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MEAN FIELD VARIATION IN RANDOM MEDIA*

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Abstract. We consider here the basic equation

$$\frac{\partial}{\partial x_i}\left[\epsilon(\mathbf{x})\,\frac{\partial\phi}{\partial x_i}(\mathbf{x})\right] = \rho(\mathbf{x}),$$

where $\epsilon(\mathbf{x})$ is a random function of position and $\rho(\mathbf{x})$ is a prescribed source term. A formal equation is derived that governs $\{\phi(\mathbf{x})\}$, where the braces indicate an ensemble average. The equation, which depends on the boundary conditions of the stochastic problem, is presented in terms of an infinite sequence of correlation functions associated with $\epsilon(\mathbf{x})$. The equation is investigated first for the case of an infinite dielectric where isotropy may be assumed. An impulse response function is obtained and an explicit form of this response function is presented for the limit of small perturbations. Further, it is shown that the equation governing $\{\phi(x)\}$ is greatly simplified for the case in which all characteristic lengths associated with $\{\phi(\mathbf{x})\}$ are large compared to all correlation lengths l_i associated with the $\epsilon(\mathbf{x})$ field. The question of boundary conditions is next considered and as an example a spherical boundary (radius R) is studied. It is demonstrated, in this case, that if $R \gg l_i$ the effects of the boundary conditions on the equation governing $\{\phi(x)\}$ are negligible except at points within a thin layer near the boundary. The relationship between the ensemble average and the local volume average is also discussed.

1. Introduction. The problem to be discussed involves the operator equation

$$Lu = f, \tag{1}$$

where L is a random differential operator and f may be a random forcing term. It is desired to obtain an equation governing the mean of the random field quantity u. This may be formally accomplished as follows. Take the average of Eq. (1). This gives

$$\{L\}\{u\} + \{L'u'\} = \{f\} \tag{2}$$

where $\{\ \}$ indicates the ensemble average of the relevant quantity and the superscript ($'$) denotes the fluctuating part of the indicated quantity about this ensemble average. Subtract Eq. (2) from Eq. (1). This gives the following equation for u':

$$[\{L\} + (I - P)L']u' = -L'\{u\} + f'. \tag{3}$$

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In Eq. (3), I represents the identity operator and P represents the averaging operator, i.e. $Pu = \{u\}$. Solving Eq. (3) for u', operating on this result by L' and averaging gives

$$\{L'u'\} = \{L'[\{L\} + (I - P)L']^{-1}f'\} - \{L'[\{L\} + (I - P)L']^{-1}L'\}\{u\}. \tag{4}$$

Substituting Eq. (4) in Eq. (2) gives the desired equation on $\{u\}$.

$$[\{L\} - \{L'[\{L\} + (I - P)L']^{-1}L'\}]\{u\} = \{f\} - \{L'[\{L\} + (I - P)L']^{-1}f'\}. \tag{5}$$

The study presented in this report is an investigation of the operator equation (5). In doing this, it was found necessary to express the inverse operator, $[\{L\} + (I - P)L']^{-1}$, by its binomial expansion. I.e.

$$[\{L\} + (I - P)L']^{-1} = \sum_{n=0}^{\infty} (-1)^n [\{L\}^{-1}(I - P)L']^n \{L\}^{-1} . \tag{6}$$

Thus, the only inversion that must be carried out is for the operator $\{L\}$. Use of the binomial expansion limits the validity of a result to a certain class of problems for which this expansion converges. Unfortunately, it is not possible for us to define precisely the class of problems for which the results are valid. For a specific problem, it is usually possible to offer some justification on physical grounds but the ultimate justification will be the usefulness of the equations obtained in predicting results which may be experimentally justified. It may be noted, however, that since the series in Eq. (6) is not truncated in the main body of the report, the solution is not a perturbation solution about the operator $\{L\}$.

In the next section, the above formulism is carried out for the equation

$$\frac{\partial}{\partial x_i} \epsilon(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial x_i} = \rho(x), \tag{7}$$

where $\epsilon(\mathbf{x})$ is a random function of position and $\rho(\mathbf{x})$ is a source term which is taken for convience to be nonrandom. This equation represents the field equation for a number of physical problems. For example, $E_i(x) = \partial\phi/\partial x_i$, might denote the electric field in a medium with a variable permittivity, or ϕ might represent the temperature in a medium with a variable conductivity. An explicit expression is obtained for the operator appearing in the equation of $\{\phi\}$. The equation is in terms of an infinite sequence of correlation functions associated with $\epsilon(\mathbf{x})$.

In Sec. 3 we consider an infinite medium, invoke statistical isotropy and show how this allows us to obtain a solution to our equation governing $\{\phi\}$ in transform space (i.e. a three-dimensional Fourier transform space). Using this solution it is easy to see how the solution may be simplified for the case in which all correlation lengths associated with the $\epsilon(\mathbf{x})$ field are small compared to all characteristic lengths associated with the problem. We explicitly discuss the small perturbation case.

In Sec. 4, we consider the validity and meaning of equating the ensemble average $\{\phi\}$ with a local volume average. It is further shown that the governing equation on the local volume average of ϕ is of the same form as the governing equation on ϕ for a homogeneous medium if one introduces the idea of nonlocal effects.

Finally, attention is paid to the problem of a bounded medium. The difficulties introduced by adding the boundary are discussed. It is shown for the case in which all characteristic lengths of the boundary are large compared to all correlation lengths that the infinite medium field equations are valid in the region interior to a small layer at the boundary.

