

ON CONTROLLABLE HEAT FLUX FIELDS AND THE DETERMINATION OF TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITIES*

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1. Introduction. It is known that there is no single non-uniform temperature field that is possible in all heat conductors which have a temperature-dependent thermal conductivity [8]. We say then that there are no controllable states of temperature associated with such materials. We shall show below, however, that controllable heat flux fields do exist for this class of materials, and we shall demonstrate how the knowledge of such fields enables one to design experimental programs which are universally applicable to the determination of nonlinear thermal conductivities. The controllable states will be seen to be derivable from a potential, but, unlike the classical linear case, the potential will be a scalar field distinct from the temperature field in the heat conductor.

2. Nonlinear heat conduction. In the typical boundary-value problem in continuum physics one wishes to determine states of stress, temperature, etc., within a body when conditions of loading, temperature, etc., are specified on the surface of the body. Physical principles such as balance of momentum and conservation of energy provide the differential field equations to be solved, while experimental measurements, experiential inferences, or assumptions provide the boundary data. It is often the case, however, that the physical quantities which appear as dependent variables in the field equations are not the same as those which are available as boundary data. For example, in steady-state heat flow the balance-of-energy equation involves the divergence of the heat flux vector, while the most readily available boundary data may be temperature values. Thus, before a standard mathematical boundary-value problem can be presented, there must be some supplemental information, and this information is provided by equations which characterize the constitution of the body's material.

One such constitutive equation functionally relates the heat flux, the temperature, and the temperature gradient at a point in a heat conducting medium. The determination of constitutive relations like this is essential to the practical use of any physical theory, but since that determination must be an experimental one, a very tedious program of data collection may be necessary in theories of nonlinear material behavior.

Consider the case of steady heat conduction in stationary, nondeformable bodies when no heat supplies are present. The field equation which governs such processes is simply**

$$\operatorname{div} \mathbf{h} = 0 \tag{2.1}$$

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** We shall use Gibbs' vector notation throughout this work.

where $\mathbf{h} = \hat{\mathbf{h}}(\mathbf{x})$ is the heat flux vector per unit area per unit time at position \mathbf{x} in the body. Consider further the class of homogeneous and isotropic materials discussed by Coleman and Mizel [1] which is defined by constitutive equations of the form

$$\mathbf{h} = -\hat{K}(\theta, \gamma) \nabla \theta \quad (2.2)$$

where $\theta = \hat{\theta}(\mathbf{x})$ is the temperature field, $\gamma = |\nabla \theta|^2$, and $K = \hat{K}(\theta, \gamma) > 0$ is the thermal conductivity which must be determined by experiment on a material specimen.

Now, in general, if one knows the surface temperature distribution and wishes to determine the temperature field $\theta = \hat{\theta}(\mathbf{x})$ within a heat conductor of the type (2.2), he must either convert his temperature data to heat flux data by means of (2.2) or use (2.2) in (2.1) to rewrite the field equation in terms of temperature before a standard mathematical boundary-value problem presents itself. The former alternative requires not only knowledge of the function \hat{K} but also information about the temperature gradient. The latter alternative results in the following nonlinear differential equation for the temperature field:

$$\gamma \frac{\partial \hat{K}(\theta, \gamma)}{\partial \theta} + \nabla \gamma \cdot \nabla \theta \frac{\partial \hat{K}(\theta, \gamma)}{\partial \gamma} + \nabla^2 \theta \hat{K}(\theta, \gamma) = 0. \quad (2.3)$$

Still, knowledge of the function \hat{K} is required before the analysis can proceed.

The problem of determining the response function \hat{K} is conceptually simple. If one is willing to instrument a specimen in such a way that the temperature, the temperature gradient, and the heat flux at a point (or, equivalently, the heat flux and temperature profile along a line) can be measured, then by (2.2) the response function

$$\hat{K}(\theta, \gamma) = |\mathbf{h}|/\gamma^{1/2} \quad (2.4)$$

may be ascertained in principle to any degree of accuracy for any range of values (θ, γ) if enough data are collected. However, besides being inelegant, this direct method is unattractive for the amount of data required for any characterization of \hat{K} over a reasonably wide range of (θ, γ) values.

One often-mentioned alternative to such direct methods of determining material response functions is the use of controllable-state experiments. In the context of our heat conduction problem the idea is as follows. If there is a temperature field $\theta = \theta_c(\mathbf{x})$ which satisfies (2.3) regardless of the specific form of the constitutive function \hat{K} then such a field must be realizable in *all* materials of the kind (2.2) under the application of suitable surface temperature distributions. Such temperature fields are known as controllable states of the class of materials defined by (2.2). For a given specimen of such a material then, one knows *a priori* that he could adjust the boundary temperature distribution until it is consistent with the unique solution $\theta = \theta_c(\mathbf{x})$ of (2.3) and measure the corresponding heat flux. These data, including the temperature gradient data derived from $\theta = \theta_c(\mathbf{x})$, could then be used in (2.2) to give the function \hat{K} . The advantage gained is obvious when one considers one-dimensional heat flow along a rod. Whereas the direct method gives a single point on the surface $K = \hat{K}(\theta, \gamma)$ for each point (θ, γ, h) of data, the controllable-state method gives a whole line on the surface $K = \hat{K}(\theta, \gamma)$ for each measurement of the rod's end temperatures and the associated heat flux.

Unfortunately, it turns out that the only temperature field which satisfies (2.3) identically in \hat{K} is a constant temperature field [8]. Thus for all practical purposes the

class of heat conductors (2.2) has no controllable temperature states which can be exploited to simplify determinations of \tilde{K} .

