extend b to infinity. (It can be done, however.) The hypothesis (iii) is the obvious growth restriction so that (19) converges at the upper limit.

We break (19) into two parts, integrating from $-b$ to x and x to ∞. Consider the second part. Given $\epsilon > 0$, by (iv) there is an $\eta > 0$ such that

$$\left|h^{-1}\int_x^{x+h}[f(t)-f(x^+)]\,dt\right| < \epsilon \qquad 0 < h \leq \eta$$

for $-\delta < x < \delta$. We then write

$$\int_x^\infty f(t)h_\nu(t-x)\,dt \equiv f(x^+)\int_x^{x+\eta}h_\nu(t-x)\,dt$$
$$+\int_x^{x+\eta}[f(t)-f(x^+)]h_\nu(t-x)\,dt \tag{22}$$
$$+\int_{+\eta}^\infty f(t)h_\nu(t-x)\,dt.$$

The first term will give the desired answer and the second two, in the limit $\nu \to \infty$, we shall show are small. To show this we look at (6)

$$h_\nu(t-x) = A(\nu)\sqrt{t-x}\,K_{1/m}\left(\frac{2\nu}{m}(t-x)^{m/2}\right)$$

The change of variables $(t - x) = z$ makes the first integral in (22), $f(x^+) \int_0^\eta h_\nu(z)\,dz$. Then by virtue of Lemma 1, we have

$$\lim_{\nu\to\infty} f(x^+)\int_0^\eta h_\nu(z)\,dz = \frac{1}{2(1+\cos\pi/m)} f(x^+). \tag{23}$$

From the well-known properties of the modified Bessel functions $h_\nu(t-x)$ is a positive decreasing function of its argument for positive argument. Thus we may apply a theorem in Natanson [6, p. 16], such that, if (ii) and (iv) are satisfied and $h_\nu(t-x)$ has the decreasing property above, then

$$\left|\int_x^{x+\eta} [f(t) - f(x^+)]h_\nu(t-x)\,dt\right| < \epsilon\int_0^\eta h_\nu(t)\,dt < \frac{\epsilon}{2(1+\cos\pi/m)}, \tag{24}$$

for all $\eta > 0$ and all x in $[-b, a]$. As for the third term we use (6) again and the asymptotic properties of $h_\nu(z)$. The change of variables $(t - x) = t\nu^{-2/m}$ yields

$$\left|\int_{x+\eta}^\infty f(t)h_\nu(t-x)\,dt\right| = \left|\nu^{-3/m}\int_{\eta\nu^{2/m}}^\infty f(x+z\nu^{-2/m})A(\nu)\sqrt{z}\,K_{1/m}\left(\frac{2}{m}z^{m/2}\right)dz\right|. \tag{25}$$

Since for z sufficiently large we may take $K_{1/m}(2/mz^{m/2}) \leqq Cz^{-1/2}e^{-2/m}z^{m/2}$, it follows from (iii) and (7) that ν may be taken so large that (25) is less than

$$\frac{\epsilon}{2}\left(\frac{1+2\cos\pi/m}{1+\cos\pi/m}\right).$$

From (23), (24) and (25) we have

$$\lim_{\nu\to\infty}\int_x^\infty f(t)h_\nu(t-x)\,dt = \frac{f(x^+)}{2(1+\cos\pi/m)}. \tag{26}$$

(The condition (iii) on $f(x)$, for large x, may be weakened to that of Muldoon [1, Th. 2.1], with considerably more effort in the proof.)

For the negative half range we start the same way. Given $\epsilon > 0$ there is, by (iv), an $\eta > 0$ such that

$$\left|\eta^{-1}\int_{x-\eta}^x [f(t) - f(x^-)]\,dt\right| < \epsilon.$$

Then we take

$$\begin{aligned}\int_{-b}^x f(t)h_\nu(t-x)\,dt &= f(x^-)\int_{x-\eta}^x h_\nu(t-x)\,dt \\ &\quad + \int_{x-\eta}^x [f(t) - f(x^-)]h_\nu(t-x)\,dt \\ &\quad + \int_{-b}^{x-\eta} f(t)h_\nu(t-x)\,dt.\end{aligned}$$

