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A STEERING PROBLEM*

BY

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Abstract. If two circular gear-wheels of different sizes engage, this may be regarded as a mechanism which generates a linear relationship between the angles turned through, the ratio of the angles being constant. In this paper it is shown that, if the circular gear wheels are replaced by suitably shaped oval wheels (or cams) which engage without slipping, it is possible to generate an arbitrary functional relationship between the angles turned through. It is further shown how this mechanism might be used in the steering of a four-wheeled vehicle with theoretically perfect satisfaction of the condition that, at any instant, the vehicle as a whole has a unique centre of rotation, so that no side-slip of the wheels occurs. The angle through which the plane of the inner front wheel is turned might be as great as a right angle, this limit being much greater than that attainable with the usual Ackermann linkage.

1. Mechanism to generate an arbitrary functional relationship between two angles. Consider two oval wheels (or cams) which lie in a plane and can turn about fixed points, A and B (Fig. 1). Initially the cams are in contact and a rolling contact is maintained by means of fine teeth on the edges of the cams, or in some other way. If the initial position is as in Fig. 1, no such motion is possible. For in rolling contact the particles

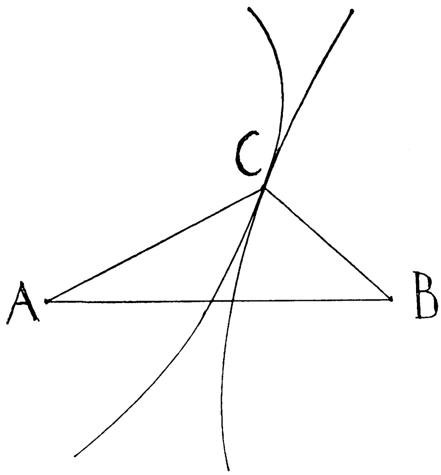

FIG. 1. Two cams in contact, but rolling impossible since C does not lie on AB.

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of the two cams at the point of contact C receive the same infinitesimal displacement, and this must be perpendicular to both AC and BC. In fact, for rolling to be possible, the point of contact C must lie on the line AB, as in Fig. 2.

This figure shows one configuration of the two cams. If rolling takes place, this establishes a one-to-one correspondence between the points on the edges of the two cams, corresponding points being those which at some time are brought into contact. In Fig. 2, C is self-correspondent, P is any point on the cam which turns about A, and Q is its correspondent on the cam which turns about B. Polar coordinates, (r, θ) for P and (ρ, ϕ) for Q, are as shown in Fig. 2.

Now let the cam on the left be rotated clockwise through the angle θ. This brings the particle shown at P in Fig. 2 down onto the line AB, and so it becomes a contact-point. But, since P and Q are correspondents, this contact must be with the particle of the right cam which was at Q, and so that particle is brought down also onto the line AB. The contact does not, of course, occur at the point C of Fig. 2, but elsewhere on the line AB.

Since the cams are supposed rigid, the particles which were at P and Q in Fig. 2 are brought down into coincidence at a point on AB distant r from A and ρ from B. Since AB is a constant ($AB = c$, say), we have then

$$r + \rho = c. \tag{1.1}$$

That is our first equation. To obtain the second, we note that, in rolling contact, corresponding arcs are equal, and so

$$\text{arc } CP = \text{arc } CQ. \tag{1.2}$$

Thus, if we hold the cams fixed in the position of Fig. 2 but mentally move the points P and Q down to coincide at C, we have

$$\int_C^P (dr^2 + r^2\, d\theta^2)^{1/2} = \int_C^Q (d\rho^2 + \rho^2\, d\phi^2)^{1/2}. \tag{1.3}$$

Changing P and Q infinitesimally, but still in correspondence, and squaring the result, we get

$$dr^2 + r^2\, d\theta^2 = d\rho^2 + \rho^2\, d\phi^2. \tag{1.4}$$

This is our second equation.

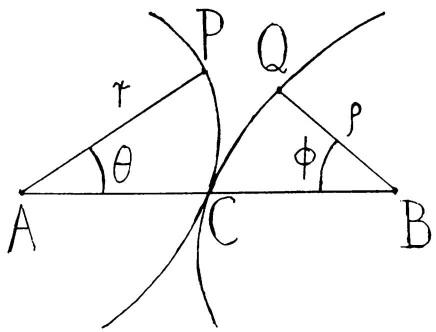

FIG. 2. Cams with contact at C, turning about A and B. P and Q are corresponding points.