2. Derivation of equation on $\{\phi(\mathbf{x})\}$. In this section the formulism presented in the introduction is to be applied to the differential equation

$$\frac{\partial}{\partial x_i}\left[\epsilon(\mathbf{x})\frac{\partial\phi}{\partial x_i}\right] = \rho(\mathbf{x}), \tag{8}$$

where $\epsilon(\mathbf{x})$ is a statistically homogeneous random function of position and $\rho(\mathbf{x})$ is a source term which is taken to be nonrandom. Here and hereafter the summation convention is employed. Referring to the operator Eq. (1), we can immediately identify

$$L = \frac{\partial}{\partial x_i}\left[\epsilon(\mathbf{x})\frac{\partial}{\partial x_i}\right] \tag{9}$$

and $\{f\} = \rho(\mathbf{x})$. The assumption that $\epsilon(\mathbf{x})$ is statistically homogeneous allows us to write

$$\{L\} = \{\epsilon\}\ \partial^2/\partial x_i\ \partial x_i\ , \tag{10}$$

while

$$L' = \frac{\partial}{\partial x_i}\left[\epsilon'(\mathbf{x})\frac{\partial}{\partial x_i}\right]. \tag{11}$$

The equation on the mean field $\{\phi(\mathbf{x})\}$ is now given by Eq. (5). Using the iterated expression for the inverse of the operator, $[\{L\} + (I - P)L']^{-1}$, the only operator that we need invert to obtain an explicit equation on $\{\phi(\mathbf{x})\}$ is $\{L\}$. The inverse of the Laplacian operator may be carried out by the introduction of a Green's function (see Courant and Hilbert [4, p. 261]). The appropriate Green's function to introduce will, of course, depend on the boundary conditions. In this treatment the boundary conditions on ϕ are taken as nonrandom; hence ϕ' satisfies homogeneous conditions. Denoting the appropriate Green's function by $G(\mathbf{x}, \mathbf{x}')$, where $\mathbf{x}$ locates the field point and $\mathbf{x}'$ locates the source point, we write

$$\{L\}^{-1}u(\mathbf{x}) = \frac{1}{\{\epsilon\}}\int G(\mathbf{x}, \mathbf{x}')u(\mathbf{x}')\ d\mathbf{x}' \tag{12}$$

where $u(\mathbf{x})$ is a generic function of position and the integration is to be carried out over the entire volume.

With Eq. (12), the manipulations involved in obtaining an explicit equation on $\{\phi(\mathbf{x})\}$ are straightforward although extremely cumbersome. Consider the second term on the left-hand side of Eq. (5). We write

$$\{L'[\{L\} + (I - P)L']^{-1}L'\}\{\phi(\mathbf{x})\} = \frac{\partial}{\partial x_i}\{\epsilon'(\mathbf{x})E_i'(\mathbf{x})\}, \tag{13}$$

where

$$E_i'(\mathbf{x}) = \frac{\partial}{\partial x_i}[\{L\} + (I - P)L']^{-1}\frac{\partial}{\partial x_{1_k}}[\epsilon'(\mathbf{x}_1)\{E_k(\mathbf{x}_1)\}]. \tag{14}$$

In Eq. (14)

$$\{E_k(\mathbf{x})\} = \frac{\partial}{\partial x_k}\{\phi(\mathbf{x})\}. \tag{15}$$

As discussed in the introduction, the inversion of the operator appearing in Eq. (14) is to be carried out by formally expanding it as an infinite series. The resulting series may be put in an understandable form by first introducing the integrodifferential operator, $A_{ij}(\mathbf{x}, \mathbf{x}')$, defined by

$$A_{ij}(\mathbf{x}, \mathbf{x}')u_j(\mathbf{x}') \equiv \frac{1}{\{\epsilon\}} \int \frac{\partial G}{\partial x_i}(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial x_j'} [\epsilon'(\mathbf{x}')u_j(\mathbf{x}')]\, d\mathbf{x}' , \tag{16}$$

where $u_j(\mathbf{x})$ is a generic vector field. Thus, we write

$$\begin{aligned} E_i'(\mathbf{x}) = {} & A_{ij}(\mathbf{x}, \mathbf{x}_1)\{E_j(\mathbf{x}_1)\} \\ & + (I - P)A_{ij}(\mathbf{x}, \mathbf{x}_1)A_{jk}(\mathbf{x}_1, \mathbf{x}_2)\{E_k(\mathbf{x}_2)\} \\ & + (I - P)A_{ij}(\mathbf{x}, \mathbf{x}_1)(I - P)A_{jk}(\mathbf{x}_1, \mathbf{x}_2)A_{kl}(\mathbf{x}_2, \mathbf{x}_3)\{E_l(x_3)\} \\ & + \cdots + \\ & \cdot(I - P)A_{ij}(\mathbf{x}, \mathbf{x}_1)(I - P)A_{jk}(\mathbf{x}_1, \mathbf{x}_2) \cdots (I - P)A_{pq}(\mathbf{x}_{n-2}, \mathbf{x}_{n-1}) \\ & \qquad A_{qr}(\mathbf{x}_{n-1}, x_n)\{E_r(\mathbf{x}_n)\} \\ & + \cdots . \end{aligned} \tag{17}$$

This may be written in the form

$$E_i'(\mathbf{x}) = H_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}, \tag{18}$$

where $H_{ij}(\mathbf{x}, \mathbf{x}')$ is an integrodifferential operator given by

$$\begin{aligned} H_{ij}(\mathbf{x}, \mathbf{x}') = {} & A_{ij}(\mathbf{x}, \mathbf{x}_1) \\ & + (I - P)A_{ik}(\mathbf{x}, \mathbf{x}_1)A_{kj}(\mathbf{x}_1, \mathbf{x}') \\ & + (I - P)A_{ik}(\mathbf{x}, \mathbf{x}_1)(I - P)A_{kl}(\mathbf{x}_1, \mathbf{x}_2)A_{lj}(\mathbf{x}_2, \mathbf{x}') \\ & + \cdots . \end{aligned} \tag{19}$$

We next multiply both sides of Eq. (18) by $\epsilon'(\mathbf{x})$ and average. We find

$$\{\epsilon'(\mathbf{x})E_i'(\mathbf{x})\} = K_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}, \tag{20}$$

where

$$K_{ij}(\mathbf{x}, \mathbf{x}') = \{\epsilon'(\mathbf{x})H_{ij}(\mathbf{x}, \mathbf{x}')\}. \tag{21}$$

The equation on $\{\phi(\mathbf{x})\}$ may now be written using Eqs. (5), (13), and (20) and is given by

$$\{\epsilon\} \frac{\partial}{\partial x_i} \{E_i(\mathbf{x})\} + \frac{\partial}{\partial x_i} [K_{ij}(\mathbf{x}, \mathbf{x}')\{E_{ij}(\mathbf{x}')\}] = \rho(\mathbf{x}), \tag{22}$$

where it is to be recalled that $\{E_i(\mathbf{x})\} = \partial\{\phi(\mathbf{x})\}/\partial x_i$.

