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A BOUND ON THE ERROR IN REISSNER'S THEORY OF PLATES*

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Introduction. In a recent paper [1] we derived an expression for the mean square error in stresses obtained from the solution of a boundary-value problem in the classical theory of plates with respect to the solution of a corresponding problem in the theory of elasticity. Further, the relative mean square error was bounded by a quantity proportional to the plate thickness. The derivation in [1] employs the hypersphere theorems of Prager and Synge [2], [3] in elasticity which lead to the equations of plate theory via energy minimization techniques.

Subsequently, Simmonds [4] bounded the relative mean square error by a quantity proportional to the plate thickness squared for isotropic plates. The derivation of this improved bound is based on a direct application of the main hypersphere theorem [2] with a specially constructed kinematically admissible stress field of a more elaborate form than in [1].

In the present note we derive an expression for the mean square error and a bound on the relative error for Reissner's theory of plates [5] which includes the effect of shear deformation. The derivation is similar to that of [1], although we employ a kinematically admissible stress field of nearly the same form as in [4]. The relative error is again bounded by a quantity proportional to plate thickness squared. This lack of essential improvement over the error bound for classical theory is not surprising in view of the well-known fact that Reissner's theory offers improvement over classical theory only near the edge of the plate.

Of particular interest is the value obtained here for the numerical constant in the constitutive equation for the shear stress resultant, namely $\frac{5}{6}$. This value is in agreement with Reissner's original derivation [5] and most subsequent derivations,[1] e.g., the direct derivation of Green [7]. In the present derivation the value $\frac{5}{6}$ follows from minimization of both the potential energy and the complementary energy whose sum is the mean square error in stress [1], [3]. Therefore, $\frac{5}{6}$ can be regarded as the best value for the shear constant within the context of mean square error minimization for static problems. Also, we find that the constitutive equation for stress couple need not include a term proportional to the normal surface load, as in [5] and [7]. Finally, the error bound of Simmonds [4] for classical theory of isotropic plates is extended to anisotropic plates with midsurface elastic symmetry.

For brevity we do not repeat Secs. 1 and 2 of [1] which deal, respectively, with function space concepts in elasticity and the statement of a class of boundary value

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[1] An exception is the value $\pi^2/12$ proposed by Mindlin [6] for dynamic problems. The present results are restricted to statics by the nature of the function space concepts in [2], [3].

problems for plates in elasticity. Instead reference will be made to results from [1] as needed in what follows.

Potential energy. The equations of Reissner's theory will be derived here by minimization of potential energy using a kinematically admissible stress field.

Guided by solutions to plate problems in elasticity [8], we begin with a displacement field of the form

$$\begin{aligned} u_\alpha &= \beta_\alpha x_3 + f_\alpha(x_3^3 - \tfrac{3}{5}h^2 x_3), \\ u_3 &= w + g(x_3^2 - \tfrac{1}{5}h^2), \end{aligned} \tag{1}$$[2]

where β_α , f_α , w and g are independent of the coordinate x_3 normal to the middle surface. The x_3 coefficients in (1) are chosen in view of the relations

$$\int_{-h}^{h} x_3(x_3^3 - \tfrac{3}{5}h^2 x_3)\, dx_3 = \int_{-h}^{h} (1 - x_3^2/h^2)(x_3^2 - \tfrac{1}{5}h^2)\, dx_3 = 0. \tag{2}$$

Thus, β_α and w may be interpreted as resultant displacements[3] since

$$\frac{3}{2h^3}\int_{-h}^{h} u_\alpha x_3\, dx_3 = \beta_\alpha\ , \qquad \frac{3}{4h}\int_{-h}^{h} (1 - x_3^2/h^2) u_3\, dx_3 = w. \tag{3}$$

In order to be kinematically admissible, the displacement field (1) must meet the boundary conditions of the elasticity problem on the portion of the boundary S_u . For this to be possible, a slight modification of the displacement boundary conditions in [1] is required. Specifically, (2.4) and (2.5) of [1] must be replaced by

$$u_\alpha^* = \beta_\alpha^* x_3 + f_\alpha^*(x_3^3 - \tfrac{3}{5}h^2 x_3), \qquad u_3^* = w^* + g^*(x_3^2 - \tfrac{1}{5}h^2). \tag{4}$$

