# A NOTE ON THE DECOMPOSITION OF AN ABSOLUTE-VALUE LINEAR PROGRAMMING PROBLEM*

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**Introduction.** There occurs in structural design a dual linear programming problem [1]

$$\text{primal problem: minimize } \varphi = \sum_i |F_i| \, \Delta_i^a \text{ subject to } \tilde{N}F = P;$$

$$\text{dual problem: maximize } \psi = \tilde{P}\,\delta \text{ subject to } |N\delta| \leq \Delta^a. \tag{1}$$

In Eq. (1) the problem is to find the matrices $F$ and $\delta$ given the matrices $\Delta^a$, $P$, and $N$. ($F$ and $\Delta^a$ are $b \times 1$ matrices, $\delta$ and $P$ are $j \times 1$ matrices, and $N$ is a $b \times j$ matrix.) Briefly, the primal problem is concerned with finding for a truss a set of bar forces $F$ which satisfy the joint equilibrium equations and minimize $\varphi$ which is proportional to the weight of the truss; the dual problem attempts to find a set of joint displacements $\delta$ which maximize $\psi$ which is proportional to the work done by the joint loads $P$ while restricting the absolute value of the length change of each bar, i.e. $|(N\delta)_i| \leq \Delta_i^a$. When the same truss is subjected to two independent sets of joint loads [2] $P^1$ and $P^2$, Eq. (1) generalizes to

primal problem:

$$\text{minimize } \varphi = \sum_i \Delta_i^a \max \{|F_i^1|, |F_i^2|\}, \qquad \text{subject to } \tilde{N}F^1 = P^1 \text{ and } \tilde{N}F^2 = P^2;$$

dual problem:

$$\text{maximize } \psi = \tilde{P}^1\,\delta^1 + \tilde{P}^2\,\delta^2, \qquad \text{subject to } |N\delta^1| + |N\delta^2| \leq \Delta^a. \tag{2}$$

The superscripts in Eq. (2) refer to loading conditions in a rather obvious manner. Here it is necessary to find the matrices $F^1$, $F^2$, $\delta^1$ and $\delta^2$ given the matrices $P^1$, $P^2$, $\Delta^a$, and $N$. Eq. (2) is remarkable in that it decomposes into two independent problems, each of which has the form of Eq. (1). They may be written as

sum problem:

$$\begin{aligned} &\text{minimize } \varphi^S = \sum_i |F_i^S| \, \Delta_i^a \text{ subject to } \tilde{N}F^S = P^S; \\ &\text{maximize } \psi^S = \tilde{P}^S\,\delta^S \text{ subject to } |N\delta^S| \leq \Delta^a; \end{aligned} \tag{3}$$

difference problem:

$$\begin{aligned} &\text{minimize } \varphi^D = \sum_i |F_i^D| \, \Delta_i^a \text{ subject to } \tilde{N}F^D = P^D; \\ &\text{maximize } \psi^D = \tilde{P}^D\,\delta^D \text{ subject to } |N\delta^D| \leq \Delta^a \end{aligned} \tag{4}$$

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in which

$$P^S = \tfrac{1}{2}(P^1 + P^2), \qquad P^D = \tfrac{1}{2}(P^1 - P^2),$$
$$F^S = \tfrac{1}{2}(F^1 + F^2), \qquad F^D = \tfrac{1}{2}(F^1 - F^2), \tag{5}$$
$$\delta^S = \delta^1 + \delta^2, \qquad \delta^D = \delta^1 - \delta^2.$$

In order to obtain Eqs. (3) and (4) from Eq. (2) it is only necessary to use the relationships

$$\max \{|x|, |y|\} = \tfrac{1}{2}|x + y| + \tfrac{1}{2}|x - y| \tag{6}$$

and

$$|x| + |y|1 \leq 1 \Leftrightarrow |x + y| \leq 1 \qquad |x - y| \leq 1 \tag{7}$$

together with some recombinations of terms. Having gotten this far it is natural to ask about three or even $n$ loading conditions. It is to these cases that this note is directed.

**Three loading conditions.** The generalization of Eq. (2) to the case of three loading conditions is simply

primal problem:

$$\text{minimize } \varphi = \sum_i \Delta_i^a \max \{|F_i^1|, |F_i^2|, |F_i^3|\}$$
$$\text{subject to } \tilde{N}F^1 = P^1, \tilde{N}F^2 = P^2, \text{ and } \tilde{N}F^3 = P^3; \tag{8}$$

dual problem:

$$\text{maximize } \psi = \tilde{P}^1\delta^1 + \tilde{P}^2\delta^2 + \tilde{P}^3\delta^3 \text{ subject to } |N\delta^1| + |N\delta^3| + |N\delta^3| \leq \Delta^a.$$

In Eq. (8) the problem is to find $F^1$, $F^2$, $F^3$, $\delta^1$, $\delta^2$, and $\delta^3$ given $P^1$, $P^2$, $P^3$, $\Delta^a$, and $N$. Of interest here is the question of the decomposition of this system.