As before, from Lemma 2,

$$\lim_{\nu\to\infty} f(x^-) \int_{x-\eta}^{x} h_\nu(t-x)\,dt = f(x^-)\,\frac{1+2\cos\pi/m}{2(1+\cos\pi/m)}. \tag{27}$$

For the second integral, since f is of bounded variation we may take η so small that $f(t) - f(x^-)$ is of one sign. Then

$$\left|\int_{x-\eta}^{x} [f(t) - f(x^-)]h_\nu(t-x)\,dt\right| \leqq \max|f(t) - f(x^-)|\,\left|\int_{x-\eta}^{x} h_\nu(t-x)\,dt\right|.$$

But by (10) and the known properties of Bessel functions for real arguments the latter integral is bounded for all ν, say by M, and thus

$$\left|\int_{x-\eta}^{x} [f(t) - f(x^-)]h_\nu(t-x)\,dt\right| < \epsilon M. \tag{28}$$

For the third integral we make the change of variables $-(t-x) = \tau^{2/m}$, use formula (10) and the asymptotic properties of the Bessel functions $J_{1/m}$, to obtain

$$\left|\lim_{\nu\to\infty}\int_{-b}^{x-\eta} f(t)h_\nu(t-x)\,dt\right|$$

$$= \left|\lim_{\nu\to\infty} K\nu^{3/m-1/2}\int_{-(b+x)^{m/2}}^{-\eta^{m/2}} f(x+\tau^{2/m})\tau^{-3(1/m-1/2)}g(\nu\tau)\,d\tau\right| \tag{29}$$

where K is composed of a number of irrelevant factors, and

$$g(\nu\tau) = \cos 2/n(\nu\tau + \varphi)\{1 + O(1/\nu)\}.$$

The integral is, since the range of integration is fixed, the Fourier coefficient of the function $f(x+\tau^{2/m})(1+O(1/\nu))\tau^{3(1/m-1/2)}$, which is of bounded variation for $\tau \neq 0$. By a theorem in Titchmarsh [7, p. 427], this coefficient is $O(1/\nu)$ and thus the right-hand side of (29) vanishes as $\nu^{-3(1/2-1/m)}$, since $m \geqq 3$. Thus by (27), (28) and (29) we have

$$\lim_{\nu\to\infty}\int_{-b}^{x} f(t)h_\nu(t-x)\,dt = \frac{1+2\cos\pi/m}{2(1+\cos\pi/m)}. \qquad \text{Q.E.D.} \tag{30}$$

With some more careful estimates the lower limit of integration can be extended to $-\infty$. We shall not make that extension here.

For the kernels with m even, the corresponding theorem is just the natural modifica-ion of Theorem 3, taking for the hypotheses on f, for positive x, the mirror images f the hypotheses on f for negative x.

It is interesting to toy with the question of other convenient kernels and the resultant eighting factors. We shall state some of the results of calculations with other Bessel nctions. Muldoon has tried the Airy function $Bi(z)$ and finds an infinite "weighting ctor" on the positive side and zero on the negative. For m even we tried $J_{-1/m}$ and und the weights were 1/2, independent of m.

Adding these two results to the results of our theorems, it appears that the controlling tor is the analytic continuation of the kernel. For z negative all of the kernels in-tigated are oscillatory. The feature which distinguishes the $Bi(z)$ kernel is that its lytic continuation to the positive real axis is unbounded. This, of course, gives the infinite "weight factor" for $f(x^+)$. We conjecture that it also accounts for the zero for $f(x^-)$. The result with $J_{-1/m}(z)$ comes, of course, from the fact that the analytic continuation of $J_{-1/m}$ is just $e^{-i\pi/m}$ times itself. Even though the range of integration is from, say, $-\infty$ to 0, in the limit as $\nu \to \infty$ the essential contribution to the integrand comes from the origin. Since these kernels are analytic in any region not completely enclosing the origin it is well known that their behavior in such a region is determined by their behavior on any line in the region. It is on this basis we conjecture that, in order to determine the usefulness of one of these kernels on any *half* line the analytic continuation *through* the origin must be considered.

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