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ASYMPTOTIC STABILITY AND INSTABILITY CRITERIA FOR SOME ELASTIC SYSTEMS BY LIAPUNOV'S DIRECT METHOD*

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1. Introduction. Sufficient conditions for asymptotic stability and instability are derived here for some elastic systems with dissipation. The systems are assumed to be governed by autonomous partial differential equations which may be non-selfadjoint. The method of analysis is Liapunov's direct method as generalized by Zubov [1] and Movchan [2].

For elastic systems under conservative loading the energy is typically chosen as a Liapunov functional. If the energy is positive definite, then the system may be shown to be stable. With dissipation present the time derivative of the energy often becomes negative semi-definite, and it is sometimes claimed in the literature that this implies asymptotic stability. Such a conclusion is not warranted and requires verification. This question is resolved here for some cases with the use of a new Liapunov functional whose time derivative is negative definite. In addition, instability criteria are obtained for some of these dissipative systems.

Consider a continuous system which occupies a bounded domain R in one-, two- or three-dimensional space $\{x\}$, and let C denote the boundary of R. Designate by $w(x, t)$ the displacement of the system from an equilibrium state which for simplicity is taken as $w(x, t) \equiv 0$, where $t \geq 0$ represents the time. For stability analysis, this displacement is assumed to be governed by a linear partial differential equation of the form

$$m(x)w_{tt} + \mathfrak{D}w_t + \mathfrak{L}_1 w + \mathfrak{L}_2 w = 0, \qquad x \in R, \quad t \geq 0, \tag{1}$$

where $m_1 \geq m(x) \geq m_0 > 0$, with homogeneous boundary conditions

$$\mathfrak{B}w = 0, \qquad x \in C \tag{2}$$

and initial conditions

$$w(x, 0) = w^0(x), \qquad w_t(x, 0) = w_t^0(x), \qquad x \in R. \tag{3}$$

Subscripts denote partial differentiation, $m(x)$ represents the density of the system, and $\mathfrak{D}$, $\mathfrak{L}_1$, $\mathfrak{L}_2$, and $\mathfrak{B}$ are linear, time-independent, spatial differential operators with $\mathfrak{L}_1$ self-adjoint and $\mathfrak{L}_2$ non-self-adjoint.

The functional space $\mathfrak{U}$ is defined as the space composed of real vector elements

$$u = \begin{bmatrix} w \\ w_t \end{bmatrix} \tag{4}$$

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whose components w and w_t satisfy the boundary conditions (2) and certain smoothness conditions. The equilibrium state under consideration is represented by the element $u = 0$, the initial state is given by u^0 which has components w^0 and w_t^0, and the resulting motion of the system (i.e., the solution of Eqs. (1), (2), and (3)) is denoted by $u(t, u^0)$, $t \geq 0$. A metric ρ is defined on $\mathfrak{U}$, and $\rho(u, 0)$ gives the metric distance between a state u and the equilibrium state (see [3] for details).

Stability, asymptotic stability and instability are defined as follows [1, 2]:

Definition. The equilibrium state $u = 0$ is said to be *stable with respect to* ρ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(u^0, 0) < \delta$ implies that $\rho[u(t, u^0), 0] < \varepsilon$ for all $t > 0$. If in addition $\rho[u(t, u^0), 0] \to 0$ as $t \to \infty$, then $u = 0$ is said to be *asymptotically stable with respect to* ρ.

Definition. The equilibrium state $u = 0$ is said to be *unstable with respect to* ρ if there exists an $\varepsilon > 0$ such that for any $\delta > 0$, no matter how small, there exists a $u^0 \in \mathfrak{U}$ for which $\rho(u^0, 0) < \delta$ and $\rho[u(t, u^0), 0] \geq \varepsilon$ for some $t > 0$.

From the work of Zubov [1] and Movchan [2] one can state the following two theorems.

Stability Theorem. *The equilibrium state $u = 0$ is stable with respect to ρ if there exists a functional $V(u)$ having the following properties for $u \in \mathfrak{U}$, $u^0 \in \mathfrak{U}$:*

$$\text{a) } \beta^2\rho^2(u, 0) \geq V(u) \geq \alpha^2\rho^2(u, 0) \qquad \text{for some constants } \alpha, \beta; \tag{5}$$

$$\text{b) } (dV/dt)[u(t, u^0)] \leq 0 \qquad \text{for} \quad t \geq 0. \tag{6}$$