In explicit form, we may write

$$\begin{aligned} K_{ij}(\mathbf{x}, \mathbf{x}') = {} & \{\epsilon'(\mathbf{x})A_{ij}(\mathbf{x}, \mathbf{x}')\} \\ & + \{\epsilon'(\mathbf{x})(I - P)A_{ik}(\mathbf{x}, \mathbf{x}_1)A_{kj}(\mathbf{x}_1, \mathbf{x}')\} \\ & + \{\epsilon'(\mathbf{x})(I - P)A_{ik}(\mathbf{x}, \mathbf{x}_1)(I - P)A_{kl}(\mathbf{x}_1, \mathbf{x}_2)A_{lj}(\mathbf{x}_2, \mathbf{x}')\} \\ & + \cdots . \end{aligned}$$

We note that $\{\epsilon'(\mathbf{x})PQ(\mathbf{x}, \mathbf{x}')\} = 0$. Therefore since

$$\epsilon'(\mathbf{x}_i)A_{kl}(\mathbf{x}_j, \mathbf{x}_{j+1}) = A_{kl}(\mathbf{x}_j, \mathbf{x}_{j+1})\epsilon'(\mathbf{x}), \qquad i \leq j$$

we find

$$\begin{aligned} K_{ij}(\mathbf{x}, \mathbf{x}') = {} & \{A_{ij}(\mathbf{x}, \mathbf{x}')\epsilon'(\mathbf{x})\} \\ & + \{A_{ik}(\mathbf{x}, \mathbf{x}_1)A_{kj}(\mathbf{x}_1, \mathbf{x}')\epsilon'(\mathbf{x})\} \\ & + \{A_{ik}(\mathbf{x}, \mathbf{x}_1)A_{kl}(\mathbf{x}_1, \mathbf{x}_2)A_{lj}(\mathbf{x}_2, \mathbf{x}')\epsilon'(\mathbf{x})\} + \cdots \\ & - \{A_{ik}(\mathbf{x}, \mathbf{x}_1)\epsilon'(\mathbf{x})\}\{A_{kl}(\mathbf{x}_1, \mathbf{x}_2)A_{lj}(\mathbf{x}_2, \mathbf{x}')\} - \cdots . \end{aligned} \tag{23}$$

If we now define the operator $B_{ij}(\mathbf{x}, \mathbf{x}')$ by

$$B_{ij}(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') = \frac{1}{\{\epsilon\}} \int_V G_i(\mathbf{x}, \mathbf{x}') \frac{\partial u(\mathbf{x}')}{\partial x_j'} \, d\mathbf{x}' , \tag{24}$$

where $u(\mathbf{x})$ is a generic function of position, we have finally

$$\begin{aligned} K_{ij}(\mathbf{x}, \mathbf{x}') = {} & B_{ij}(\mathbf{x}, \mathbf{x}')C(\mathbf{x}, \mathbf{x}') \\ & + B_{ik}(\mathbf{x}, \mathbf{x}_1)B_{kj}(\mathbf{x}_1, \mathbf{x}')C(\mathbf{x}, \mathbf{x}_1, \mathbf{x}') \\ & + \cdots + B_{ik}(\mathbf{x}, \mathbf{x}_1)B_{kl}(\mathbf{x}_1, \mathbf{x}_2) \cdots B_{pq}(\mathbf{x}_{n-1}, \mathbf{x}_n)B_{qr}(\mathbf{x}_n, \mathbf{x}') \\ & \cdot C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}'), \\ & + \cdots \end{aligned} \tag{25}$$

where

$$\begin{aligned} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}') \equiv {} & \{\epsilon'(\mathbf{x})\epsilon'(\mathbf{x}_1) \cdots \epsilon'(\mathbf{x}_n)\epsilon'(\mathbf{x}')\} \\ & - \sum (-1)^p \{\epsilon'(\mathbf{x}) \cdots \epsilon'(\mathbf{x}_j)\}\{\epsilon'(\mathbf{x}_{j+1}) \cdots \epsilon'(\mathbf{x}_l)\} \cdots \{\epsilon'(\mathbf{x}_q) \cdots \epsilon'(\mathbf{x}')\}. \end{aligned}$$

The sum is over all combinations of p products of $\{\epsilon'(\mathbf{x}_j) \cdots \epsilon'(\mathbf{x}_l)\}$. The order of the arguments must be preserved.

$K_{ij}(\mathbf{x}, \mathbf{x}')$ is thus an integrodifferential operator determined by the Green's function $G_i(\mathbf{x}, \mathbf{x}')$ and the infinite set of correlation functions

$$C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}'); \qquad n \to \infty .$$

Before proceeding to investigate the integrodifferential equation on the mean field it is well to point out that Eq. (22) (using Eq. (25)) is dependent on the boundary conditions applied to the original stochastic problem. Thus, it would perhaps be inappropriate to term such an equation a field equation on $\{\phi(\mathbf{x})\}$. It is important to note that this is not a result of the derivation of this equation but rather is a result of the nature of the problem. In the last section of this paper the effects of boundary conditions are studied in some detail and it shall be shown, for the case of a large spherical boundary, that the contribution of the boundary conditions to the equation of $\{\phi(\mathbf{x})\}$ is significant only within a layer of the boundary. The thickness of this layer is a function of the correlation lengths associated with the fluctuations in $\epsilon(\mathbf{x})$. From this we may conclude that the equation on $\{\phi(\mathbf{x})\}$ is independent of the boundary conditions, and thus may be appropriately termed a field equation, except for points lying within a boundary layer.

3. Infinite space. In this section we consider an infinite space with a local source $\rho(\mathbf{x})$ confined essentially to some finite volume. We assume $\phi(\mathbf{x})$ vanishes at infinity. We choose the statistics of the $\epsilon(\mathbf{x})$ field to be homogeneous and isotropic. In this space $G_i(\mathbf{x}, \mathbf{x}')$ is the free space Green's function and is a function only of $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. We have

$$G_i(\mathbf{x}, \mathbf{x}') = -\frac{1}{(4\pi)} \frac{r_i}{r^3}, \tag{26}$$

and

$$B_{ij}(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') = -\frac{1}{4\pi\{\epsilon\}} \int_{V_\infty} \frac{r_i}{r^3} \frac{\partial u(\mathbf{x}')}{\partial x_j} \, d\mathbf{x}' . \tag{27}$$

