QUARTERLY OF APPLIED MATHEMATICS
JULY, 1973

POINCARÉ-LIGHTHILL AND LINEAR-TIME-SCALES METHODS FOR LINEAR PERTURBATION PROBLEMS*

BY

A. KLIMAS (*Aeronautical Research Associates of Princeton, Inc., Princeton*)

F. X. MURPHY, JR. (*City College of the City University of New York*)

G. SANDRI (*Aeronautical Research Associates of Princeton, Inc., Princeton, and Hunter College of the City University of New York*)

Summary. For a class of multidimensional linear perturbation problems of considerable significance in applications, the Poincaré–Lighthill technique is shown to give first-order expansion terms which grow unbounded relative to the leading term (secular behavior), while the method of linear time scales leads to well-behaved expansion terms. A solvable example is introduced for comparison with the exact solution.

1. Introduction. The problem of obtaining uniformly valid approximate solutions for certain classes of differential equations (given domains) has been attacked with a variety of weapons, among which have been the Poincaré–Lighthill (P–L) method [1–4] and the linear time scale (L–T–S) technique [5–8]. Several alternative techniques have also been used [9–15]. The specific formalism adopted here for L–T–S is that of [5]. For the P–L approach, sufficient conditions have been found for convergence of the expansion [16] and necessary conditions have been obtained for the boundedness of the first-order terms in multidimensional problems (based on the integrated form of Lighthill's perturbation equations) [17–19]. However, general criteria for establishing the uniformity of a perturbation expansion approximation (asymptotic in a small parameter ϵ), obtained by a specific technique, are still lacking, and the discovery of such criteria is a major challenge. It is widely thought that certain techniques, specifically P–L, are of more general applicability than others, and they are thus of primary concern in the search for criteria of applicability. This note shows that, for a broad class of multidimensional problems, the P–L method fails, even in first order, to give a uniform approximation to the exact result, while the L–T–S technique succeeds.

The equation of concern is

$$(\partial F/\partial t) + H_0 F = -\epsilon H_1 F, \tag{1}$$

for $t \geq 0$. F is taken to be a vector and H_0 and H_1 are linear operators (time-independent for definiteness). The initial condition imposed on (1) is

$$F(0) \quad \text{is a } \textit{given} \text{ vector.} \tag{2}$$

* Received November 7, 1971; revised version received December 15, 1971. Some of the work reported here was supported by the National Aeronautics and Space Administration under Contract No. NAS5-23020.

If H_0 and H_1 are (finite) matrices, Eqs. (1) and (2) form a special case of those considered by Usher [18] (although they are quite different in spirit from the usual P–L problem). Eqs. (1) and (2) are of great significance for applications. The Liouville equation of nonequilibrium statistical mechanics and the Schrödinger equation of quantum mechanics take the form of Eqs. (1) and (2) for many physical problems.

The P–L technique approaches the equation by introducing a "stretched" time coordinate, s, with

$$t = s + \epsilon\zeta(s) + \cdots, \tag{3}$$

so that Eq. (1) becomes

$$(\partial F/\partial s) = -(H_0 + \epsilon H_1)(1 + \epsilon\dot{\zeta} + \cdots)F. \tag{4}$$

Expanding F,

$$F = F^0 + \epsilon F^1 + \cdots, \tag{5}$$

and equating coefficients of the powers of ϵ leads to

$$(\partial F^0/\partial s) + H_0F^0 = 0, \tag{6}$$

$$(\partial F^1/\partial s) + H_0F^1 = -(\dot{\zeta}H_0 + H_1)F^0, \cdots. \tag{7}$$

There is in this last equation a possible nonuniformity (i.e. relative unboundedness of F^1) originating in the H_1F^0 term, and it must be removed by the $\dot{\zeta}H_0F^0$ term. Clearly, then, H_0F^0 must not be zero if the technique is to work (agreeing with Wasow's criterion [16] when the problem is a scalar one and satisfying Usher's conditions [18]); but this necessary condition is far from sufficient to guarantee that the singularity in F^1 is removed. In more detail, Eq. (7) can be rewritten as

$$\exp(-H_0s)\frac{\partial}{\partial s}(\exp(H_0s)F^1) = -(\dot{\zeta}H_0 + H_1)F^0, \tag{8}$$

using the matrix exponential. Introducing the solution to Eq. (6) and integrating yields

$$F^1(s) = \exp(-H_0s)F^1(0) - \exp(-H_0s)\left[[\zeta(s) - \zeta(0)]H_0 + \int_0^s d\tau \exp(H_0\tau)H_1\exp(-H_0\tau)\right]\cdot F^0(0). \tag{9}$$

