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CONVERGENT INTEGRALS OF SOLUTIONS TO A LINEAR DIFFERENTIAL SYSTEM*

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If q is positive, continuous, monotone and unbounded on $[0, \infty)$, then every nontrivial solution of the second-order differential equation

$$y'' + q(t)y = 0 \qquad (' = d/dt) \tag{1}$$

is oscillatory and the corresponding conjugate energy

$$E(t, y) = q(t)^{-1}y'(t)^2 + y(t)^2$$

is nonincreasing. Although $\lim_{t\to\infty} E(t, y_0) \equiv E(\infty, y_0) = 0$ for at least one nontrivial solution y_0, it might be the case that $E(\infty, y_1) > 0$ for some solution y_1. A more complete discussion can be found in [1, p. 85], [2, p. 510], [3], [4]. In light of this possible behavior, Hartman and Wintner have shown that

$$\int_0^\infty y = \lim_{t\to\infty} \int_0^t y \tag{2}$$

exists for every solution of (1). Their proof was largely geometric and the details were outlined by Hartman in [2, p. 513]. This result, although plausible if q is of sufficiently regular growth so that asymptotic integration techniques are available, is interesting since $\int_0^\infty y_1$ might exist and yet $\lim_{t\to\infty} \sup |y_1(t)|$ might be positive for a certain solution y_1. A similar phenomenon is encountered when one studies the Fresnel integrals since $\int_0^\infty \sin t^2\, dt$ exists and $\lim_{t\to\infty} \sup |\sin t^2| = 1$. To continue, Hartman and Wintner did not observe that

$$\int_0^\infty q^{-1/2}y' \tag{3}$$

exists for every solution of (1). This follows from the identity

$$\int_0^t q^{-1/2}y' = q^{-1/2}y \,\Big|_0^t - \int_0^t y\, dq^{-1/2}$$

since y is bounded, $q^{-1/2}$ is of bounded variation and q is unbounded on $[0, \infty)$. If we write (1) as a first-order system $x' = A(t)x$ where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix}, \qquad x = \begin{bmatrix} y \\ y' \end{bmatrix},$$

and define

$$\Gamma(t) = \begin{bmatrix} 1 & 0 \\ 0 & q(t)^{-1/2} \end{bmatrix}$$

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then the limits (2), (3) can be written in vector form as

$$\int_0^\infty \Gamma x. \tag{4}$$

Our purpose in this note is to discuss the limit (4) as a special case of a similar property of solutions of a linear system

$$x' = F(t)x. \tag{5}$$

This approach, which contains the previously mentioned work of Hartman and Wintner, is straightforward and does not rely on the geometry of the solution curves. We assume throughout that

$$F(t) = A(t) + B(t) \tag{6}$$

where the $d \times d$ matrices $A(t)$, $B(t)$ are continuous and $A(t)$ is nonsingular on $[0, \infty)$. The matrix function $F(t)$ is possibly complex-valued, and hence solutions of (5) are generally complex-valued.

Before giving the main result some notation must be explained. For convenience, we take the norm of any matrix M to be the sum of the absolute values of its components and denote this sum by $|M|$. This should not cause confusion since it will be apparent from the subsequent formulas how one should interpret the various norms. If $M(s) = (M_{ij}(s))$ and $N(s)$ are $m \times n$ and $n \times p$ matrix functions defined on $[0, \infty)$, respectively, then

$$\int_0^t d(M)N = \left(\int_0^t \sum_{k=1}^n N_{kj}(s)\, dM_{ik}(s)\right), \qquad \int_0^\infty d(M)N = \lim_{t\to\infty} \int_0^t d(M)N,$$

provided that all indicated Riemann-Stieltjes integrals and limits exist.

THEOREM. *Let F and a nonsingular matrix function Γ be such that $|\Gamma(t)x(t)|$ is bounded for every solution x of* (5). *Assume that a $d \times d$ matrix function D can be chosen so that the following conditions hold*:

(i) $|D\Gamma A^{-1}B\Gamma^{-1}|$ *is integrable on* $[0, \infty)$;

(ii) $|D\Gamma A^{-1}\Gamma^{-1}| \to 0$ *as* $t \to \infty$;

(iii) *the matrix function $D\Gamma A^{-1}$ is continuous, locally of bounded variation and $\int_0^t d(D\Gamma A^{-1})\Gamma^{-1}$ is of bounded variation on* $[0, \infty)$. *Then*

$$\int_0^\infty D\Gamma x = \lim_{t\to\infty} \int_0^t D\Gamma x \tag{7}$$

exists for every solution of (5).