If in addition

$$\text{c) } V[u(t, u^0)] \to 0 \quad \text{as} \quad t \to \infty \tag{7}$$

then $u = 0$ is asymptotically stable with respect to ρ. A sufficient condition for (c) *is given by*

$$\text{c}') \ (dV/dt)[u(t, u^0)] \leq -\nu^2\rho^2(u, 0) \qquad \text{for some constant } \nu. \tag{8}$$

Instability Theorem. *The equilibrium state $u = 0$ is unstable with respect to ρ if there exists a functional W having the following properties for $u \in \mathfrak{U}$, $u^0 \in \mathfrak{U}$:*

$$\text{a) for any } \delta > 0 \text{ there exists a } u^0 \in \mathfrak{U} \text{ for which } 0 < \rho(u^0, 0) < \delta \text{ and } W(u^0) > 0; \tag{9}$$

$$\text{b) } |W(u)| \leq \mu^2\rho^4(u, 0) \text{ for some constant } \mu; \tag{10}$$

$$\text{c) } (dW/dt)[u(t, u^0)] > 0 \text{ whenever } W[u(t, u^0)] > 0. \tag{11}$$

These theorems provide sufficient conditions for stability, asymptotic stability and instability which are written in a form convenient for application to the systems under consideration.

2. Asymptotic stability. The operator $\mathfrak{L}_1$ is self-adjoint so that, for example,

$$\int_R w_t\mathfrak{L}_1 w \, dx = \int_R w\mathfrak{L}_1 w_t \, dx \tag{12}$$

where w and w_t are components of a vector $u \in \mathfrak{U}$. The energy E for the conservative part of the system is defined by

$$E(u) = \frac{1}{2}\int_R [mw_t^2 + w\mathfrak{L}_1 w] \, dx. \tag{13}$$

With the use of integration by parts and Eqs. (1) and (12), one can show that the time rate of change of E during motion of the system is given by

$$\frac{dE}{dt}[u(t, u^0)] = -\int_R w_t[\mathfrak{D}w_t + \mathfrak{L}_2 w]\, dx. \tag{14}$$

Even if the functional E satisfies the stability conditions (5) and (6) for some metric ρ, where $\rho(u, 0)$ depends both on w and w_t, its derivative given by Eq. (14) is at best negative semi-definite and cannot be negative definite. It is not obvious, therefore, that condition (7) for asymptotic stability is satisfied. In some cases one can show that Eq. (7) does indeed hold by applying sophisticated mathematical techniques, such as the invariance principles derived by Hale [4] and Slemrod [5]. An alternate procedure for demonstrating asymptotic stability involves construction of a new functional for which conditions (5), (6) and (8) are satisfied. This approach will be used here. The new functional depends on the operator $\mathfrak{D}$, and two cases will be treated.

A. For the first case the damping is assumed to have the form

$$\mathfrak{D}w_t = 2\xi(x)w_t, \qquad \xi_1 \geq \xi(x) \geq \xi_0 > 0, \tag{15}$$

and it is assumed that for all $u \in \mathfrak{U}$

$$\int_R w\mathfrak{L}_1 w\, dx \geq c_1^2 \int_R w^2\, dx \qquad \text{for some constant } c_1. \tag{16}$$

The functional to be considered is

$$V(u) = \frac{1}{2}\int_R [mw_t^2 + w\mathfrak{L}_1 w + 2\xi w_t w + 2(\xi^2/m)w^2]\, dx \tag{17}$$

which reduces to the energy E if no damping is present. Let the metric ρ be defined by

$$\rho^2 = E. \tag{18}$$

Any metric equivalent to ρ also could be used [3].

It can easily be verified that condition (6) is satisfied, and the time rate of change of V is given by

$$\frac{dV}{dt}[u(t, u^0)] = -\int_R [\xi w_t^2 + w_t\mathfrak{L}_2 w + (\xi/m)w(\mathfrak{L}_1 w + \mathfrak{L}_2 w)]\, dx. \tag{19}$$

For $\mathfrak{L}_2 = 0$ it is seen that Eq. (8) is satisfied, and inequality (16) is then a sufficient condition for asymptotic stability. If the system is non-self-adjoint and $\mathfrak{L}_2 \neq 0$, however, additional conditions may be required in order to satisfy Eq. (8). This is illustrated in the following example.