When $\zeta \equiv 0$, this is a standard formula in perturbation theory [19]. Now, as a *simple example*, if H_1 is the sum of matrices which satisfy

$$H_1 = H + C, \quad [H_0, C] \equiv H_0C - CH_0 = 0,$$

and if $\int_0^s d\tau \exp(H_0\tau)H\exp(-H_0\tau)$ is bounded in the limit as $s \to \infty$, then the commuting part, C, leads to a singularity in F^1 at $s = \infty$ which cannot be removed by the stretching function ζ,

$$F^1(s) = \exp(-H_0s)F^1(0) - \exp(-H_0s)\left[(\zeta(s) - \zeta(0))H_0 + sC + \int_0^s d\tau \exp(H_0\tau)H\exp(-H_0\tau)\right]\cdot F^0(0), \tag{10}$$

unless C is essentially the same operator as H_0 or $F^0(0)$ is an eigenvector of both H_0 and C. In these cases, Eqs. (1) and (2) would not, in effect, require uniformization. (For this problem, the linear-time-scale method gives

$$\hat{F}^1(\tau_0) = \exp(-H_0\tau_0)\hat{F}^1(0) - \exp(-H_0\tau_0)\left[\left(\frac{\partial}{\partial\tau_1} + C\right)\tau_0 + \int_0^{\tau_0} d\tau \exp(H_0\tau)H\exp(-H_0\tau)\right]\hat{F}^0(\tau_0 = 0) \tag{10'}$$

where the notation is explained below. The secularity is removed by requiring

$$\frac{\partial\hat{F}^0}{\partial\tau_1} + C\hat{F}^0 = 0.$$

Examples of operators C and H satisfying our requirements can be readily constructed when iH_0 and iH_1 are finite-dimensional Hermitian matrices and iH_0 is not degenerate. In this case, iH_0 has a complete set of eigenvectors $| i\rangle$,

$$H_0 |i\rangle = -iE_i| i\rangle,$$

which may, in view of the commutation of H_0 and C, also be taken to be eigenvectors of C. In the basis of the $| i\rangle$'s, both H_0 and C are diagonal. Now, if C is chosen to be the diagonal elements of H_1 on this basis, and zero off-diagonal, then the integral term is nonsecular since

$$\int_0^s d\tau \exp(H_0\tau)H\exp(-H_0\tau) = \sum_{i,j}\int_0^s d\tau \exp[i(E_j - E_i)\tau]H_{ij}\, |i\rangle\langle j|$$

and $H_{ij} = (H_1 - C)_{ij} = 0$ if $i = j$ so that the diagonal elements are zero and

$$\int_0^s d\tau \exp[i(E_j - E_i)\tau]H_{ij}\, |i\rangle\langle j| = \frac{\exp[i(E_j - E_i)s] - 1}{i(E_j - E_i)} H_{ij}\, |i\rangle\langle j|$$

is bounded for nondegenerate iH_0 in the limit $s \to \infty$; finiteness of the dimensionality then guarantees boundedness of the summation.

The obvious conclusion of this discussion is that the P–L technique works only for very restricted cases in the multidimensional problem.

2. Exactly solvable example. Choosing $\mathbf{F}(t)$ to be a three-dimensional vector with components F_i $(i = 1, 2, 3)$ in a Cartesian coordinate system, an example of Eq. (1) which is exactly solvable is

$$(\partial\mathbf{F}/\partial t) + \boldsymbol{\beta} \times \mathbf{F} = -\epsilon\nu\mathbf{F} \tag{11}$$

where $\boldsymbol{\beta}$ is a fixed three-vector of unit length. This equation arises as a model in the study of energetic particle beams in the presence of turbulent magnetoplasmas. The vector $\boldsymbol{\beta}$ represents the mean magnetic field in the plasma and ν represents the overall effect of the random part of the field. It is closely related to an equation studied in great detail by L–T–S methods [8]. Defining the matrices Ω, N, P,

$$\Omega_{ij} = \epsilon_{ijk}\beta_k\,, \qquad P_{ij} = \beta_i\beta_j\,, \qquad N_{ij} = \delta_{ij} - \beta_i\beta_j\,,$$

the exact solution to Eq. (11) written as

$$(\partial \mathbf{F}/\partial t) + \Omega \cdot \mathbf{F} = -\epsilon\nu \mathbf{F} \tag{11'}$$

may be shown to be

$$\mathbf{F} = \exp(-\epsilon\nu t)[P + N \cos t - \Omega \sin t] \cdot \mathbf{F}(0). \tag{12}$$