A typical term of the product $K_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}$ is

$$I_i^{(n)}(\mathbf{x}) = \frac{(-1)^{n+1}}{(4\pi\{\epsilon\})^{n+1}} \int_{V_\infty} \frac{r_i(\mathbf{x}, \mathbf{x}_1)}{r^3(\mathbf{x}, \mathbf{x}_1)} \frac{\partial}{\partial \mathbf{x}_{1k}} \int_{V_\infty} \frac{r_k(\mathbf{x}_1, \mathbf{x}_2)}{r^3(\mathbf{x}_1, \mathbf{x}_2)} \frac{\partial}{\partial \mathbf{x}_{2l}} \cdots \int_{V_\infty} \frac{r_p(\mathbf{x}_{n-1}, \mathbf{x}_n)}{r^3(\mathbf{x}_{n-1}, \mathbf{x}_n)}$$
$$\times \frac{\partial}{\partial x_{nq}} \int_{V_\infty} \frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3(\mathbf{x}_n, \mathbf{x}')} \frac{\partial}{\partial \mathbf{x}'_j} [C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\}] \, d\mathbf{x}' \, d\mathbf{x}_n \cdots d\mathbf{x}_1. \tag{28}$$

Consider the last integral

$$Q = \int_{V_\infty} \frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3(\mathbf{x}_n, \mathbf{x}')} \frac{\partial}{\partial \mathbf{x}'_j} [C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\}] \, d\mathbf{x}' . \tag{29}$$

Q may be written in the form

$$Q = \int_{V_\infty} \frac{\partial}{\partial \mathbf{x}'_j} \left[\frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3 \, \mathbf{x}_n, \mathbf{x}')} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\} \right] dx'$$
$$- \int_{V_\infty} \left[\frac{\partial}{\partial \mathbf{x}'_j} \frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3(\mathbf{x}_n, \mathbf{x}')} \right] C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\} \, d\mathbf{x}'. \tag{30}$$

Using the divergence theorem, the first integral is equal to

$$\int_S \frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3(\mathbf{x}_n, \mathbf{x}')} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\} n_j \, dS.$$

S may be taken to be an infinite sphere around the point $\mathbf{x}_n$ and a small sphere around this point $\mathbf{x}$ to allow evaluation of the integral in the neighborhood of the singularity $1/r^3(\mathbf{x}_n, \mathbf{x}')$. We assume that $C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')$ vanishes rapidly enough as $|\mathbf{x}_n - \mathbf{x}'| \to \infty$ so that the infinite sphere gives no contribution. The singularity gives the contribution

$$-(4\pi/3)E_j(\mathbf{x}_n)C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}_n).$$

Noting that

$$\frac{\partial}{\partial x_j} \frac{r_q(\mathbf{x}_n, \mathbf{x}')}{r^3(\mathbf{x}_n, \mathbf{x}')} = \frac{1}{r^3(\mathbf{x}_n, \mathbf{x}')} \left[\delta_{jq} - \frac{3r_j r_q}{r^2} \right] \equiv \frac{\alpha_{jq}(\mathbf{x}_n, \mathbf{x}')}{r^3},$$

Q may be written as[2]

$$Q = -\frac{4\pi}{3} E_j(\mathbf{x}_n)C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}_n)$$

[2] In this form the second integral exists only in a Cauchy principal value sense.

$$- \int_{V_\infty} \frac{1}{r^3(\mathbf{x}_n, \mathbf{x}')} \alpha_{jq}(\mathbf{x}_n, \mathbf{x}') C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\} \, d\mathbf{x}. \tag{31}$$

Substituting Eq. (31) into Eq. (28) yields

$$I_i^{(n)}(\mathbf{x}) = \frac{(-1)^n}{[4\pi\{\epsilon\}]^n} \frac{1}{3\{\epsilon\}} \int_{V_\infty} \frac{r_i(\mathbf{x}, \mathbf{x}_1)}{r^3(\mathbf{x}, \mathbf{x}_1)} d\mathbf{x}_1 \frac{\partial}{\partial x_{1k}} \int \frac{r_k(\mathbf{x}_1, \mathbf{x}_2)}{r^3(\mathbf{x}_1, \mathbf{x}_2)} d\mathbf{x}_2 \frac{\partial}{\partial x_{2l}}$$

$$\cdot \int_{V_\infty} \frac{r_p(\mathbf{x}_{n-2}, \mathbf{x}_{n-1})}{r^3(\mathbf{x}_{n-2}, \mathbf{x}_{n-1})} d\mathbf{x}_{n-1} \frac{\partial}{\partial x_{(n-1)q}} \int_{V_\infty} \frac{r_q(\mathbf{x}_{n-1}, \mathbf{x}_n)}{r^3(\mathbf{x}_{n-1}, \mathbf{x}_n)} \frac{\partial}{\partial \mathbf{x}_{nj}} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}_n)\{E_j(\mathbf{x}_n)\} \, d\mathbf{x}_u$$

$$+ \frac{(-1)^n}{[(4\pi)\{\epsilon\}]^{n+1}} \frac{r_i(\mathbf{x}, \mathbf{x}_1)}{r^3(\mathbf{x}, \mathbf{x}_1)} \frac{\partial}{\partial \mathbf{x}_{1k}} \int_{V_\infty} \frac{r_k(\mathbf{x}_1, \mathbf{x}_2)}{r^3(\mathbf{x}_1, \mathbf{x}_2)} \frac{\partial}{\partial \mathbf{x}_{2l}} \cdots$$

$$\cdot \int \frac{r_p(\mathbf{x}_{n-1}, \mathbf{x}_n)}{r^3(\mathbf{x}_{n-1}, \mathbf{x}_n)} \frac{\partial}{\partial x_{nq}} \int_{V_\infty} \frac{1}{r^3(\mathbf{x}_n, \mathbf{x}')} \alpha_{qj}(\mathbf{x}_n, \mathbf{x}') \, C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\} \, d\mathbf{x}' . \tag{32}$$

We note that the first integral of Eq. (32) is the same as $I_i^{(n-1)}$ given by this equation if $C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}, \mathbf{x}')$ is replaced by $(1/3\{\epsilon\})C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}, \mathbf{x}', \mathbf{x}')$. The sequence of steps Eq. (29)–Eq.(32) can thus be performed for $I_i^{(n-1)}$ where $C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}, \mathbf{x}')$ is replaced by

$$C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}, \mathbf{x}') + \frac{1}{3\{\epsilon\}} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}, \mathbf{x}', \mathbf{x}').$$

Let us define the following function

$$C_\Sigma(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_j) = C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_j) + \frac{1}{3\{\epsilon\}} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_j, \mathbf{x}_j)$$
$$+ \cdots + \left[\frac{1}{3\{\epsilon\}}\right]^{m-1} C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_j, \overset{m}{\cdots}, \mathbf{x}_j) + \cdots \tag{33}$$

where $\mathbf{x}_j, \overset{m}{\cdots}, \mathbf{x}_j$ means the coordinate $\mathbf{x}_j$ is repeated m times.