By (1.1) we have $dr = -d\rho$, and so (1.1) and (1.4) together give

$$r + \rho = c, \qquad r\, d\theta = \rho\, d\phi, \tag{1.5}$$

c being a given constant. These two equations have been deduced from the condition that the cams turn about A and B, rolling on one another without slipping.

We shall now show that *the shape of one cam can be chosen arbitrarily, but then the shape of the other is determined.*

Let us define the shape of the cam which turns about A by the polar equation $r = f(\theta)$ in the notation of Fig. 2. We have then, from (1.5), the following equations:

$$\begin{aligned} \rho &= c - r = c - f(\theta), \\ d\phi &= r\, d\theta/\rho = (c - r)^{-1} r\, d\theta = [c - f(\theta)]^{-1} f(\theta)\, d\theta, \\ \phi &= \int_0^\theta [c - f(x)]^{-1} f(x)\, dx. \end{aligned} \tag{1.6}$$

Elimination of θ from the first and last of these equations gives an equation of the form $F(\rho, \phi) = 0$, and this determines the form of the second cam.

As a simple check on the above, suppose the first cam to be a circle of radius $\frac{1}{2}c$ and A its centre. Then $f(\theta) = \frac{1}{2}c$, $\rho = \frac{1}{2}c$, $\phi = \theta$; the second cam is a circle of radius $\frac{1}{2}c$ and B is its centre; the two cams turn through equal angles.

The main result to be established is as follows:

Given the distance c between the points A and B about which the cams turn, it is possible to choose their forms so that an arbitrarily chosen functional relationship

$$\phi = G(\theta) \tag{1.7}$$

is generated when they roll on one another without slipping.

The fact, already established, that one arbitrary function may be used in the mechanism, suggests intuitively that the above statement is true. To prove it, we note that, when we insert (1.7) in the last of (1.6), we get for the function f the integral equation

$$G(\theta) = \int_0^\theta [c - f(x)]^{-1} f(x)\, dx. \tag{1.8}$$

Differentiation gives

$$G'(\theta) = dG(\theta)/d\theta = [c - f(\theta)]^{-1} f(\theta), \tag{1.9}$$

and so

$$f(\theta) = c[1 + G'(\theta)]^{-1} G'(\theta). \tag{1.10}$$

The shape of one cam is then given by $r = f(\theta)$, this function being determined by (1.10). As for the other cam, its equations read

$$\rho = c - f(\theta), \qquad \phi = G(\theta), \tag{1.11}$$

θ being regarded as a parameter.

2. The steering problem and its solution. Fig. 3 shows a car cornering without any side-slip of the wheels. Let us assume for simplicity that the planes of all four wheels are vertical. Then the condition to be satisfied is this: the perpendiculars to the planes

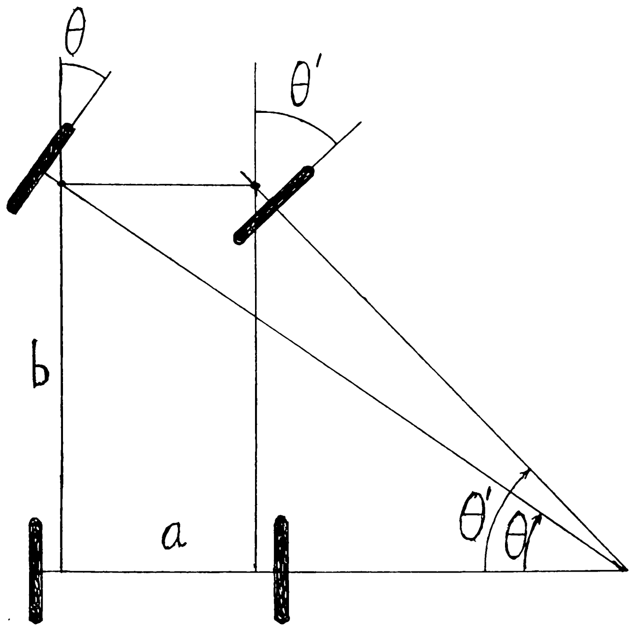

FIG. 3. Car cornering with unique centre of rotation on the back axle produced.

of the front wheels, drawn through their centres, must meet at a point on the back axle produced. Thus, if θ and θ' are the angles through which the planes of the front wheels are turned, the condition to be satisfied reads

$$b \cot \theta - b \cot \theta' = a, \tag{2.1}$$

or

$$\cot \theta - \cot \theta' = k = a/b, \tag{2.2}$$

where a is the width of the car (measured between the knuckles about which the front wheels turn when steered) and b the length of the car (measured from those knuckles to the back axle).