Then (1) satisfies the boundary conditions on S_u provided that

$$\beta_\alpha = \beta_\alpha^*\ , \qquad w = w^* \quad \text{on} \quad C_u\ , \tag{5}$$

$$f_\alpha = f_\alpha^*\ , \qquad g = g^* \quad \text{on} \quad C_u\ . \tag{6}$$

Proceeding from (1) as in [1] we obtain the following expression for the potential energy:

$$V_p = V_p^{(0)} + \int_s \left(\frac{4h^7}{175} B_{\alpha\beta\gamma\delta} f_{\alpha.\beta} f_{\gamma.\delta} - \tfrac{4}{5}h^2 pg\right) dx_1\, dx_2\ , \tag{7}$$

where

$$\begin{aligned} V_p^{(0)} = \int_s \{ &\tfrac{1}{3}h^3 B_{\alpha\beta\gamma\delta}\beta_{\alpha.\beta}\beta_{\gamma.\delta} + \tfrac{4}{3}h^3 B_{\alpha\beta33}\beta_{\alpha.\beta} g \\ &+ \tfrac{4}{3}h^3 B_{3333} g^2 + B_{\alpha3\beta3}[h(\beta_\alpha + w_{.\alpha})(\beta_\beta + w_{.\beta}) + \tfrac{4}{15}h^3(\beta_\alpha + w_{.\alpha})(3f_\beta + g_{.\beta}) \\ &+ \tfrac{8}{75}h^5(3f_\alpha + g_{.\alpha})(3f_\beta + g_{.\beta})] - pw\}\, dx_1\, dx_2 - \int_{C_\sigma} (\beta_\alpha M_\alpha^* + wQ^*)\, ds. \end{aligned}$$

On minimization of V_p^0 by the calculus of variations, the Euler equations can be written

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[2] This form for the displacement field differs only slightly from that of Simmonds [4]. The essential difference from the displacement field of [1] is the f_α term. Here Greek indices take values 1 and 2 and repeated indices imply summation.

[3] These same resultants or weighted displacements appear in the direct derivation of Reissner's theory given by Green [7].

as

$$g = -\frac{B_{\alpha\beta33}}{2B_{3333}}\beta_{\alpha,\beta}, \qquad f_\alpha = -\tfrac{1}{3}g_{,\alpha} - \frac{5}{12h^2}(\beta_\alpha + w_{,\alpha}) \tag{8}$$

and

$$M_{\alpha\beta,\beta} - Q_\alpha = 0, \qquad -Q_{\alpha,\alpha} = p, \tag{9}$$

where

$$M_{\alpha\beta} = \frac{t^3}{12}\tilde{B}_{\alpha\beta\gamma\delta}\beta_{\gamma,\delta}, \qquad Q_\alpha = \tfrac{5}{6}tB_{\alpha3\beta3}(\beta_\beta + w_{,\beta}),$$

$$t = 2h, \qquad \tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - B_{\alpha\beta33}B_{\gamma\delta33}/B_{3333}. \tag{10}$$

Here t is the thickness of the plate. Further, the natural boundary conditions are

$$M_{\alpha\beta}n_\beta = M_\alpha^*, \qquad Q_\alpha n_\alpha = Q^* \quad \text{on} \quad C_\sigma. \tag{11}$$

Eqs. (9), (10), (5) and (11) are the equilibrium equations, constitutive relations and boundary conditions of Reissner's theory of plates [5]. The functions f_α^* and g^* in the displacement boundary conditions (4) of the elasticity problem are determined by (6) and (8) *after* solution of the plate problem. From a practical viewpoint this is a minor defect since resultant displacements β_α^* and w^* are specified on C_u.

The kinematically admissible stresses corresponding to (1) can be written as

$$\begin{aligned}
\sigma''_{\alpha\beta} &= \tilde{B}_{\alpha\beta\gamma\delta}\beta_{\gamma,\delta}x_3 + B_{\alpha\beta\gamma\delta}f_{\gamma,\delta}(x_3^3 - \tfrac{3}{5}h^2x_3),\\
\sigma''_{33} &= B_{33\gamma\delta}f_{\gamma,\delta}(x_3^3 - \tfrac{3}{5}h^2x_3),\\
\sigma''_{\alpha3} &= \tfrac{5}{4}B_{\alpha3\beta3}(\beta_{,\beta} + w_{,\beta})(1 - x_3^2/h^2),
\end{aligned} \tag{12}$$

where, by (8), (9) and (10),

$$f_{\alpha,\beta} = \frac{B_{\gamma\delta33}}{6B_{3333}}\beta_{\gamma,\delta\alpha\beta} - \tfrac{2}{3}A_{\alpha3\eta3}\tilde{B}_{\eta\xi\gamma\delta}\beta_{\gamma,\delta\xi\beta}.$$

Complementary energy. The equations of Reissner's theory are derived here by minimization of complementary energy using a statically admissible stress field. The approach is similar to that of Reissner [5].