(This may be derived by showing that

$$(\partial/\partial t) \exp(\epsilon\nu t)\mathbf{F} = -\Omega \cdot \exp(\epsilon\nu t)\mathbf{F}$$

and by expansion of the resulting matrix exponential using $\Omega \cdot \Omega = -N$, $N \cdot \Omega = \Omega \cdot N = \Omega$.)

(a) The *direct perturbation expansion* solution to Eq. (11′) follows from expansion of $\mathbf{F}$:

$$\mathbf{F}(t) = \mathbf{F}^0(t) + \epsilon \mathbf{F}^1(t) + \epsilon^2 \mathbf{F}^2(t) + \cdots .$$

Equating the coefficients of the powers of ϵ:

$$(\partial \mathbf{F}^0/\partial t) + \Omega \cdot \mathbf{F}^0 = 0,$$
$$(\partial \mathbf{F}^1/\partial t) + \Omega \cdot \mathbf{F}^1 = -\nu \mathbf{F}^0, \quad \text{etc.} \tag{13}$$

The solution to the zeroth-order equation is then

$$\mathbf{F}^0(t) = \exp(-\Omega t) \cdot \mathbf{F}^0(0) = [P + N \cos t - \Omega \sin t] \cdot \mathbf{F}^0(0), \tag{14}$$

which is a reasonable result for leading-order behavior, in view of Eq. (12). In first order, though, a secular term arises:

$$\begin{aligned} \mathbf{F}^1(t) &= \exp(-\Omega t) \cdot \mathbf{F}^1(0) - \nu \exp(-\Omega t) \int_0^t \exp(\Omega \tau) \cdot \mathbf{F}^0(\tau)\, d\tau \\ &= \exp(-\Omega t) \cdot \mathbf{F}^1(0) - \nu \exp(-\Omega t) \int_0^t \mathbf{F}^0(0)\, d\tau \\ &= \exp(-\Omega t) \cdot \mathbf{F}^1(0) - \nu t \exp(-\Omega t) \cdot \mathbf{F}^0(0) \\ &= \exp(-\Omega t) \cdot \mathbf{F}^1(0) - \nu t \mathbf{F}^0(t). \end{aligned} \tag{15}$$

This result also follows from the Taylor expansion of Eq. (12) in powers of ϵ. The second term here is secular (growing without bound for increasing t) and ruins the uniformity of the approximation. Higher-order terms also are secular, with factors arising from the expansion of $\exp(-\epsilon\nu t)$.

(b) The *linear time scales* procedure begins with the observation of the simple time dependence of the secularities. The time scales $\tau_n = \epsilon^n t$ $(n = 0, 1, 2, \cdots)$ are introduced as the independent variables for the problem, and an "extension" of Eq. (11′) is written. The function $\hat{F}(\tau_0, \tau_1, \tau_2, \cdots)$ will be called an extension of $\mathbf{F}(t)$ if it reduces to $\mathbf{F}(t)$ on the time scale trajectory, $\tau_n = \epsilon^n t$; in brief, if

$$\mathbf{F}(t) = \hat{F}(t, \epsilon t, \epsilon^2 t, \cdots). \tag{16}$$

Furthermore, along the trajectory, the time derivative operator may be extended as

$$\partial/\partial t = \partial/\partial \tau_0 + \epsilon(\partial/\partial \tau_1) + \epsilon^2(\partial/\partial \tau_2) + \cdots .$$

More general extensions are possible but the L–T–S given above suffices for the problems considered in this paper. Thus, the extension of Eq. (11′) is

$$\frac{\partial \hat{F}}{\partial \tau_0} + \epsilon \frac{\partial \hat{F}}{\partial \tau_1} + \epsilon^2 \frac{\partial \hat{F}}{\partial \tau_2} + \cdots + \Omega \cdot \hat{F} = -\epsilon\nu\hat{F}. \tag{17}$$