It is convenient to introduce an auxiliary angle ϕ and write the above condition in two equations:

$$\begin{aligned} \cot \theta - \cot \phi &= \tfrac{1}{2}k, \\ \cot \theta' - \cot \phi &= -\tfrac{1}{2}k. \end{aligned} \tag{2.3}$$

We are now to consider how to shape two cams so that they generate, as in Sec. 1, the first of these functional relationships, when they rotate in contact without slipping. But instead of using the formulae (1.8–11), it is simpler to go back to the basic formulae (1.5):

$$r + \rho = c, \qquad r\, d\theta = \rho\, d\phi. \tag{2.4}$$

We are to solve these with

$$\cot \theta - \cot \phi = \tfrac{1}{2}k. \tag{2.5}$$

Differentiation gives

$$\operatorname{cosec}^2 \theta \, d\theta - \operatorname{cosec}^2 \phi \, d\phi = 0, \tag{2.6}$$

and so, by (2.4),

$$(c - r) \operatorname{cosec}^2 \theta = r \operatorname{cosec}^2 \phi. \tag{2.7}$$

But by (2.5)

$$\operatorname{cosec}^2 \phi = 1 + (\cot \theta - \tfrac{1}{2}k)^2. \tag{2.8}$$

On substituting this in (2.7), we get a relationship between r and θ, which gives, after a short trigonometric calculation,

$$r = \tfrac{1}{2}c \cos^2 \gamma [1 - \sin \gamma \sin (2\theta + \gamma)]^{-1}, \tag{2.9}$$

where

$$\tan \gamma = k/4 = a/(4b). \tag{2.10}$$

We are to interpret (2.9) as the polar equation of the cam in Fig. 2 which turns about A. As for the shape of the other cam, we do not have to repeat the calculation: for Eqs. (2.4) and (2.5) retain the same form if we interchange r and ρ, at the same time interchanging θ and ϕ, and replacing k by $-k$, or, equivalently, γ by $-\gamma$. This gives for the second cam the polar equation

$$\rho = \tfrac{1}{2}c \cos^2 \gamma \, [1 + \sin \gamma \sin (2\phi - \gamma)]^{-1}. \tag{2.11}$$

We now have, in (2.9) and (2.11), a pair of cams which generate the functional relationship (2.5), which is the first of (2.3). As for the second of (2.3), let us, for notational reasons, write ϕ' for ϕ, so that it reads

$$\cot \theta' - \cot \phi' = -\tfrac{1}{2}k. \tag{2.12}$$

This is the same as (2.5) except that k is replaced by $-k$, and so, to generate (2.12), we need a left cam with equation

$$r' = \tfrac{1}{2}c \cos^2 \gamma \, [1 + \sin \gamma \sin (2\theta' - \gamma)]^{-1} \tag{2.13}$$

and a right cam with equation

$$\rho' = \tfrac{1}{2}c \cos^2 \gamma \, [1 - \sin \gamma \sin (2\phi' + \gamma)]^{-1}. \tag{2.14}$$

On examining (2.9), (2.11), (2.13) and (2.14), we see that only two shapes of cam are involved. But indeed there is actually only *one* shape involved, because if, in (2.9), we replace r by r' and θ by $-\theta'$, we obtain (2.13).

To study this unique shape we may use (2.9), in which θ is measured from the line AB (Fig. 2). We can simplify the equation by measuring the angle, say ψ, from $\theta = -\tfrac{1}{2}\gamma - \pi/4$, so that

$$2\theta + \gamma = 2\psi - \pi/2; \tag{2.15}$$

at the same time let us write

$$\tfrac{1}{2}c \cos^2 \gamma = C. \tag{2.16}$$

Thus (2.9) becomes

$$r = C[1 + \sin\gamma \cos 2\psi]^{-1}. \tag{2.17}$$

This expression is invariant under the transformations $\psi \to -\psi$ and $\psi \to \pi - \psi$. Thus (2.17) represents an oval curve with the symmetries of an ellipse, the semi-axes being

$$\begin{aligned} &\text{for} \quad \psi = 0, \qquad r = C(1 + \sin\gamma)^{-1}, \\ &\text{for} \quad \psi = \tfrac{1}{2}\pi, \qquad r = C(1 - \sin\gamma)^{-1}. \end{aligned} \tag{2.18}$$