Guided by solutions to plate problems in elasticity [8], we begin with a stress field of the usual form

$$\begin{aligned}
\sigma'_{\alpha\beta} &= \frac{3x_3}{2h^3}M_{\alpha\beta}, \qquad \sigma'_{\alpha3} = \frac{3}{4h}\left(1 - \frac{x_3^2}{h^2}\right)Q_\alpha,\\
\sigma'_{33} &= \frac{3}{4h}\left(1 - \frac{x_3^2}{3h^2}\right)x_3p + q,
\end{aligned} \tag{13}$$

where p and q are related to the surface stresses as in [1] by

$$p = \sigma_3^+ + \sigma_3^-, \qquad q = \tfrac{1}{2}(\sigma_3^+ - \sigma_3^-).$$

This stress field is statically admissible; i.e., the equilibrium equations and the stress boundary conditions of the elasticity problem are satisfied provided that $M_{\alpha\beta}$ and Q_α satisfy (9) and (11). By (13), (4) and (1.18) of [1] the complementary energy can be written as

$$V_c = V_c^{(0)} + \int_s \left[\frac{3}{5h} A_{\alpha\beta 33} M_{\alpha\beta} p + A_{3333} h\left(\frac{17}{140} p^2 + q^2\right)\right] dx_1\, dx_2\,, \tag{14}$$

where

$$V_c^{(0)} = \int_s \left[\frac{3}{4h^3} A_{\alpha\beta\gamma\delta} M_{\alpha\beta} M_{\gamma\delta} + \frac{6}{5h} A_{\alpha 3\beta 3} Q_\alpha Q_\beta\right] dx_1\, dx_2 - \int_{c_u} [\beta_\alpha^* M_{\alpha\beta} n_\beta + w^* Q_\alpha n_\alpha]\, ds.$$

On minimization of $V_c^{(0)}$ with (9) enforced by Lagrange multipliers β_α and w, we obtain the constitutive equations (10) as Euler equations and the natural boundary conditions (5). Thus, we again have the equations of Reissner's theory of plates.

Approximate stress field and error. According to the hypersphere theorem [2], [3], if $\mathfrak{d}'$ and $\mathfrak{d}''$ are statically and kinematically admissible stress fields, respectively, for a boundary value problem in elasticity with exact stress field $\mathfrak{d}$, then the approximate stress field

$$\mathfrak{d}_A = \tfrac{1}{2}(\mathfrak{d}' + \mathfrak{d}'') \tag{15}$$

has a mean square error E; i.e.,

$$||\mathfrak{d}_A - \mathfrak{d}|| = E \tag{16}$$

where E is given by

$$E = ||\tfrac{1}{2}(\mathfrak{d}' - \mathfrak{d}'')|| \tag{17}$$

and further

$$E = V_c + V_p\,. \tag{18}$$

Here $||\mathfrak{d}||$ denotes the mean square norm defined in [1], [2] and [3]. By the bound (1.16) of [1], the relative error is of the form

$$E/||\mathfrak{d}|| = n + \mathcal{O}(n^2), \qquad n = E/(\mathfrak{d}' \cdot \mathfrak{d}'')^{1/2}, \tag{19}$$

where the inner product $(\mathfrak{d}' \cdot \mathfrak{d}'')$ is defined in [1], [2] and [3]. In the present case, the error in the approximate stress field given by (12), (13) and (15) is

$$E^2 = \int_s \left[\frac{4h^7}{175} B_{\alpha\beta\gamma\delta} f_{\alpha,\beta} f_{\gamma,\delta} + h A_{3333}\left(\frac{17}{140} p^2 + q^2\right)\right] dx_1\, dx_2\,, \tag{20}$$

which can be deduced from (17) using (12), (13) and (1.4) of [1], or from (18) using (7), (14) and the easily verified relations

$$V_p^{(0)} + V_c^{(0)} = 0, \qquad 3A_{\alpha\beta 33} M_{\alpha\beta} = 4h^3 g. \tag{21}$$ [4]

Further, by (12), (13) and (1.9) of [1]

$$\mathfrak{d}' \cdot \mathfrak{d}'' = \int_s \left[\frac{3}{4h^3} A_{\alpha\beta\gamma\delta} M_{\alpha\beta} M_{\gamma\delta} + \frac{3}{10h} A_{\alpha\beta 33} M_{\alpha\beta} p + \frac{6}{5h} A_{\alpha 3\beta 3} Q_\alpha Q_\beta\right] dx_1\, dx_2\,. \tag{22}$$

Thus, by (9), (10), (19), (21) and (22), we obtain the following estimate for the relative error:

$$E/||\mathfrak{d}|| = Ch^2 + \mathcal{O}(h^4), \tag{23}$$

[4] In view of the second of (21), the p terms in $V_p + V_c$ cancel whether or not they are included in $V_p^{(0)}$ and $V_c^{(0)}$. Therefore, it is not necessary to include these p terms in $V_p^{(0)}$ and $V_c^{(0)}$ which would lead to a p term in the constitutive equation for $M_{\alpha\beta}$ as in [5], [7].