In terms of $C_\Sigma(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_j)$ and a new operator I_{ij}, defined as

$$I_{ij}(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') = -\frac{1}{4\pi\{\epsilon\}} \int_{V_\infty} \frac{1}{r^3(\mathbf{x}, \mathbf{x}')} \alpha_{ij}(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') \, d\mathbf{x}' ,$$

$u(\mathbf{x})$ being a generic function of position, $K_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}$ is given by

$$K_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\} = -\frac{1}{3\{\epsilon\}} C_\Sigma(\mathbf{x})\{E_i(\mathbf{x})\} + I_{ij}(\mathbf{x}, \mathbf{x}')C_\Sigma(\mathbf{x}, \mathbf{x}')$$
$$\cdot \{E_j(\mathbf{x}')\} + I_{ik}(\mathbf{x}, \mathbf{x}_1)I_{kj}(\mathbf{x}_1, \mathbf{x}')C_\Sigma(\mathbf{x}, \mathbf{x}_1, \mathbf{x}')\{E_j(\mathbf{x}')\}$$
$$+ \cdots + I_{ik}(\mathbf{x}, \mathbf{x}_1)I_{kl}(\mathbf{x}_1, \mathbf{x}_2) \cdots I_{pq}(\mathbf{x}_{n-1}, \mathbf{x}_n)I_{qr}(\mathbf{x}_n, \mathbf{x}')$$
$$\cdot C_\Sigma(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')\{E_j(\mathbf{x}')\}$$
$$+ \cdots . \tag{34}$$

In infinite space Eqs. (22) and (34) constitute a formal solution to our problem.

If we define $G_{ij}(\mathbf{x}, \mathbf{x}')$ as

$$G_{ij}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi\{\epsilon\}} \frac{1}{r^3(\mathbf{x}, \mathbf{x}')} \alpha_{ij}(\mathbf{x}, \mathbf{x}') C_\Sigma(\mathbf{x}, \mathbf{x}')$$
$$-\frac{1}{4\pi\{\epsilon\}} \frac{1}{r^3(\mathbf{x}, \mathbf{x}_1)} \alpha_{ij}(\mathbf{x}, \mathbf{x}_1) I_{kj}(\mathbf{x}_1, \mathbf{x}') C_\Sigma(\mathbf{x}, \mathbf{x}_1, \mathbf{x}')$$
$$+ \cdots - \frac{1}{4\pi\{\epsilon\}} \frac{1}{r^3(\mathbf{x}, \mathbf{x}_1)} \alpha_{ij}(\mathbf{x}, \mathbf{x}_1) I_{kl}(\mathbf{x}_1, \mathbf{x}_2) \cdots I_{pq}(\mathbf{x}_{n-1}, \mathbf{x}_n)$$
$$\cdot I_{qr}(\mathbf{x}_n, \mathbf{x}') C_\Sigma(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}') + \cdots, \tag{35}$$

we may write

$$K_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\} = -\frac{1}{3\{\epsilon\}} C_\Sigma(\mathbf{x})\{E_i(\mathbf{x})\} + \int G_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}\, d\mathbf{x}' . \tag{36}$$

Since the ϵ field statistics are homogeneous and isotropic $C_\Sigma(\mathbf{x})$ is independent of position and $G_{ij}(\mathbf{x}, \mathbf{x}')$ depends only on $\mathbf{x} - \mathbf{x}'$. Thus we may write Eq. (20) in the form

$$\left[\{\epsilon\} - \frac{1}{3}\frac{C_\Sigma}{\{\epsilon\}}\right] \frac{\partial}{\partial x_i} \{E_i\} + \frac{\partial}{\partial x_i} \int G_{ij}(\mathbf{x} - \mathbf{x}')\{E_j(\mathbf{x}')\}\, d\mathbf{x}' = \rho(\mathbf{x}). \tag{37}$$

Inspection of $G_{ij}(\mathbf{x} - \mathbf{x}')$ shows that in every term there is a correlation function of the form $C(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}')$. When $|\mathbf{x} - \mathbf{x}'| \gg l$, where l is a characteristic correlation length of the field, C decreases rapidly in magnitude. We assume here that similarly G_{ij} has this property.

Let us denote a characteristic length associated with the source $\rho(\mathbf{x})$ to be L_ρ and assume for the moment that $L_\rho/l \gg 1$. In this case we may assume that $\{E_j(x')\}$ varies slowly in distances of the order of l. Thus we write

$$\int G_{ij}(\mathbf{x} - \mathbf{x}')\{E_j(\mathbf{x}')\}\, d\mathbf{x}' \approx \{E_j(\mathbf{x})\} \int G_{ij}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}' . \tag{38}$$

$\eta_{ij} = \int G_{ij}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'$ is independent of $\mathbf{x}$. In addition η_{ij} is an isotropic tensor and thus $\eta_{ij} = \eta\, \delta_{ij}$. Therefore we have finally for Eq. (37)

$$\left[\{\epsilon\} - \frac{1}{3}\frac{C_\Sigma}{\{\epsilon\}} + \eta\right] \frac{\partial}{\partial x_i} \{E_i(\mathbf{x})\} = \rho(\mathbf{x}). \tag{39}$$

Writing

$$\epsilon^* = \{\epsilon\} - \tfrac{1}{3}(C_\Sigma/\{\epsilon\}) + \eta, \tag{40}$$

we see that ϵ^* is just the effective constant of the random media defined for constant average fields (see Beran [1, Chap. 5]).