The extension of F is then expanded in powers of ϵ: $\hat{F} = \hat{F}^0 + \epsilon\hat{F}^1 + \epsilon^2\hat{F}^2 + \cdots$. Substitution and equation of coefficients of ϵ then gives

$$(\partial \hat{F}^0/\partial \tau_0) + \Omega \cdot \hat{F}^0 = 0, \tag{18}$$

$$(\partial \hat{F}^1/\partial \tau_0) + \Omega \cdot \hat{F}^1 = -(\partial \hat{F}^0/\partial \tau_1) - \nu\hat{F}^0. \tag{19}$$

The solution to Eq. (18) is, with streamlined notation,

$$\hat{F}^0(\text{all } \tau) = \exp(-\Omega\tau_0) \cdot \hat{F}^0(\tau_0 = 0). \tag{20}$$

Substitution into Eq. (19) yields

$$\hat{F}^1 (\text{all } \tau) = \exp(-\Omega\tau_0)\hat{F}^1(\tau_0 = 0) - \tau_0 \exp(-\Omega\tau_0)\left[\frac{\partial \hat{F}^0}{\partial \tau_1} + \nu\hat{F}^0\right](\tau_0 = 0). \tag{21}$$

The secularity in the second term is then removed by setting

$$(\partial \hat{F}^0/\partial \tau_1) + \nu\hat{F}^0 = 0 \quad \text{for} \quad \tau_0 = 0,$$

so that

$$\hat{F}^0(\tau_0 = 0) = \exp(-\nu\tau_1)\hat{F}^0(\tau_0 = \tau_1 = 0). \tag{22}$$

This time dependence is all that is needed to recover the exact solution. The dependence on the lowest two time scales of $\mathbf{F}^0$ yields

$$\hat{F}^0 = \exp(-\Omega\tau_0) \exp(-\nu\tau_1)\hat{F}^0(\tau_0 = \tau_1 = 0). \tag{23}$$

On the trajectory, using Eq. (16), this reads

$$\begin{aligned} \mathbf{F}^0(t) &= \exp(-\epsilon\nu t) \exp(-\Omega t)\mathbf{F}^0(0) \\ &= \exp(-\epsilon\nu t)[P + N \cos t - \Omega \sin t]\mathbf{F}^0(0) \end{aligned} \tag{24}$$

which reproduces the form of the exact solution.

In general, the homogeneous terms (the first term in Eq. (22), for example) must be related to lower-order expressions and a more detailed analysis must be made to apply linear time scales rigorously. Care must be taken with initial conditions to insure that $\mathbf{F}$ is smooth in the multidimensional ($\boldsymbol{\tau}$) space [8, 20]. In this case, the elementary nature of the example chosen here has led to a solution in zeroth order almost immediately.

(c) The *Poincaré–Lighthill "stretching" method* handles the uniformization difficulty by introducing a new time scale:

$$t = s + \epsilon\zeta(s) + \cdots. \tag{3}$$

This means that Eq. (11′) becomes

$$(\partial \mathbf{F}/\partial s) = -(1 + \epsilon\dot{\zeta} + \cdots)(\Omega + \epsilon\nu I) \cdot \mathbf{F}, \tag{25}$$

and expansion of $\mathbf{F}$ in powers of ϵ yields

$$(\partial \mathbf{F}^0/\partial s) = -\Omega \cdot \mathbf{F}^0, \tag{26}$$

$$(\partial \mathbf{F}^1/\partial s) = -\Omega \cdot \mathbf{F}^1 - (\Omega \dot{\zeta}_1 + \nu I)\cdot \mathbf{F}^0. \tag{27}$$

Substitution of the usual solution to Eq. (26) into Eq. (27) leads to

$$\mathbf{F}^1(s) = \exp(-\Omega s)\mathbf{F}^1(0) - \exp(-\Omega s) \int_0^s (\Omega \dot{\zeta}(\sigma) + \nu I)\, d\sigma \cdot \mathbf{F}^0(0)$$

$$= \exp(-\Omega s)\mathbf{F}^1(0) - (\Omega \zeta(s) + s\nu I)\cdot \mathbf{F}^0(s).$$

The $s\nu I$ term is secular, but the stretching function, ζ, cannot be chosen in any way that will cancel it (since Ω is not a multiple of I). Thus, the P–L technique fails to lead to a uniform expansion even in first order.