where

$$C^2 = \int_s \left[\frac{4}{175} B_{\alpha\beta\gamma\delta} f_{\alpha,\beta} f_{\gamma,\delta} + \left(\frac{17}{140} + \gamma^2\right) \frac{4}{9} A_{3333} \cdot (\tilde{B}_{\alpha\beta\gamma\delta} \beta_{\alpha,\beta\gamma\delta})^2\right] dx_1\, dx_2 \Big/ \int_s \frac{1}{3} \tilde{B}_{\alpha\beta\gamma\delta} \beta_{\alpha,\beta} \beta_{\gamma,\delta}\, dx_1\, dx_2\,,\ \gamma = q/p.$$

Further, an explicit bound on the relative error follows immediately from (20), (22) and (1.16) of [1].

Classical plate theory. We indicate how the foregoing derivation can be modified to obtain results for classical plate theory. The derivation from complementary energy is the same as in [1] and need not be repeated here. In the derivation from potential energy using the displacement field (1), β_α must be related to w before the potential energy is minimized, for otherwise Reissner's theory would result as in the foregoing. As suggested by Simmonds [4], a suitable relation between β_α and w follows from setting

$$\sigma''_{\alpha 3} = \sigma'_{\alpha 3} = \frac{3}{4h}(1 - x_3^2/h^2) Q_\alpha\ ; \tag{24}$$

i.e.,

$$\begin{aligned} \beta_\alpha &= -w_{,\alpha} + \frac{12}{5h} A_{\alpha 3\beta 3} Q_\beta\,, \\ f_\alpha &= -\tfrac{1}{3} g_{,\alpha} - \frac{1}{h^3} A_{\alpha 3\beta 3} Q_\beta\,. \end{aligned} \tag{25}$$

Further, in order to make σ''_{33} small we set

$$g = B_{\alpha\beta 33} w_{,\alpha\beta} / 2B_{3333}\,. \tag{26}$$ [5]

Then the equations of the classical theory of plates follow from minimization of the same $V_p^{(0)}$ as in [1]. In particular, the constitutive equations read

$$M_{\alpha\beta} = -\frac{t^3}{12} \tilde{B}_{\alpha\beta\gamma\delta} w_{,\gamma\delta}\,, \qquad Q_\alpha = -\frac{t^3}{12} \tilde{B}_{\alpha\beta\gamma\delta} w_{,\beta\gamma\delta}\,. \tag{27}$$

The kinematically admissible stress field is

$$\begin{aligned} \sigma''_{\alpha\beta} &= \sigma'_{\alpha\beta} + B_{\alpha\beta\gamma\delta}[\tfrac{12}{5} A_{\gamma 3\eta 3} Q_{\eta,\delta} x_3 + f_{\gamma,\delta}(x_3^3 - \tfrac{3}{5} h^2 x_3)], \\ \sigma''_{\alpha 3} &= \sigma'_{\alpha 3}\,, \qquad \sigma''_{33} = B_{33\gamma\delta}[\tfrac{12}{5} A_{\gamma 3\eta 3} Q_{\eta,\delta} x_3 + f_{\gamma,\delta}(x_3^3 - \tfrac{3}{5} h^2 x_3)]. \end{aligned} \tag{28}$$

By (17), (25), (27) and (28), the mean square error in the approximate stress field given by (13), (15) and (28) is

$$E^2 = h^7 \int_s \left[\frac{64}{75} B_{\alpha\beta\gamma\delta} \psi_{\alpha,\beta} \psi_{\gamma,\delta} + \frac{4}{175} B_{\alpha\beta\gamma\delta} f_{\alpha,\beta} f_{\gamma,\delta} + A_{3333}\left(\frac{17}{140} + \gamma^2\right) \frac{4}{9} (B_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta})^2\right] dx_1\, dx_2\,, \tag{29}$$

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[5] This equation can also be obtained as an Euler equation on minimization of part of the potential energy.

where

$$\psi_\alpha = A_{\alpha 3\beta 3}\tilde{B}_{\beta\eta\gamma\delta}w_{,\eta\gamma\delta} .$$

Further

$$\mathbf{\delta}' \cdot \mathbf{\delta}'' = \frac{h^3}{3}\int_s \tilde{B}_{\alpha\beta\gamma\delta}w_{,\alpha\beta}w_{,\gamma\delta}\, dx_1\, dx_2 , \tag{30}$$

and thus an estimate for relative mean square error of the form (23) follows from (19), (29) and (30). The foregoing results for classical theory of plates agree in essence with the results of Simmonds [4] for isotropic plates.

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