Consider an elastic column of length l, mass per unit length $m(x)$, and bending stiffness $s(x)$, $s_1 \geq s(x) \geq s_0 > 0$, which is subjected to a compressive load P. The equation for the displacement $w(x, t)$ is assumed to be

$$m(x)w_{tt} + 2\xi(x)w_t + s(x)w_{xxxx} + Pw_{xx} = 0, \qquad 0 \leq x \leq l, \quad t \geq 0. \tag{20}$$

Under certain boundary conditions the system is conservative when $\xi = 0$; for instance, one end may be clamped ($w = w_x = 0$) or simply-supported ($w = sw_{xx} = 0$) while the other is either clamped, simply-supported, or sliding ($w_x = (sw_{xx})_x = 0$). One then obtains

$$\begin{aligned}
\mathfrak{L}_1 w &= s(x)w_{xxxx} + Pw_{xx}\,, \ \mathfrak{L}_2 = 0,\\
\rho(u, 0) &= \left\{\frac{1}{2}\int_0^l [mw_t^2 + sw_{xx}^2 - Pw_x^2]\,dx\right\}^{1/2},\\
V(u) &= \rho^2(u, 0) + \int_0^l [\xi w_t w + (\xi^2/m)w^2]\,dx,\\
\frac{dV}{dt}[u(t, u^0)] &= -\int_0^l \{\xi w_t^2 + (\xi/m)[sw_{xxxx} + Pw_{xx}]w\}\,dx,
\end{aligned} \tag{21}$$

and the equilibrium state $w(x, t) \equiv 0$ is asymptotically stable with respect to ρ if for all $u \in \mathfrak{U}$

$$\int_0^l [sw_{xx}^2 - Pw_x^2]dx \geq c_1^{\,2} \int_0^l w^2\,dx \qquad \text{for some constant } c_1\,. \tag{22}$$

If the column is clamped at one end and free at the other, with the load P acting tangentially to the column at the free end so that $sw_{xx} = (sw_{xx})_x = 0$ there, the system is nonconservative. In this case

$$\begin{aligned}
\mathfrak{L}_1 w &= s(x)w_{xxxx}\,, \ \mathfrak{L}_2 w = Pw_{xx}\,,\\
\rho(u, 0) &= \left\{\frac{1}{2}\int_0^l [mw_t^2 + sw_{xx}^2]\,dx\right\}^{1/2},\\
V(u, 0) &= \rho^2(u, 0) + \int_0^l [\xi w_t w + (\xi^2/m)w^2]\,dx,
\end{aligned} \tag{23}$$

and

$$\frac{dV}{dt}[u(t, u^0)] = -\int_0^l [\xi w_t^2 + Pw_t w_{xx} + (\xi/m)(sw_{xxxx} + Pw_{xx})w]\,dx. \tag{24}$$

With the calculus of variations one can show that

$$\int_0^l w_{xx}^2\,dx \geq (0.125\pi^4/l^4) \int_0^l w^2\,dx \tag{25}$$

for these boundary conditions, so that condition (16) is satisfied and condition (8) becomes the governing one. Inequality (25) can be used with Eq. (24) to bound dV/dt from above by an integral involving a quadratic form in w_{xx} and w_t and a quadratic form in w_{xx} and w. It then can be shown that these quadratic forms are negative definite and Eq. (8) is satisfied if

$$P < \min_{0\leq x\leq l}\, (2\xi/\pi^2 m)[(8l^4\xi^2 + \lambda\pi^4 sm)^{1/2} - 8^{1/2}l^2\xi], \tag{26}$$

where $\lambda = (m/\xi)\{\max_{0\leq x\leq l}\,(\xi/m)\}$. Thus Eq. (26) gives a sufficient condition for asymptotic stability with respect to ρ. Since V and ρ do not involve the load P, one can see that the analysis is also valid if P varies with time and if Eq. (26) is satisfied at all times $t \geq 0$.

B. For the second case consider a one-dimensional system governed by the equation

$$m(x)w_{tt} + 2\eta w_{txxxx} + \mathfrak{L}_1 w = 0, \qquad 0 \leq x \leq l, \quad t \geq 0 \tag{27}$$

where η is a positive constant. Here $\mathfrak{L}_2 = 0$ and $\mathfrak{D} = 2\eta\, \partial^4/\partial x^4$. Assume that for all $u \in \mathfrak{U}$

$$\int_0^l w\mathfrak{L}_1 w\, dx \geq c_2^2 \int_0^l w_{xx}^2\, dx \qquad \text{for some constant } c_2\,, \tag{28}$$

$$\int_0^l w_t w_{xxxx}\, dx = \int_0^l w_{txx} w_{xx}\, dx, \tag{29}$$

and that one has an inequality

$$\int_0^l w_{xx}^2\, dx \geq (c_3^2 + \tfrac{1}{2}) \int_0^l w^2\, dx \qquad \text{for some constant } c_3 \tag{30}$$

for w and therefore also for w_t. The functional V is chosen to be

$$V(u) = \frac{1}{2} \int_0^l [mw_t^2 + w\mathfrak{L}_1 w + 2\eta w_t w + 2(\eta^2/m) w_{xx}^2]\, dx \tag{31}$$

and the metric is defined by Eq. (18).