Proof. Multiply (5) by $D\Gamma A^{-1}$ and integrate by parts to obtain

$$D\Gamma A^{-1}x \,\big|_0^t = \int_0^t D\Gamma x + \int_0^t (D\Gamma A^{-1}B\Gamma^{-1})\Gamma x + \int_0^t d(M)\Gamma x \tag{8}$$

where $M(t) = \int_0^t d(D\Gamma A^{-1})\Gamma^{-1}$. The second and third integrals in the right member of (8) have finite limits at infinity since $|\Gamma x|$ is bounded, $|D\Gamma A^{-1}B\Gamma^{-1}|$ is integrable and M is of bounded variation on $[0, \infty)$. The term in the left member has a limit at infinity since $|D\Gamma A^{-1}x| \le |D\Gamma A^{-1}\Gamma^{-1}|\,|\Gamma x| \to 0$. The proof is complete.

In order to illustrate this theorem we will study one particular case. If the function

q is real-valued and locally of bounded variation on $[0, \infty)$, then $q(t) = q(0) + q_+(t) - q_-(t)$ denotes the Jordan decomposition of q. Further details concerning the Jordan decomposition are given in any text on analysis, but we do mention that q_+, q_- are nondecreasing.

COROLLARY. *In the linear differential equation*

$$y'' + (q(t) + f(t))y = 0 \tag{9}$$

let f be complex-valued and continuous on $[0, \infty)$ and let q be positive, continuous and locally of bounded variation on $[0, \infty)$.

(a) *Let α, β be constants which satisfy $\alpha, \beta < \frac{1}{2}$. If $\lim_{t\to\infty} q(t) = \infty$ and*

$$\int_0^\infty q^{-1}\, dq_- < \infty, \qquad \int_0^\infty |f|\, q^{-1/2} < \infty, \tag{10}$$

then the improper integrals

$$\int_0^\infty q^\alpha y, \qquad \int_0^\infty q^{\beta-1/2} y' \tag{11}$$

converge (possibly conditionally) for every solution of (9).

(b) *Let α, β satisfy $\alpha, \beta > \frac{1}{2}$. If $\lim_{t\to\infty} q(t) = 0$ and*

$$\int_0^\infty q^{-1}\, dq_+ < \infty, \qquad \int_0^\infty |f|\, q^{-1/2} < \infty \tag{12}$$

then the improper integrals

$$\int_0^\infty q^{\alpha+1/2} y \qquad \int_0^\infty q^\beta y' \tag{13}$$

converge (possibly conditionally) for every solution of (9).

We recover the original results of Hartman and Wintner by taking $\alpha = \beta = 0$ and $f \equiv 0$ in part (a) of the corollary. It should be mentioned that a certain type of duality is exhibited in the statement of these results. This duality is most evident when one compares the range of the constants α, β and the limiting behavior of q. Other duality relationships were discussed by Hartman in a slightly different context [2, p. 512]. The improper Riemann-Stieltjes integrals in (10), (12) need some explanation. Since

$$q(0)q(t)^{-1} = \exp\left(\int_0^t q^{-1}\, dq_-\right) \exp\left(-\int_0^t q^{-1}\, dq_+\right) \tag{14}$$

and q_+, q_- are nondecreasing, we see that q^{-1} is of bounded variation on $[0, \infty)$ and $\lim_{t\to\infty} q(t) (\leq \infty)$ exists if $\int_0^\infty q^{-1}\, dq_-$ is finite. In the same manner one can show that q is of bounded variation on $[0, \infty)$ and $\lim_{t\to\infty} q(t) (\geq 0)$ exists if $\int_0^\infty q^{-1}\, dq_+$ is finite. If $\delta > 0$, then q^δ is of bounded variation on $[0, \infty)$ if $\int_0^\infty q^{-1}\, dq_+$ is finite and $q^{-\delta}$ is of bounded variation on $[0, \infty)$ if $\int_0^\infty q^{-1}\, dq_-$ is finite. This follows from (14).

Before giving the proof of the corollary we need to state a useful lemma which isolates certain properties of solutions of (9).

LEMMA. *The functions*

$$E(t) \exp\left\{-\int_0^t q^{-1}\, dq_- - \int_0^t |f|\, q^{-1/2}\right\} \tag{15}$$

and

$$G(t)\exp\left\{-\int_0^t q^{-1}\,dq_+ - \int_0^t |f|\,q^{-1/2}\right\}, \tag{16}$$

where

$$E(t) = q(t)^{-1}\,|y'(t)|^2 + |y(t)|^2, \qquad G(t) = q(t)E(t), \tag{17}$$

are nonincreasing on $[0, \infty)$ *for every solution* y *of* (9).