In general it is useful to define an effective constant even when the ratio L_ρ/l is arbitrary. This allows the equation to assume a useful form as $L_l/l \to \infty$. In addition ϵ^* is a measurable quantity. Thus in general we may write for Eq. (41)

$$\epsilon^* \frac{\partial}{\partial x_i} \{E_i(x)\} + \frac{\partial}{\partial x_i} \int \left[G_{ij}(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}') \int G_{ij}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'\right]$$
$$\{E_j(\mathbf{x}')\}\, d\mathbf{x}' = \rho(\mathbf{x}). \tag{41}$$

Finally, writing

$$M_{ij}(\mathbf{x} - \mathbf{x}') \equiv G_{ij}(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}') \int G_{ij}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}' \ , \tag{42}$$

we have

$$\epsilon^* \frac{\partial}{\partial x_i} \{E_i(\mathbf{x})\} + \frac{\partial}{\partial x_i} \int M_{ij}(\mathbf{x} - \mathbf{x}')\{E_j(\mathbf{x}')\}\, d\mathbf{x}' = \rho(\mathbf{x}), \tag{43}$$

or remembering $\{E_i(\mathbf{x})\} = (\partial/\partial x_i)\{\phi(\mathbf{x})\}$,

$$\epsilon^* \nabla^2 \{\phi(\mathbf{x})\} + \frac{\partial}{\partial x_i} \int M_{ij}(\mathbf{x} - \mathbf{x}') \frac{\partial}{\partial x_i} \{\phi(\mathbf{x}')\}\, d\mathbf{x}' = \rho(\mathbf{x}). \tag{44}$$

Eq. (44) is the equation governing $\{\phi(\mathbf{x})\}$. Because of the random medium, the free space equation

$$\epsilon \nabla^2 \phi(\mathbf{x}) = \rho(\mathbf{x}) \tag{45}$$

has been modified in two ways:

(1) ϵ is replaced by ϵ^* the effective constant;
(2) The term

$$\frac{\partial}{\partial x_i} \int M_{ij}(\mathbf{x} - \mathbf{x}') \frac{\partial}{\partial x_i} \{\phi(\mathbf{x}')\}\, d\mathbf{x}'$$

has been added.
Both ϵ^* and $M_{ij}(\mathbf{x} - \mathbf{x}')$ may be calculated in principle if all the correlation functions $C(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ were known. An alternate procedure is to measure ϵ^* and $M_{ij}(\mathbf{x}-\mathbf{x}')$.

ϵ^* is easily measurable and presents no difficulty. $M_{ij}(\mathbf{x} - \mathbf{x}')$ has not been measured to our knowledge but before we discuss its measurability we wish to perform further manipulations with Eq. (44) and present a solution for $\{\phi(\mathbf{x})\}$ for a reasonable choice of $M_{ij}(\mathbf{x} - \mathbf{x}')$. First let us take the Fourier transform of both sides of Eq. (44). This is easily done since the integral appearing in the equation is of the Faltung type. We find

$$-\epsilon^* k^2 (\hat{\phi}(\mathbf{k})\} - k_i \hat{M}_{ij}(\mathbf{k}) k_j \{\hat{\phi}(\mathbf{k})\} = \hat{\rho}(\mathbf{k}). \tag{46}$$

$\hat{M}_{ij}(\mathbf{k})$ is an isotropic tensor in $\mathbf{k}$ space and thus has the form

$$\hat{M}_{ij}(\mathbf{k}) = \hat{M}_1(k)\, \delta_{ij} + \hat{M}_2(k) k_i k_j \ . \tag{47}$$

Writing

$$k^2 \hat{M}(k) \equiv k_i k_j \hat{M}_{ij}(\mathbf{k}) = k^2 \hat{M}_1(k) + \hat{M}_2(k) k^4, \tag{48}$$

we have

$$\{\hat{\phi}(\mathbf{k})\} = -\frac{\hat{\rho}(\mathbf{k})}{k^2[\epsilon + k^2 M(\mathbf{k})]}. \tag{49}$$

$\hat{M}(k)$ depends on all the correlation functions associated with the ϵ field. In the case of small perturbations, however, we may confine our attention to only the two-point correlation function. As an example we now consider this case.

In the small perturbation case we find from Eq. (35) (see in this limit Bourret [2])

$$G_{ij}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi\{\epsilon\}} \frac{1}{r^3(\mathbf{x}, \mathbf{x}')} \alpha_{ij}(\mathbf{x}, \mathbf{x}')C(\mathbf{x}, \mathbf{x}'). \tag{50}$$

Since $C(\mathbf{x}, \mathbf{x}') = C(|\mathbf{x} - \mathbf{x}'|)$ direct calculation shows that $\int G_{ij}(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' = 0$. Thus we have

$$M_{ij}(\mathbf{x} - \mathbf{x}') = -\frac{1}{4\pi\{\epsilon\}} \frac{1}{r^3(\mathbf{x}, \mathbf{x}')} \alpha_{ij}(\mathbf{x}, \mathbf{x}')C(\mathbf{x}, \mathbf{x}'), \tag{51}$$

and

$$k_i k_j \hat{M}_{ij}(\mathbf{k}) = -\frac{k_i k_j}{4\pi\{\epsilon\}} \int_{V_r} \frac{1}{r^3} \alpha_{ij}(\mathbf{r})C(r)e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}, \tag{52}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$.

Manipulation then gives

$$\hat{M}(k) = -\frac{2}{\{\epsilon\}} \int_0^\infty \frac{C(r)}{r} \left[\frac{\sin kr}{kr}\left(\frac{3}{(kr)^2} - 1\right) - \frac{3}{k^2r^2}\cos kr\right] dr. \tag{53}$$

Here we note

$$\hat{M}(k)\Big|_{k\to 0} \to -\frac{2}{15}\frac{(kl)^2}{\{\epsilon\}} \int_0^\infty \left(\frac{r}{l}\right) C\left(\frac{r}{l}\right) d\left(\frac{r}{l}\right) \to 0,$$

and

$$\hat{M}(k)\,|_{k\to\infty} \to -\frac{2}{3}\frac{C(0)}{\{\epsilon\}}. \tag{54}$$

Thus if the characteristic length of $\rho(\mathbf{x})$, L_ρ, is very large compared to l then the medium responds as if it were homogeneous with effective constant $\epsilon^* = \{\epsilon\} - \frac{1}{3}C(0)/\{\epsilon\}$. When the characteristic length of $\rho(\mathbf{x})$ is very small compared to l, $\hat{\rho}(k)$ is a constant until $k \sim 1/L_\rho$. For $k \geq 1/l$, $\{\hat{\phi}(\mathbf{k})\}$ is determined by the average value $\{\epsilon\} - C(0)/\{\epsilon\}$. This is the small perturbation limit of $1/\{1/\epsilon\}$. Since the values $k \to \infty$ determine $\rho(\mathbf{x})$ as $|\mathbf{x}| \to 0$ this is the expected result. $\rho(\mathbf{x})$; $|\mathbf{x}| \to 0$ experiences only one value of ϵ in each member of the ensemble.