Condition (6) is satisfied and one can write

$$\frac{dV}{dt}[u(t, u^0)] = -\eta \int_0^l [2w_{txx}^2 - w_t^2 + (1/m) w\mathfrak{L}_1 w]\, dx. \tag{32}$$

It follows from Eq. (30) that condition (8) is satisfied, and therefore inequality (28) is sufficient for asymptotic stability. For example, if $\mathfrak{L}_1$ and ρ are given by Eqs. (21) for a column with conservative loading and if Eq. (27) governs the motion, then the equilibrium state $w(x, t) \equiv 0$ is asymptotically stable with respect to ρ if for all $u \in \mathfrak{U}$

$$\int_0^l [sw_{xx}^2 - Pw_x^2]\, dx \geq c_2^2 \int_0^l w_{xx}^2\, dx \qquad \text{for some constant } c_2\,. \tag{33}$$

3. Instability. In this section assume that $\mathfrak{L}_2 = 0$ and that $\mathfrak{D}$ is self-adjoint and positive semi-definite. Generalizing an example of Movchan [2], define

$$g(u) = \int_R [mw_t w + \tfrac{1}{2} w\mathfrak{D}w]\, dx \tag{34}$$

and

$$\begin{aligned} W(u) &= -E(u)g(u) \quad &&\text{if} \quad E(u) < 0, \\ &= 0 &&\text{if} \quad E(u) \geq 0 \end{aligned} \tag{35}$$

where E is defined by Eq. (13). Let there be an initial displacement w^0 for which

$$\int_R w^0 \mathfrak{D} w^0\, dx > 0, \qquad \int_R w^0 \mathfrak{L}_1 w^0\, dx < 0. \tag{36}$$

Then for $u^0 = (\zeta w^0, 0)^T$ one has $E(u^0) < 0$, $g(u^0) > 0$, and $W(u^0) > 0$, so that condition (9) of the instability theorem is satisfied by a sufficiently small choice of ζ. The metric ρ is chosen so that inequality (10) holds. For condition (11), when $W[u(t, u^0)] > 0$ one has $E[u(t, u^0)] < 0$, $g[u(t, u^0)] > 0$, and

$$(dW/dt)[u(t, u^0)] = -g[u(t, u^0)](dE/dt)[u(t, u^0)] - E[u(t, u^0)](dg/dt)[u(t, u^0)]. \tag{37}$$

From Eq. (14) with $\mathfrak{L}_2 = 0$ one sees that $dE/dt \leq 0$ and the first term of dW/dt is non-negative. For the second term, one can show that

$$(dg/dt)[u(t, u^0)] = \int_R [mw_t^2 - w\mathfrak{L}_1 w]\, dx \tag{38}$$

which must be positive when E is negative. Therefore dW/dt is positive along solutions when W is positive. It follows that inequalities (36) give a sufficient condition for instability with respect to ρ.

As an example, consider again a column with

$$m(x)w_{tt} + 2\xi w_t + 2\eta w_{txxxx} + s(x)w_{xxxx} + Pw_{xx} = 0, \qquad 0 \leq x \leq l, \quad t \geq 0, \tag{39}$$

where ξ and η are positive constants and the loading is conservative. Then

$$\begin{aligned} E(u) &= \frac{1}{2}\int_0^l [mw_t^2 + sw_{xx}^2 - Pw_x^2]\, dx, \\ g(u) &= \int_0^l [mw_t w + \xi w^2 + \eta w_{xx}^2]\, dx, \end{aligned} \tag{40}$$

and one can choose

$$\rho(u, 0) = \left\{\int_0^l [mw_t^2 + sw_{xx}^2]\, dx\right\}^{1/2}. \tag{41}$$

The equilibrium state $w(x, t) \equiv 0$ is unstable with respect to ρ if there is a displacement w^0 for which

$$\int_0^l [s(w_{xx}^0)^2 - P(w_x^0)^2]\, dx < 0. \tag{42}$$

With the choice of any admissible w^0 one obtains a sufficient condition for instability. For instance, if one end of the column is clamped and the other is sliding, then the choice

$$w^0 = 1 - \cos(\pi x/l) \tag{43}$$

yields the sufficient instability condition

$$P > 2(\pi/l)^2 \int_0^l s \cos^2(\pi x/l)\, dx.$$

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