Probably the most direct way to approach the case of two loading conditions, Eq. (2), is through the dual problem using Eq. (7) which shows that the region indicated in Fig. 1

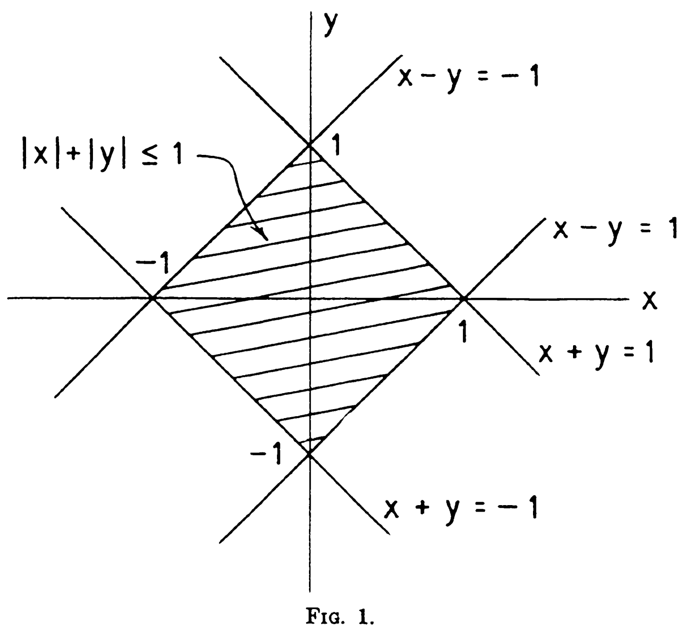


Fig. 1.

can be described as bounded by four straight lines. In a similar vein, Fig. 2 indicates that the region inside the octahedron can be described as bounded by eight planes, i.e.

$$\begin{aligned} |x| + |y| + |z| \leq 1 &\Leftrightarrow |x + y + x| \leq 1 \\ &\Leftrightarrow |-x + y + z| \leq 1 \\ &\Leftrightarrow |-x - y + z| \leq 1 \\ &\Leftrightarrow |x - y + z| \leq 1 \end{aligned} \tag{9}$$

This motivates rewriting the dual problem of Eq. (8) as

dual problem:

$$\text{maximize } \psi = \bar{p}^1 d^1 + \bar{p}^2 d^2 + \bar{p}^3 d^3 + \bar{p}^4 d^4$$

$$\text{subject to } |Nd^1| \leq \Delta^a, \quad |Nd^2| \leq \Delta^a, \quad |Nd^3| \leq \Delta^a, \quad |Nd^4| \leq \Delta^a, \tag{10}$$

in which

$$\begin{aligned} p_1 &= \tfrac{1}{4}(P^1 + P^2 + P^3), & d_1 &= \delta^1 + \delta^2 + \delta^3, \\ p_2 &= \tfrac{1}{4}(-P^1 + P^2 + P^3), & d_2 &= -\delta^1 + \delta^2 + \delta^3, \\ p_3 &= \tfrac{1}{4}(-P^1 - P^2 + P^3), & d_3 &= -\delta^1 - \delta^2 + \delta^3, \\ p_4 &= \tfrac{1}{4}(P^1 - P^2 + P^3), & d_4 &= \delta^1 - \delta^2 + \delta^3. \end{aligned} \tag{11}$$

Since neither the $p$s nor the $d$s are independent, Eq. (10) does not decompose directly as does the case of two loading conditions. As an alternative procedure, it can, however, be embedded in a system in which they are considered independent (i.e. the decompo-

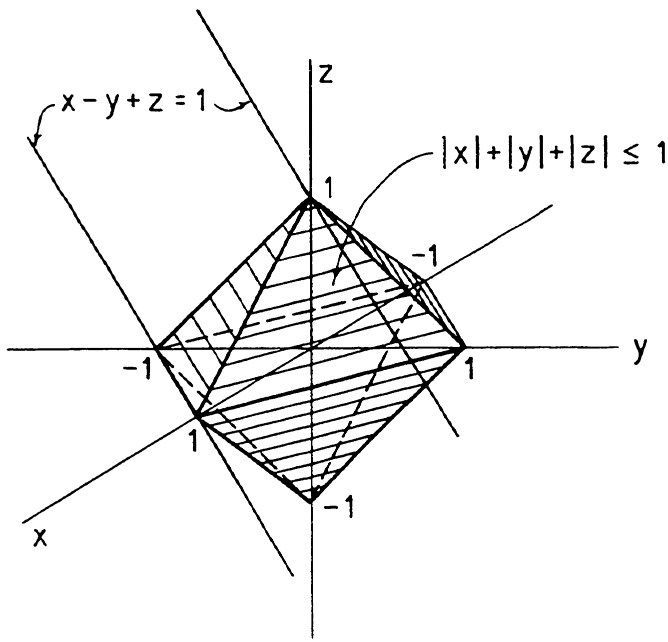


Fig. 2.

sition of Eq. (10) into four problems of the form of Eq. (1) can be forced) which involves the relaxation of constraints and results in an upper bound on $\psi$. Since the solution of these four problems is a relatively simple matter compared to the direct solution of Eq. (8) and since, given these four solutions, it is a relatively simple matter to combine them to produce lower bounds using the dual problem directly, this embedding has interesting potential. As an aside, attempting to carry the form of Eq. (10) through to the primal leads to the inequality

$$\max \{|x|, |y|, |z|\} \leq \tfrac{1}{4}[|x + y + z| + |-x + y + z| + |-x - y + z| + |x - y + z|] \tag{12}$$

which follows directly from the fact that

$$\tfrac{1}{4}[|x + y + z| + |-x + y + z| + |-x - y + z| + |x - y + z|] = \tfrac{1}{2}\{\max [|x + y|, |z|] + \max [|x - y|, |z|]\}$$

using the fact that $\max [|x + y|, |z|] \geq |z|$ and the symmetry of the right-hand side of Eq. (12) in $x$, $y$, $z$.

The case of $n$ loading conditions is now fairly obvious but since the relationship which corresponds to the right-hand side of Eq. (9) contains $\frac{1}{2}(2^n)$ terms, the algebraic difficulties compound themselves rapidly.

## References

[1] W. R. Spillers and John Farrell, *On the analysis of structural design*, J. Math. Anal. Appl. 25, 285–295 (1969)

[2] W. R. Spillers and O. Lev, *Design for two loading conditions*, submitted to Proc. Amer. Society Civil Engineers.