Recalling that γ is given by (2.10), we note that for a very long vehicle γ is small, and the cams are nearly circular; while for a very short vehicle γ approaches $\frac{1}{2}\pi$ and the second expression in (2.18) tends to infinity—the cam becomes very long and thin.

To see what would be required in practice, let us take

$$a/b = \tfrac{1}{2} \tag{2.19}$$

as the width/length ratio for the vehicle. Then by (2.10)

$$\begin{aligned} \tan\gamma &= 1/8 = 0.125, \qquad \gamma = 7^\circ\, 7'\, 30'', \\ \sin\gamma &= 0.12403, \qquad \cos\gamma = 0.99228. \end{aligned} \tag{2.20}$$

Since γ is a fairly small angle, it is clear that the cam is not far from circular. Its shape is shown in Fig. 4 for the above numerical value of γ. The semi-axes are

$$\psi = 0, \qquad r = C \times 0.890; \qquad \psi = \tfrac{1}{2}\pi, \qquad r = C \times 1.142. \tag{2.21}$$

By (2.15) $\theta = 0$ corresponds to

$$\psi = \tfrac{1}{2}\gamma + 45^\circ = 48^\circ\, 33'. \tag{2.22}$$

This direction, and its reflection in a principal axis, are shown in Fig. 4.

In Fig. 5 we look down on the front of a car, travelling in the direction of the arrow. The two front wheels are shown in the straight-ahead position. Four cams, each with the shape shown in Fig. 4, are mounted on the front axle. Except in so far as they are connected, these four cams can turn freely about vertical axes through their centres, A, B, B', A'. The wheels are attached to the cams A, A' so that their (vertical) planes turn with the cams; the outer halves of those two cams are not used and may be omitted—they are included in the diagram merely to make whole arrangement easier to understand.

The four cams are so placed that the front axle coincides with the oblique lines of Fig. 4, and there is contact between cams A and B, and also between cams B' and A'. The edges of the cams have fine teeth, or some other connection such as flexible bands, so that rolling takes place when one of them is turned. The cams B and B' are so connected that, when one turns, the other turns through the same angle in the same sense. This connection is indicated as a band passing round two equal circular discs attached to the cams B and B', but this might be replaced by some more sophisticated mechanism.

As stated, Fig. 5 shows the wheels in the straight-ahead position. By virtue of the connections, the whole mechanism has only one degree of freedom. If the plane of the left front wheel is turned clockwise through an angle θ, the cam A turns clockwise through that same angle. This makes the cam B turn counterclockwise through an angle ϕ, given [cf. (2.3)] by