We expect $\hat{M}(k)$ to behave similarly when the fluctuations in $\epsilon(\mathbf{x})$ are large. That is, we expect

$$\hat{M}(k)\,|_{k\to 0} \to ak^2 \to 0,$$

$$\hat{M}(k)\,|_{k\to\infty} \to \frac{1}{\{1/\epsilon\}} - \epsilon^*. \tag{55}$$

Here a is a constant with dimensions length^2. For example for some materials a useful expression might be

$$\hat{M}(k) = \left[\frac{1}{\{1/\epsilon\}} - \epsilon^*\right][1 - \exp -l_c^2k^2]$$

where l_c is some characteristic correlation length.

If we consider $l/L_\rho \ll 1$ but wish a first correction to this limit, Eq. (53) becomes

$$\{\hat{\phi}(\mathbf{k})\} = \frac{-\hat{\rho}(\mathbf{k})}{k^2[\epsilon^* + ak^2]}. \tag{56}$$

In coordinate space this yields

$$\epsilon^* \nabla^2 \{\phi(\mathbf{x})\} - a\nabla^4 \{\phi(\mathbf{x})\} = \rho(\mathbf{x}). \tag{57}$$

Eq. (57) may also be derived from Eq. (43) by transforming to the coordinate $r = \mathbf{x} - \mathbf{x}'$ and expanding $\{E_i(\mathbf{x} + \mathbf{r})\}$ in a Taylor series about $\mathbf{x}$. Eq. (57) allows us to take into account slowly varying fields ($l/L_\rho \ll 1$). To do this it is only necessary to know the additional constant a.

4. Discussion of ensemble and volume averaging—interpretation of Eq. (44). The meaning of $\{E_i(\mathbf{x})\}$ as defined in this paper is unambiguous. We consider a family of dielectrics for which there is an associated probability distribution for the field $\epsilon(\mathbf{x})$. Each member of the family is, in turn, subjected to an identical excitation $\rho(\mathbf{x})$ and the value of the electric field vector is either measured or calculated at the identical point denoted by $\mathbf{x}$. $\{E_i(\mathbf{x})\}$ is the result of averaging all of the values so measured (or calculated).

It is not so clear how to infer any information from $\{E_i(\mathbf{x})\}$ that is useful for discussing the response of a single dielectric, the permittivity of which varies with position in space in a manner about which we have only limited information. If the random vector field $E_i(\mathbf{x})$ is defined over an infinite region of space and if it is statistically homogeneous, i.e. $\{E_i(\mathbf{x})\}$ is not a function of position, then it is possible to invoke an ergodic hypothesis. The ergodic hypothesis equates the ensemble average with a spatial volume average. Thus, one may view $\{E_i(\mathbf{x})\}$ as a spatial average of the electric field vector that exists in a single dielectric.

In the case in which the random vector field is not statistically homogeneous the conditions justifying the invoking of an ergodic hypothesis are not present. Still, one could argue that if $\{E_i(\mathbf{x})\}$ varied "slowly" with position in space then the conditions necessary for an ergodic hypothesis to be valid are approximately present. In such a case it could be hoped that some information of the response of the single medium problem might be inferred from $\{E_i(\mathbf{x})\}$.

To give some meaning to $\{E_i(\mathbf{x})\}$ varying "slowly" with position and what we might infer from $\{E_i(\mathbf{x})\}$, one might imagine that the variations in $E_i(\mathbf{x})$ are seen over two scales. On one scale one could discern details of the variation of the permittivity. On this scale (the inner scale) the overall dimensions of the dielectric and any characteristic length associated with the forcing of the dielectric appear to be infinitely large. On the second scale (the outer scale) one can make measurements of the overall dimensions of the dielectric and of characteristic lengths associated with the forcing of the dielectric. On this scale the fluctuations in the permittivity with position in space are too rapid to be discernible. The variations of $E_i(\mathbf{x})$ with distance measured on the inner scale are variations due to the variations in the permittivity. The variations of $E_i(\mathbf{x})$ with distance measured on the outer scale arise due to the finiteness of the dielectric and/or the finiteness of all characteristic lengths associated with the forcing. If $\{E_i(\mathbf{x})\}$ does not vary appreciably with a change in position of any length measured on the inner scale, then the conditions for the justification of an ergodic hypothesis are present on this scale. Hence, $\{E_i(\mathbf{x})\}$ may be associated with a local spatial volume average over a region with dimensions very large compared to the inner scale.

In a specific problem, the length defining the inner scale, which we may denote by l_i, will be given by some correlation length associated with the variations in the permittivity, for example, the correlation length associated with $C(r)$. The length defining the outer scale, which we may denote by L_0, has already been defined as the smallest characteristic length that can either be associated with the overall geometry or with the forcing mechanism (i.e. L_ρ). If $l_i/L_0 \ll 1$, one can equate $\{E_i(\mathbf{x})\}$ to a local volume average taken over a region which is large compared to l_i but small compared to L_0.

For problems in which two clearly discernable length scales are not present it is not possible to extract any deterministic information regarding the response of a single medium from statistical averages such as $\{E_i(\mathbf{x})\}$.

In general Eq. (44) only makes sense if viewed from an ensemble point of view. If, however, $l_i/L_\rho \ll 1$ (e.g. $l/L_\rho \ll 1$) the problem may be viewed from either an ensemble averaged or volume averaged point of view. Eq. (57) admits of either interpretation.

5. Effect of boundaries. The analysis of the bounded dielectric differs from that of the infinite dielectric only in the form of the Green's function that is used to invert the operator, $\{L\} = \{\epsilon\}\nabla^2$, which acts on a vector field. In the case of an infinite dielectric the appropriate Green's function is

$$G_{\infty i}(\mathbf{x}, \boldsymbol{\xi}) = (x_i - \xi_i)/4\pi\, |x_i - \xi_i|^3, \tag{58}$$

whereas in the case of the bounded dielectric the appropriate Green's function is

$$G_{Bi}(\mathbf{x}, \boldsymbol{\xi}) = G_{\infty i}(\mathbf{x}, \boldsymbol{\xi}) + W_i(\mathbf{x}, \boldsymbol{\xi}), \tag{59}$$

where $G_{Bi}(\mathbf{x}, \boldsymbol{\xi})$ is to satisfy the condition specified for $\mathbf{x}$ on the boundary and W is continuous both within the dielectric and on the boundary and is regular within the dielectric. (See Courant and Hilbert [4, p. 262].)

