UNIFORM BOUNDEDNESS THEOREM FOR A NONLINEAR MATHIEU EQUATION*

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Bolotin [1, Ch. 3–Ch. 6] has derived the equation

$$f''(t) + 2\epsilon f'(t) + \Omega^2(1 - 2\mu \cos \theta t)f(t) + \psi(f(t), f'(t), f''(t)) = 0 \qquad (1)$$

to describe the motion of a parametrically excited pin-ended elastic column. The function $f(t)$ is related to the displacement at time t of the column from its undisturbed position as follows. Let l be the length of the column and x the displacement at height h. Then $x = f(t) \sin (\pi h/l)$. In a recent paper Genin and Maybee [2], using an energy function technique, have shown that all solutions are bounded when (1) is of the form

$$f'' = -(2\epsilon f' + \Omega^2(1 - 2\mu \cos \theta t)f + \gamma f^3 + 2\epsilon_l f'f^2 + 2\kappa f'^2 f)/(1 + 2\kappa f^2) \qquad (2)$$

and certain restrictions are placed on the parameters ϵ, Ω, μ, θ, γ, ϵ_l, and κ. In this note a theorem is presented which shows that every solution of (2) has a bound which is independent of the solution's initial values. This result is obtained under less restrictive conditions than those used in [2].

THEOREM. *There is a bounded region A in the xy-plane such that if $f(t)$ satisfies* (2), *$f(t_0) = f_0$, and $f'(t_0) = f_0'$, then $(f(t), f'(t)) \in A$ for all $t \geq T$ where T is a finite value which depends on t_0, f_0, and f_0'. The conditions placed on the parameters of* (2) *are*:

$$\epsilon, \gamma > 0; \epsilon_l, \kappa \geq 0; \quad \textit{and} \quad \epsilon_l > 0 \quad \textit{if} \quad \kappa > 0.$$

Proof. Let

$$\Phi(x, y) = (x^2 + bxy + cy^2)(1 + 2\kappa x^2) + (b\epsilon + c\Omega^2 - 1)x^2 + \tfrac{1}{2}(b\epsilon_l + c\gamma - 4\kappa)x^4$$

where b and c are chosen so that $b > 1/\epsilon$ and $c > \max (b^2/4, b/(4\epsilon), (4\kappa - b\epsilon_l)/\gamma, \delta)$ where

$$\delta = 0 \qquad \text{if} \quad \kappa = 0,$$
$$= b\kappa/\epsilon_l \quad \text{if} \quad \kappa > 0.$$

Since $4c > b^2$, $x^2 + bxy + cy^2$ is a positive definite form. Also $b\epsilon + c\Omega^2 - 1 > 0$ and $b\epsilon_l + c\gamma - 4\kappa > 0$. Hence Φ is a positive definite function of x and y. It is easily seen that, for any constant $C > 0$, the contour line $\Phi(x, y) = C$ is a single simple closed curve about the origin.

Let $f(t)$ satisfy (2). Then

$$\frac{d}{dt}[\Phi(f(t), f'(t))] = \frac{\partial \Phi}{\partial f} f' + \frac{\partial \Phi}{\partial f'} f'' = -(4c\epsilon - b)f'^2 - 4(c\epsilon_l - b\kappa)f'^2 f^2 - b\Omega^2 f^2 - b\gamma f^4 + 2\Omega^2 \mu \cos \theta t(bf^2 + 2cf'f).$$

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Using $4c\epsilon - b > 0$ we have

$$\frac{d\Phi}{dt} = -\{(4c\epsilon - b)f'^2 + 4\Omega^2\mu c \cos(\theta t)f'f + (4\Omega^4\mu^2c^2/(4c\epsilon - b))f^2\}$$
$$-\{4(c\epsilon_l - b\kappa)f'^2 + b\gamma f^2 - [b\Phi^2(2\mu \cos\theta t - 1) + 4\Omega^4\mu^2c^2/(4c\epsilon - b)]\}f^2.$$

Since $|\cos\theta t| \leq 1$,

$$\frac{d\Phi}{dt} \leq -\{(4c\epsilon - b)^{1/2}\,|f'| - (2\Omega^2\mu c/(4c\epsilon - b)^{1/2})\,|f|\}^2$$
$$-\{4(c\epsilon_l - b\kappa)f'^2 + b\gamma f^2 - [b\Omega^2(2\mu - 1) + 4\Omega^4\mu^2c^2/(4c\epsilon - b)]\}f^2. \tag{3}$$

By examining the possibilities $c\epsilon_l - b\kappa = 0$ or $c\epsilon_l - b\kappa > 0$ and $\Omega^2\mu = 0$ or $\Omega^2\mu \neq 0$, it follows from (3) that $d\Phi/dt$ may be nonnegative only in a bounded region of the xy-plane. Let A be the region bounded by a contour line of Φ which encloses in its interior all points at which $d\Phi/dt \geq 0$. Since $d\Phi/dt$ is negative and bounded away from 0 for points not in A, it follows that after some finite time, depending on t_0, f_0, and f_0', the point $(f(t), f'(t))$ will enter and remain in A.

It may also be seen from (3) that the trivial solution has global asymptotic stability when $b\Omega^2(2\mu - 1) + 4\Omega^4\mu^2c^2/(4c\epsilon - b) \leq 0$. This inequality is achieved when $|\mu|$ is sufficiently small.

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References

[1] V. V. Bolotin, *The dynamic stability of elastic systems*, Holden-Day, San Francisco, Calif., 1964
[2] J. Genin and J. S. Maybee, *Boundedness theorem for a nonlinear Mathieu equation*, Quart. Appl. Math. 28, 450–453 (1970)