ON AN INTEGRAL EQUATION APPROACH TO DISPLACEMENT PROBLEMS OF CLASSICAL ELASTICITY*

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1. Introduction. In a recent paper Kanwal [1] has established an integral equation method for solving displacement problems of elasticity. The method is based on the generalization of Green's procedure in potential theory. But it appears to have gone unnoticed by Kanwal that his formulation has a direct bearing on Betti's method of elasticity [2], which has long been known as a tensor counterpart of Green's procedure.

The purpose of the present note is to point this fact out and to show further that the result obtained by Kanwal is identical with that of Betti. The note deals only with the static case, but the analysis may be extended easily to the dynamic case.

2. Integral equation method. In the absence of body forces the Navier-Cauchy equations in elastostatics are

$$(\lambda + \mu) \operatorname{grad} \vartheta + \mu \nabla^2 \mathbf{u} = 0, \qquad \vartheta = \operatorname{div} \mathbf{u}, \tag{1}$$

where $\mathbf{u}$ (u_i ; $i = 1, 2, 3$) is the displacement vector and λ, μ are Lamé constants of the material medium. The components u_i are functions of Cartesian coordinates x_i . Let the region under consideration be denoted by V and its bounding surface by S. The unit normal $\mathbf{n}$ (n_i ; $i = 1, 2, 3$) will be directed outward to S. A brief description of Kanwal's procedure in deriving the solutions of Eqs. (1) follows.

Choose a tensor function $\mathbf{U}$ (U_{ij} ; $i, j = 1, 2, 3$) such that

$$U_{ij} = \frac{1}{8\pi\mu(\lambda + 2\mu)} \left\{ (\lambda + 3\mu) \frac{\delta_{ij}}{|\mathbf{P} - \mathbf{q}|} + (\lambda + \mu) \frac{(x_i - x_i^0)(x_j - x_j^0)}{|\mathbf{P} - \mathbf{q}|^3} \right\}, \tag{2}$$

which corresponds to the ith component of the displacement at a point $\mathbf{P}(x_i)$ by a unit force applied in the jth direction at a point $\mathbf{q}(x_i^0)$; δ_{ij} is the Kronecker delta. From U_{ij} we can compute dilatations θ_i which are given by

$$\theta_i = \frac{\partial U_{ik}}{\partial x_k} = -\frac{1}{4\pi(\lambda + 2\mu)} \frac{x_i - x_i^0}{|\mathbf{P} - \mathbf{q}|^3}. \tag{3}$$

To find the displacement $\mathbf{u}$ at an interior point $\mathbf{P}$ we surround the point $\mathbf{P}$ with an infinitesimal sphere S_σ (since θ_i and U_{ij} are singular at $\mathbf{P} = \mathbf{q}$) of radius σ so that the sphere lies entirely in V. Let the volume enclosed by the sphere be denoted by V_σ . Applying Green's second identity to the functions u_i , U_{ij} , we obtain

$$\int_{V-V_\sigma} \{u_i \nabla^2 U_{ij} - U_{ij} \nabla^2 u_i\} \, dV = \int_{S+S_\sigma} \left\{ u_i \frac{\partial U_{ij}}{\partial n} - U_{ij} \frac{\partial u_i}{\partial n} \right\} ds, \tag{4}$$

* Received August 10, 1970; revised version received December 25, 1970.

whence it can be derived that

$$\int_{S+S_\sigma} \left[\mathbf{u} \cdot \left\{ \mu \frac{\partial \mathbf{U}}{\partial n} + (\lambda + \mu) \theta \mathbf{n} \right\} - \mathbf{U} \cdot \left\{ \mu \frac{\partial \mathbf{u}}{\partial n} + (\lambda + \mu) \vartheta \mathbf{n} \right\} \right] ds = 0. \tag{5}$$

Evaluating the integral over S_σ and letting $\sigma \to 0$, we may derive the displacement at any point $\mathbf{P}$ ($\subset V$) as

$$\mathbf{u}(\mathbf{P}) = -\int_S \left[\mathbf{u} \cdot \left\{ \mu \frac{\partial \mathbf{U}}{\partial n} + (\lambda + \mu) \theta \mathbf{n} \right\} - \mathbf{U} \cdot \left\{ \mu \frac{\partial \mathbf{u}}{\partial n} + (\lambda + \mu) \vartheta \mathbf{n} \right\} \right] \cdot \tag{6}$$

Now we show that Kanwal's formulation (6) is identical with the classic result of Betti, which expresses the displacement at the point $\mathbf{P}$ as

$$u_i(\mathbf{P}) = \int_S \{ t_j U_{ij} - u_j T_{ij} \} \, ds, \tag{7}$$

better known as Betti's second identity [3]. The boundary tractions $\mathbf{t}$ (t_i ; $i = 1, 2, 3$) and $\mathbf{T}$ (T_{ij} ; $i, j, = 1, 2, 3$) are computed from the displacements $\mathbf{u}$ and $\mathbf{U}$ respectively. Eq. (5) may be rewritten as

$$\int_{S+S_\sigma} \{ \mathbf{u} \cdot \mathbf{T} - \mathbf{t} \cdot \mathbf{U} \} \, ds + \mu \int_{S+S_\sigma} \{ \mathbf{u} \cdot (\mathbf{a} \wedge \mathbf{U}) - \mathbf{U} \cdot (\mathbf{a} \wedge \mathbf{u}) \} \, ds = 0, \tag{8}$$

where $\mathbf{a} = \nabla \wedge \mathbf{n}$. The second integral in Eq. (8) vanishes identically by Gauss' divergence theorem, and thus Eq. (8) is identical with Eq. (2.3) of Rizzo [4] from which Betti's second identity (7) follows immediately.

References

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