In this section we should like to consider the effect of modifying the Green's function in this manner. This will be accomplished by considering the special case of a spherical dielectric with the boundary condition that ϕ is zero on the surface of the sphere. This particular configuration is, of course, chosen since the Green's function for the sphere is known. The result of considering this special case will be a demonstration that the contribution of the boundary to the equation on $\{E_i\}$ when $l/R \ll 1$ (R is the radius of the sphere) is small except for a region in the immediate vicinity of the boundary. Thus, one might conclude that the equation derived for $\{E_i\}$ based on the infinite dielectric is an approximation to the field equation on $\{E_i\}$ which is valid for bounded media except for a small boundary layer. For the boundary layer itself, the equation on $\{E_i\}$ depends on the configuration of the body and on the boundary conditions.

For a sphere of radius R subject to the indicated boundary conditions the appropriate Green's function is

$$G_B(\vec{x}, \vec{\xi}) = -\frac{r_i}{4\pi r^3} + \frac{R}{4\pi\xi}\frac{r_{Ri}}{r_R^3} \tag{60}$$

where $r_i = x_i - \xi_i$ and $r_{Ri} = x_i - (R/\xi)^2\xi_i$. r_i is the vector from the field point to the source point and r_{Ri} is the vector from the field point to the reflected image of the source point. (The reflection is about the surface of the sphere.) The expression for $\{\epsilon' E_i'\}$ may be written, if no iteration is performed,

$$\{\epsilon'(\mathbf{x})E_i'(\mathbf{x})\} = -\frac{1}{4\pi\{\epsilon\}}\int_{\vec{\xi}}\frac{r_i}{r^3}\left[\frac{\partial}{\partial\xi_i}\{\epsilon'(\mathbf{x})\epsilon'(\xi)E_i(\xi)\}\,d\vec{\xi}\right] + \frac{R}{4\pi\{\epsilon\}}\int_{\vec{\xi}}\frac{r_{Ri}}{r_R^3}\left[\frac{\partial}{\partial\xi_i}\{\epsilon'(\mathbf{x})\epsilon'(\xi)E_i(\xi)\}\,d\vec{\xi}\right]. \tag{61}$$

We assume now that $\{\epsilon'(\mathbf{x})\epsilon'(\xi)E_i(\xi)\}$ falls appreciably to zero when $r = |\mathbf{x} - \xi|$ is greater than l. In this entire development we have only considered media for which all order correlation functions $\{\epsilon'(\mathbf{x}_1) \cdots \epsilon'(\mathbf{x}_n)\}$ are effectively zero beyond some distance which we take to be of order l. This does not allow us to state, however, that the above three-point correlation function behaves similarly, but consideration of Eq. (29) leads us to this inference. It is clearly true in the first iteration wherein $\{\epsilon'(\mathbf{x})\epsilon'(\xi)E_i(\xi)\}$ is replaced by $\{\epsilon'(\mathbf{x})\epsilon'(\xi)\}\{E_i(\xi)\}$, and the repeated integrations necessary in the terms with higher-order correlation functions do not appear to change the character of the decay. Moreover on physical grounds it is difficult to imagine that $E_i'(\xi)$ is correlated to $\epsilon'(\mathbf{x})$ when $|\mathbf{r}| \gg l$. $E_i'(\xi)$ is determined by contributions within the sphere of radius r and these contributing elements are uncorrelated when they are separated by distances greater than l.

Subject to above assumption we consider the relative magnitudes of the two terms in Eq. (61). We want to show that for points away from the surface $|\mathbf{x}| = R$ the second term is small compared to the first term. If this is so then in the region away from the surface we may use the free space Green's function and the analysis given in Sec. 3 is applicable.

Essentially we wish to compare the magnitude of the factors r_i/r^3 and $Rr_{Ri}/\xi r_R^3$. Let us consider $\vec{x} = 0$ to be the origin of the sphere. Choose a point $\mathbf{x}_p$ with radial coordinate r_p where $R - r_p > \alpha l$ where α is a number $\gg 1$. We assume, however, that R is large enough to meet the condition $R \gg \alpha l$:

$$O\left\{\frac{r_i}{r^3}\right\} = \frac{1}{r^2} = \frac{1}{l^2},$$

$$O\left(\frac{Rr_{Ri}}{\xi r_R^3}\right) = \frac{1}{R^2}. \tag{62}$$

Thus when $R - r_p > l$ the second term is smaller than the first term by a factor of $(l/R)^2$. We note that there is no singularity in the second integral and the singularity in the first integral is only apparent ($d\vec{\xi} = r^2\,d\Omega$). Moreover we assume no singularities in the function

$$\partial/\partial\xi_i\{\epsilon'(\mathbf{x})\epsilon'(\xi)E_i(\xi)\}.$$

If we neglect the second term in Eq. (61) the analysis given in Sec. 3 is applicable and the governing equation within the sphere ($r < R - \alpha l$) is given by Eq. (43).

The above analysis was restricted to a spherical surface. On physical grounds we assume, however, that if all radii of curvature in the boundary are $\gg \alpha l$ then the same conclusions hold for this surface.

6. Conclusions. We have determined, by iteration, an equation governing $\{\phi(\mathbf{x})\}$. This equation is determined by the sequence of correlation functions associated with the ϵ-field. In infinite space, where isotropy may be invoked, we showed how the equation may be simplified by use of an impulse response function (Eq. 44). A solution in transform space is noted in Eq. (49) and for the case of small perturbations an explicit expression is found for the transform of the impulse response function (Eq. 53). When the characteristic length associated with the source, L_ρ, is large compared to a characteristic correlation length, l, a simplified equation results (Eq. (56) or Eq. (57)). This equation is not restricted to small perturbations and contains only one unknown constant, a.

The case of a finite boundary was also considered using a spherical surface as an example. It was shown that outside of a boundary layer near the surface of the sphere the free space analysis given in Sec. 3 is valid and Eq. (43) may be used.

The elastic properties of a random medium may be treated by procedures analogous to those given in this paper. In a subsequent paper we shall study the variation of the average displacement field in a medium in which the elastic constants may be treated stochastically.

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