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COMPUTATION OF ROTATIONAL FLOWS*

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1. Introduction. It is well known that vector fields may be resolved into irrotational and solenoidal parts

$$\mathbf{q} = \nabla\phi + \text{curl } \mathbf{A} \tag{1}$$

where also

$$\text{div } \mathbf{A} = 0 \tag{2}$$

and so

$$\nabla^2\phi = \text{div } \mathbf{q}, \tag{3}$$

$$\nabla^2\mathbf{A} = -\text{curl } \mathbf{q}. \tag{4}$$

ϕ and A are respectively called scalar and vector potentials of $\mathbf{q}$. In seeking to calculate $\mathbf{q}$ from given distributions of div $\mathbf{q}$ and curl $\mathbf{q}$ by the use of (3) and (4) the boundary conditions on ϕ and $\mathbf{A}$ become complicated [1]. This paper presents a simpler alternative method for computation of rotational flows.

The resolution (1) is not unique, unless suitable separate boundary conditions for ϕ and $\mathbf{A}$ are given. For example, a field $\mathbf{q}$ that is both irrotational and solenoidal may be expressed either wholly as $\nabla\phi$ or wholly as curl $\mathbf{A}$. Such a choice exists for irrotational, incompressible flows in the Cartesian x, y plane which may be represented in terms of a velocity potential ϕ or alternatively by a stream function ψ where $\mathbf{A} = \psi\, \mathbf{i}_z$. In polar coordinates we may take $\mathbf{A} = -\psi r^{-1}\mathbf{i}_\theta$ where ψ is now Stokes' stream function for a suitable axisymmetric flow. In these cases the alternatives are effected by appropriate boundary conditions: $\partial\phi/\partial n = \mathbf{q}\cdot\mathbf{n}$, the normal velocity, and $\mathbf{A} = 0$ for expression in terms of ϕ only; or $\phi = 0$ and $\psi = \int \mathbf{q}\cdot\mathbf{n}\, ds$ integrated along the boundary from some fixed point for expression in terms of ψ only.

The situation for more general fields is described by the following for a three-dimensional domain D with suitably smooth boundary surface D^*.

2. Determination of the solenoidal part. The existence of a scalar potential satisfying (3) is well known for consistent boundary data for $\partial\phi/\partial n$. Subtracting the gradient of ϕ from $\mathbf{q}$ leaves a solenoidal field for which a vector potential is sought. Denoting smoothness classes of functions by $C_m(D)$ for the class of functions with continuous derivatives of order m on D, and $C_m(D + D^*)$ for functions of $C_m(D)$ with continuous one-sided normal mth derivatives and continuous tangential mth derivatives on D^*, Hirasaki and Hellums [1] give a theorem:

If $\mathbf{V}$ *is in* $C_1(D + D^*)$ *and* $C_2(D)$ *and* div $\mathbf{V} = 0$ *in* D *then there exists a vector potential* $\mathbf{A}$ *such that* $\mathbf{V} = \text{curl } \mathbf{A}$ *and* div $\mathbf{A} = 0$ *in* D.

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However, to find **A** or curl **A** is very difficult. While the theorem leads to the Poisson equation (4), where curl **q** is taken as given, the boundary conditions for **A** are obscure. It is easy to produce the correct boundary conditions for **q** by taking, say, $\mathbf{A} = 0$ and $\partial\phi/\partial n = \mathbf{q}\cdot\mathbf{n}$. Then $\mathbf{A} = 0$ gives curl $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\nabla\phi +$ curl **A** has the correct normal velocity. It is also easy to solve (4) with these boundary conditions. But the result will not satisfy div $\mathbf{A} = 0$ and so fails to give curl $(\nabla\phi + \text{curl } \mathbf{A}) =$ curl **q**. Taking the divergence of (4) gives ∇^2 div $\mathbf{A} = 0$, and in order to maintain div $\mathbf{A} = 0$ the boundary conditions on **A** must be set accordingly. Only the tangential components of **A** affect the normal component of curl **A**; these must be set to give the correct normal component of **q**. Then the normal derivative of the normal component of **A** must be set to correspond to div $\mathbf{A} = 0$ on the boundary. Where $\partial\phi/\partial n$ is set equal to $\mathbf{q}\cdot\mathbf{n}$, the tangential component of **A** may be set to zero, or otherwise it is difficult to obtain these tangential components; Hirasaki and Hellums obtain them from the solution of an elliptic partial differential equation on the boundary surface. In any case the attraction of (4) is lost because of the complication of the boundary conditions.