Proof of Lemma. Let μ denote the exponential factor in (15) and define $\lambda = E\mu$. Since

$$\lambda(t) - \lambda(\tau) = \int_\tau^t \mu\,dE + \int_\tau^t E\,d\mu, \qquad 2q^{-1/2}\,|yy'| \le E$$

and (9) holds, it is not difficult to show that $\lambda(t) \le \lambda(\tau)$ if $t \ge \tau \ge 0$. A similar computation was indicated by Hartman and we omit the details [2, p. 510]. The function defined in (16) is nonincreasing since λ is nonincreasing and (14) is an identity. The proof is complete.

Proof of Corollary. (a) In the notation of the theorem, let

$$x = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -f & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad D = \begin{pmatrix} q^\alpha & 0 \\ 0 & q^\beta \end{pmatrix}.$$

We conclude from (10) and (15) that $|\Gamma x|$ is bounded for every solution of (9). Since

$$D\Gamma A^{-1}\Gamma^{-1} = \begin{pmatrix} 0 & -q^{\alpha-1/2} \\ q^{\beta-1/2} & 0 \end{pmatrix} \tag{18}$$

we see that $|D\Gamma A^{-1}\Gamma^{-1}| \to 0$ as $t \to \infty$ if $\alpha, \beta < \frac{1}{2}$. Also,

$$D\Gamma A^{-1}B\Gamma^{-1} = \begin{pmatrix} -fq^{\alpha-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and hence $|D\Gamma A^{-1}B\Gamma^{-1}|$ is integrable since $\alpha < \frac{1}{2}$ and (10) holds. But

$$\int_0^t d(D\Gamma A^{-1})\Gamma^{-1} = \begin{pmatrix} 0 & (1-\alpha)(\alpha - \frac{1}{2})^{-1}q^{\alpha-1/2} \\ q^{\beta-1/2} & 0 \end{pmatrix} + C$$

where C is some constant matrix, and the entries in the right member are of bounded variation on $[0, \infty)$ since $\int_0^\infty q^{-1}\,dq_-$ is finite and $\alpha, \beta < \frac{1}{2}$. The proof of (a) is complete. (b) The proof of part (b) is similar if we take A, B and D as in the proof of (a) and let

$$\Gamma = \begin{pmatrix} q^{1/2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The only point one must verify is hypothesis (iii) of the theorem. But

$$\int_0^t d(D\Gamma A^{-1})\Gamma^{-1} = \begin{pmatrix} 0 & -q^{\alpha-1/2} \\ \beta(\beta - \frac{1}{2})^{-1}q^{\beta-1/2} & 0 \end{pmatrix} + C$$

for some constant matrix C, and hence the matrix function in the left member of the last formula line is of bounded variation on $[0, \infty)$ since $\alpha, \beta > \frac{1}{2}$ and $\int_0^\infty q^{-1}\,dq_+$ is finite. The proof of the corollary is complete.

One might question if the restrictions "$\alpha, \beta < \frac{1}{2}$" or "$\alpha, \beta > \frac{1}{2}$" are apparent or intrinsic in the problem The following argument shows that they are in general necessary. Returning to the equation which was discussed in the opening paragraph, let q be such that (1) has a solution y_1 with $E(\infty, y_1) > 0$. This is always possible [4]. It follows that the oscillatory solution y_1 has the property that y_1 and $q^{-1/2}y_1'$ do not have limits at infinity. Hence the matrix in the left member of (8) (see (18)) will not have a limit at infinity if $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$. It is easy to check that the second and third integrals in the right member of (8) converge in these cases. Consequently the improper integrals given in (11) do not converge in general if $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$.

The restrictions "$\alpha, \beta > \frac{1}{2}$" are in general necessary in part (b) of the corollary. Suppose that $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$. If q and y_1 are as in the previous paragraph, then $v(s) \equiv y_1'(t(s))$ is an oscillatory solution of the ordinary differential equation

$$(d^2v)/(ds^2) + Q(s)v = 0, \qquad 0 \leq s < \infty$$

where $s = s(t) = \int_0^t q$, $t(s)$ is the inverse function of $s(t)$ and $Q(s) = 1/q(t(s))$. The function Q is nonincreasing, $\lim_{s\to\infty} Q(s) = 0$ and $dv/ds = -y_1(t(s))$. This transformation is discussed by Hartman [2, p. 512]. The solution v has the property that dv/ds and $Q^{1/2}v$ do not have limits at infinity although they are bounded on $[0, \infty)$. If $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$, we can mimic the arguments used in the previous paragraph to show that the term in the left member of (8) does not have a limit at infinity. In either case the second and third integrals in the right member of (8) have limits at infinity and hence the improper integrals $\int_0^\infty Qv$, $\int_0^\infty Q^{1/2}\,dv/ds$ do not converge.

References

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