$$\cot\theta - \cot\phi = \tfrac{1}{2}k. \tag{2.23}$$

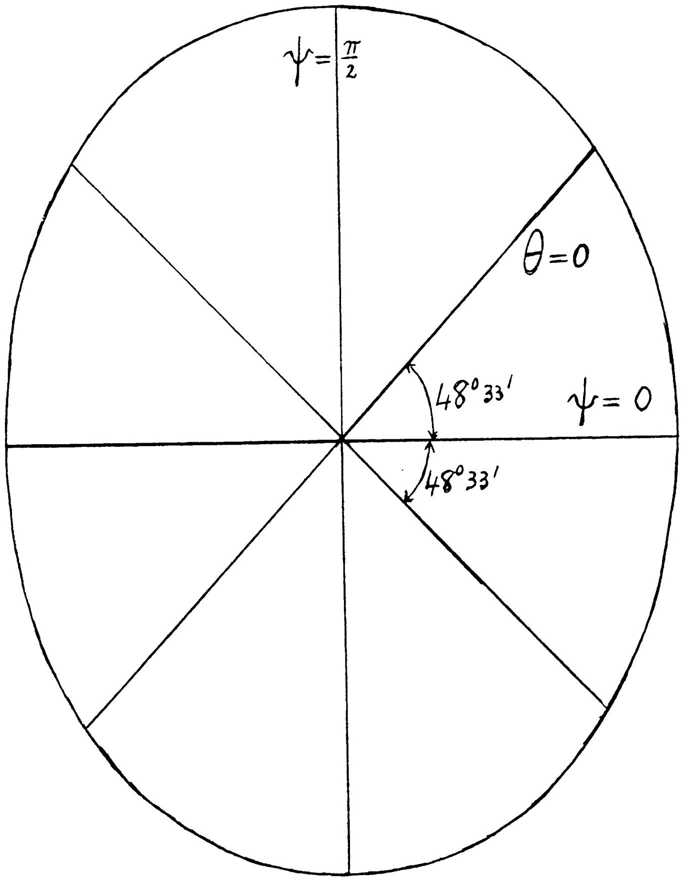

FIG. 4. A steering cam for the case $k = a/b = \frac{1}{2}$.

By virtue of their connection, the rotation of B generates in B' a rotation through the same angle ($\phi' = \phi$), and this in turn makes the cam A' turn clockwise through an angle θ' given by [cf. (2.3)]

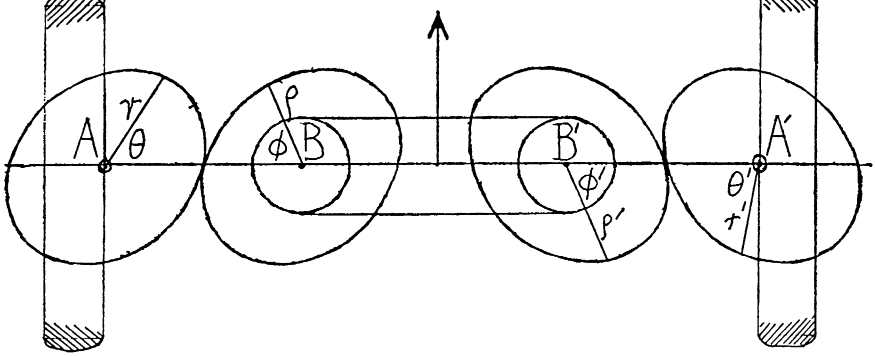

FIG. 5. Exact solution of the steering problem by means of four identical cams.

$$\cot \theta' - \cot \phi' = -\tfrac{1}{2}k. \tag{2.24}$$

Since the right-hand wheel turns also through the angle θ', we see that the basic condition (2.2) is satisfied, viz.

$$\cot \theta - \cot \theta' = k = a/b. \tag{2.25}$$

Thus, when the front wheels are turned, the perpendiculars to their planes meet on the back axle, so that the vehicle has a unique centre of rotation.

In the above operation, the radii marked r, ρ, ρ', r' are brought down or up so as to lie on the front axle of the vehicle. The diagram is drawn for the case cited above ($k = a/b = \frac{1}{2}$), and for the following values of the angles (satisfying (2.23) and (2.24));

$$\theta = 56°, \qquad \phi = \phi' = 67°, \qquad \theta' = 80°, \tag{2.26}$$

omitting a few minutes of arc. Thus the inner front wheel is locked over through an angle of 80°, which means that the centre of rotation is not far from the inner rear wheel, and the turning radius not much greater than the width of the vehicle.

It should be pointed out, however, that such a large angle of lock is not possible for the mechanism shown in Fig. 5 if the centres of the wheels lie on the line AA' because the right-hand wheel would be brought into contact with the cam B'. This difficulty may be avoided in two ways. First, the horizontal plane containing the four cams might be raised right above the wheels. Second, the wheels, while still attached to the cams A, A', might be moved farther apart.

It should also be noted that the parameter c [cf. (2.4)] is at our disposal. It determines the size of the cams, and appears in Fig. 5 as the length AB or $B'A'$. There the cams are drawn large in order that the kinematics of the mechanism may be understood. They might be much smaller.

But these would be matters for a mechanical engineer if the steering mechanism were to be realized practically. The essential underlying idea is the generation of an arbitrary functional relationship between two angles by means of cams rolling on one another. It is a pleasant accident of the steering problem that these cams do not differ too much from circular form.

The material of this paper was presented at a meeting of the Dublin University Mathematical Society on 12 November 1926. The essential idea was mentioned in the Boyle Medal Lecture delivered to the Royal Dublin Society on 28 March 1972, but otherwise the work has not been published previously.