A much simpler method is to find a solenoidal component of **q** directly by the Biot–Savart law, and then to set appropriate boundary conditions for the irrotational component. In doing this it is first necessary to extend the given vorticity distribution $\boldsymbol{\omega}$ in D piecewise continuously into the whole three-dimensional space so that div $\boldsymbol{\omega}$ exists and is zero everywhere. We will call this a solenoidal extension. This may be done either by closing the vortex lines by extending them outside D or by continuing them to infinity. The velocity induced at a point by a semi-infinite straight-line vortex $\boldsymbol{\omega}$ starting at a point with relative position **r** is obtained by direct integration as

$$-\boldsymbol{\omega} \times \mathbf{r}[1 - (\omega r)^{-1}\boldsymbol{\omega}\cdot\mathbf{r}][4\pi\,|\boldsymbol{\omega} \times \mathbf{r}|^2\,\omega^{-2}]^{-1}. \tag{5}$$

THEOREM 1. *Let* $\boldsymbol{\omega} \in C_1(D + D^*)$ *be solenoidal in* D. *Then there is a vector field* **V** *continuous on* $D + D^*$ *which is given by*

$$4\pi\mathbf{V} = -\int_\infty r^{-3}\boldsymbol{\omega} \times \mathbf{r}\, dv \tag{6}$$

where the integral is over the whole three-dimensional space on which $\boldsymbol{\omega}$ *is extended solenoidally and* **r** *is the position vector of the integration point relative to the point at which* **V** *is determined, such that* div $\mathbf{V} = 0$ *and* curl $\mathbf{V} = \boldsymbol{\omega}$ *in* D.

Proof. Since $\boldsymbol{\omega} \in C_1(D + D^*)$, a bounded, solenoidal extension exists. Define **A** by

$$4\pi\mathbf{A} = \int_\infty r^{-1}\boldsymbol{\omega}\, dv;$$

then **A** has continuous first derivatives everywhere and has second derivatives satisfying $\nabla^2\mathbf{A} = -\boldsymbol{\omega}$ wherever $\boldsymbol{\omega}$ is Hölder-continuous, in particular in D [2, p. 150 etc.].
Now

$$4\pi \operatorname{div} \mathbf{A} = \int_\infty \operatorname{div}(r^{-1}\boldsymbol{\omega})\, dv,$$

the divergence being with respect to coordinates of the point P at which **A** is determined. I.e.,

$$4\pi \operatorname{div} \mathbf{A} = \left[\int_S r^{-1}\boldsymbol{\omega}\, dS\right],$$

where integration is over suitable surfaces transverse to the vortex lines and the square brackets denote changes along the vortex lines. These surface integrals exist for bounded $\boldsymbol{\omega}$ even when the surface contains P, and tend to zero as all points of S tend to infinity since $\int_S \boldsymbol{\omega}\cdot d\mathbf{S}$ is constant. Since $\boldsymbol{\omega}$ is a solenoidal extension its lines are either closed, end where $\boldsymbol{\omega}$ is zero or run to infinity so that div $\mathbf{A} = 0$ everywhere. At points where $\boldsymbol{\omega}$ is Hölder-continuous, curl curl $\mathbf{A}$ exists and

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \nabla \operatorname{div}\mathbf{A} - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A} = \boldsymbol{\omega}.$$

Now

$$4\pi \operatorname{curl}\mathbf{A} = \int_\infty \operatorname{curl}_P (r^{-1}\boldsymbol{\omega})\, dv = -\int_\infty r^{-3}\boldsymbol{\omega} \times \mathbf{r}\, dv = 4\pi\mathbf{V}$$

and the theorem is proved.

Remarks. 1. One such extension is to continue vortex lines to infinity and amend (6) by a surface integral of (5):

$$\begin{aligned} 4\pi\mathbf{V} = &-\int_D r^{-3}\boldsymbol{\omega} \times \mathbf{r}\, dv \\ &-\int_{D^*} \boldsymbol{\omega} \times \mathbf{r}[1 - (\omega r)^{-1}\boldsymbol{\omega}\cdot\mathbf{r}]\omega^2\, |\boldsymbol{\omega} \times \mathbf{r}|^{-2}\, dS, \end{aligned} \tag{7}$$

allowing overlapping of the semi-infinite extensions provided that they are smooth enough for the integral in (6) to exist.

2. The solenoidal extension will admit transverse discontinuities in $\boldsymbol{\omega}$. Where $\boldsymbol{\omega}$ lies along D^* no extension is necessary. Then at such points $\mathbf{V}$ will be continuous but not differentiable.