3. Discussion and conclusions. The perturbation problem (Eqs. (1) and (2)) has been chosen to illustrate the claim that the P–L technique and linear time scales work for essentially *distinct* classes of problems. The example has a behavior for *large t* which is difficult to approximate in the perturbation expansion (secularity), while the typical difficulty handled by P–L is that of a singularity for finite t (usually the origin) in the expansion of a *nonlinear* equation.

The secular difficulty is handled easily by linear time scales, but P–L cannot handle the singularity when it occurs, as in the example, at infinity; even transforming coordinates to move the singularity to the origin ($t' = t^{-1}$) is not helpful since it introduces at the t' origin an essential singularity which is beyond the scope of the P–L technique. (P–L can remove some singularities which appear in the perturbation expansion, but it should not be expected to remove those singularities which *belong* in the exact solution.)

It should be emphasized that, while the example given illustrates a class of problems for which P–L fails and L–T–S succeeds, there are certain problems which require more general extensions. Some have been introduced which include both P–L and L–T–S [20, 21].

References

[1] M. J. Lighthill, Phil. Mag. **40**, 1179 (1949)
[2] M. J. Lighthill, A. für Flug. **9**, 267 (1961)
[3] H. S. Tsien, *The Poincaré-Lighthill-Kuo method in advanced applied mechanics* **4**, Academic Press, Inc., N. Y., 1956, p. 281
[4] C. Comstock, U. S. Naval Postgraduate School Report No. NPS-53ZK71071A, 20 July 1971
[5] G. Sandri, Ann Phys. (NY) **24**, 332, 380 (1963)
[6] E. Frieman, J. Math. Phys. **4**, 410 (1963)
[7] J. Cole and J. Kevorkian, in *Nonlinear differential equations and nonlinear mechanics*, J. P. LaSalle and S. Lefshetz, eds., Academic Press, Inc., N. Y., 1963
[8] A. Klimas, R. V. Ramnath and G. Sandri, J. Math. Anal. Appl. **32**, 482 (1970). It is interesting to compare the quantities G, B, Γ of [8] with the quantities $\bar{g}_w$, $\bar{g}_s$, ϕ of [16]. We have $\bar{g}_w = G$, $\bar{g}_s = B$, $\phi = \Gamma$.
[9] N. Krylov and N. Bogolubov, *Introduction to nonlinear mechanics*, Princeton University Press, Princeton, N. J., 1947
[10] N. Bogolubov and Y. Mitropolsky, *Asymptotic methods in the theory of nonlinear oscillations*, Gordon and Breach, N. Y., 1961

[11] G. F. Carrier, *Boundary layer problems in applied mathematics*, CPAM **7**, 11 (1954); in *Advances in applied mechanics* III, Academic Press, N. Y., 1953

[12] S. Kaplun, *Fluid mechanics and singular perturbations*, Academic Press, N. Y., 1967

[13] M. D. Van Dyke, *Perturbation methods in fluid Mechanics*, Academic Press, Inc., N. Y., 1964

[14] J. Cole, *Perturbation methods in applied mathematics*, Blaisdell, 1967

[15] R. Bellman, *Perturbation techniques in mathematics, physics and engineering*, Holt, Rinehart and Winston, San Francisco, 1964

[16] W. Wasow, J. Rat. Mech. Anal. **4**, 751 (1955)

[17] P. D. Usher, *Coordinate stretching and interface* location, II, a new PL expansion, J. Comp. Phys. **3**, 29 (1968)

[18] P. D. Usher, Quart. Appl. Math. **28**, 463 (1971)

[19] M. F. Pritulo, *On the determination of uniformly accurate solutions of differential equations by the method of perturbation of coordinates*, P.M.M. 26, 444 (1962)

[20] F. X. Murphy, Jr. and G. Sandri, *Lectures in nonequliibrium statistical mechanics*, C.U.N.Y.mimeograph (1971) and Not. Am. Math. Soc. **18**, 928 (1971)

[21] G. V. Ramanathan and G. Sandri, J. Math. Phys. **10**, 1973 (1969)