3. The condition $\boldsymbol{\omega} \in C_1(D + D^*)$ could be relaxed provided a solenoidal extension exists.

THEOREM 2. *Let* $\mathbf{q} \in C_2(D + D^*)$. *Then there exist* ϕ *and* $\mathbf{V}$, *determined as in Theorem* 1, *such that* $\mathbf{q} = \nabla\phi + \mathbf{V}$ *in* D.

Proof. Taking $\boldsymbol{\omega} = \operatorname{curl}\mathbf{q}$, which satisfies the condition of Theorem 1, we may construct $\mathbf{V}$ so that curl $\mathbf{V} = \boldsymbol{\omega}$ and div $\mathbf{V} = 0$ in D, and $V\cdot n$ is continuous on D^*. Then

$$\begin{aligned} \nabla^2\phi &= \operatorname{div}\mathbf{q} \quad \text{in} \quad D, \\ \partial\phi/\partial n &= \mathbf{q}\cdot\mathbf{n} - \mathbf{V}\cdot\mathbf{n} \quad \text{on} \quad D^* \end{aligned} \tag{8}$$

determine a function ϕ (within an arbitrary additive constant), since div $\mathbf{q} \in C_1(D)$ and $\mathbf{q}\cdot\mathbf{n}$ is continuous on D^* and the boundary data are consistent by applying Green's theorem to $\mathbf{q} - \mathbf{V}$, as essentially follows from Kellogg ([2], p. 150 ff. and p. 314). Now consider

$$\mathbf{q}_1 = \mathbf{q} - \nabla\phi - \mathbf{V}.$$

Then in D,

$$\operatorname{curl}\mathbf{q}_1 = \operatorname{curl}\mathbf{q} - \operatorname{curl}\mathbf{V} = \mathbf{0},$$

so that $\mathbf{q}_1 = \nabla\phi_1$ and

$$\nabla^2\phi_1 = \operatorname{div}\mathbf{q}_1 = \operatorname{div}\mathbf{q} - \nabla^2\phi = 0.$$

On D^*

$$\partial\phi_1/\partial n = \mathbf{q}_1\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n} - \partial\phi/\partial n - \mathbf{V}\cdot\mathbf{n} = 0,$$

and ϕ_1 is constant in D. Thus $\mathbf{q}_1$ is zero and the theorem is proved.

Remarks. 1. The condition $\mathbf{q} \in C_2(D + D^*)$ is not minimal.

2. The solution of (7) and (8) is much simpler than the procedure of Hirasaki and Hellums and provides also for the computation of a non-solenoidal q with known div and curl.

3. Computation schemes. A suitable computation scheme would involve the iterative coupling of the calculation of irrotational and solenoidal parts. From a given approximation, the iterative step would be essentially:

1. Solve the linearised vorticity transport equation for $\boldsymbol{\omega}$.
2. Calculate a suitable solenoidal component $\mathbf{V}$ as above, using the Biot-Savart law.
3. Use boundary data $\mathbf{q}\cdot\mathbf{n} - \mathbf{V}\cdot\mathbf{n}$ to solve the continuity equation $\nabla^2\phi = 0$ for the irrotational component $\nabla\phi$.

For compressible flow part 3 involves a more complicated equation representing div $\rho\mathbf{q} = 0$, and in this case an alternative is to use a mass-flow potential Φ defined so that $\mathbf{m} = \rho\mathbf{q} = \nabla\Phi + \mathbf{M}$, div $\mathbf{M} = 0$. This retains Laplace's equation in step 3, and instead of using boundary data $\mathbf{q}\cdot\mathbf{n}$ it is generally equally realistic to set $\mathbf{m}\cdot\mathbf{n}$. A minor penalty occurs in step 2 where curl $\mathbf{M}$ = curl $\rho\mathbf{q} = \rho\boldsymbol{\omega} + \nabla\rho \times \mathbf{q}$ must be calculated from $\boldsymbol{\omega}$ and the velocity field before the Biot-Savart law is used.

Such a scheme is especially suitable for almost irrotational flows, where simplifying assumptions can realistically be made about the vorticity. For example, a flow may carry only streamwise vorticity with little viscous dissipation, with corresponding simplifications of vorticity transport and energy equations. Such a case occurs in flow in turbomachinery blade rows receiving shed vorticity from upstream blade rows.

Postscript. Hirasaki and Hellums [3] have recently simplified their previous work, recognising that "it is advantageous to use both potentials". They set $\partial\phi/\partial n = \mathbf{q}\cdot\mathbf{n}$ with boundary conditions on $\mathbf{A}$ simpler than before: both tangential components are zero and a third-order normal-derivative condition on the normal component corresponds to div $\mathbf{A} = 0$. The present note is yet simpler with explicit representation of solenoidal parts.

REFERENCES

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