To see that such a paucity of controllable states is not always the case, consider the subclass of (2.2)-type materials defined by

$$\mathbf{h} = -\tilde{K}(\gamma) \nabla \theta. \quad (2.5)$$

It has been shown [8] that the temperature field

$$\theta = a + b\phi + cz \quad (2.6)$$

in circular cylindrical coordinates (r, ϕ, z) , where a , b , and c are arbitrary constants, is controllable for (2.5)-like materials and may thus be employed in an experimental situation to determine the function \tilde{K} .

3. Controllable heat flux fields. A recent note by Laws [5] suggests that the controllable-state concept may yet provide a shortcut to measuring temperature-dependent thermal conductivities. Whereas controllable states of deformation and temperature fields had been known for elastic heat conductors [3, 7], Laws was the first to publish [5] a list of controllable states of deformation and heat flux fields for conductors with temperature-dependent properties. Most significant is the fact that many more controllable flux-deformation fields exist than controllable temperature-deformation fields.

That the same is true for rigid heat conductors will now be shown. However, while Laws' work [5] depends on the analogy between elastic heat conductors and elastic dielectrics as treated by Singh and Pipkin [9], our analysis will demonstrate a new analogy between heat flux and fluid flow.

Following Laws [5] and Singh and Pipkin [9], it will be convenient for the ensuing analysis to write the constitutive equation (2.2) in the form

$$\nabla \theta = -\hat{J}(\theta, \omega) \mathbf{h} \quad (3.1)$$

where $\omega = \mathbf{h}^2$. Now controllable states of (3.1)-type materials are heat flux fields which are solenoidal by (2.3). But to insure that there exists a corresponding scalar temperature field θ we must have

$$\text{curl} (\hat{J}(\theta, \omega) \mathbf{h}) = \mathbf{0}. \quad (3.2)$$

Expanding this, we get

$$\frac{\partial \hat{J}(\theta, \omega)}{\partial \theta} \nabla \theta \times \mathbf{h} + \frac{\partial \hat{J}(\theta, \omega)}{\partial \omega} \nabla \omega \times \mathbf{h} + \hat{J}(\theta, \omega) \text{curl} \mathbf{h} = \mathbf{0}.$$

Now $\nabla \theta \times \mathbf{h} \equiv \mathbf{0}$ because of (3.1), and $\nabla \omega = \nabla h^2 = 2h \nabla h$ by the definition of ω and the properties of ∇ . Therefore we must have

$$\frac{\partial \hat{J}(\theta, \omega)}{\partial \omega} 2h \nabla h \times \mathbf{h} + \hat{J}(\theta, \omega) \text{curl} \mathbf{h} = \mathbf{0}.$$

For materials in which \hat{J} is independent of ω controllable fields must thus be irrotational as well as solenoidal. For arbitrary materials of the class (3.1) controllable non-zero heat flux fields must satisfy the conditions

$$\begin{aligned} \text{(a)} \quad & \text{div} \mathbf{h} = 0, \\ \text{(b)} \quad & \text{curl} \mathbf{h} = \mathbf{0}, \\ \text{(c)} \quad & \nabla h \times \mathbf{h} = \mathbf{0}. \end{aligned} \quad (3.3)$$

Conditions (a) and (b) represent the equations governing the irrotational flow of an incompressible fluid if \mathbf{h} is identified as the velocity field [4]. It is well known that for such flows there exists by (b) a scalar potential $\phi = \phi(\mathbf{x})$, which we shall call the *heat flux potential*, such that

$$\mathbf{h} = -\nabla\phi, \quad (3.4)$$

where by (a) ϕ is harmonic*, i.e.,

$$\nabla^2\phi = 0. \quad (3.5)$$

Now condition (c) of (3.3) may be satisfied by nonzero \mathbf{h} fields in two possible ways:

$$(i) \quad \nabla h = \mathbf{0},$$

$$(ii) \quad \nabla h = s\mathbf{h} \text{ for some scalar } s. \quad (3.6)$$

Condition (i) means that the normal trajectories of the surfaces $\phi = \text{const.}$ are straight lines [2]. Condition (ii) means that ∇h is parallel to \mathbf{h} and implies that the surfaces $h = \text{const.}$ coincide with those for which $\phi = \text{const.}$, i.e., that there is a functional relationship

$$h = g(\phi). \quad (3.7)$$

All of the heat flux fields shown by Laws [5] to be controllable in elastic heat conductors (when coupled with suitable deformation fields) satisfy (3.6.i), (3.3.a) and (3.3.b) and are therefore controllable in rigid conductors also. These fields are:

$$(\alpha) \quad \text{in cartesian coordinates } (x, y, z): h_x = k, h_y = h_z = 0, \quad (3.8)$$

$$(\beta) \quad \text{in cylindrical polar coordinates } (r, \theta, z): h_r = k/r, h_\theta = h_z = 0, \quad (3.9)$$

$$(\gamma) \quad \text{in spherical polar coordinates } (r, \theta, \phi): h_r = k/r^2, h_\theta = h_\phi = 0, \quad (3.10)$$

where in all cases k is an arbitrary constant. Laws [5] notes that the corresponding temperature fields may be determined by integrating the elastic conductor analogue of our (3.1), but clearly this requires in the present case prior knowledge of the response function \hat{J} . Our principal purpose here is to show how the fields (3.8)–(3.10) may be exploited to measure that function in a rational way.

Since it is not clear whether fields other than (3.8)–(3.10) satisfy (3.6) and (3.3) (a) and (b), there may be additional heat flux fields which are controllable in (3.1)-type conductors. We shall not pursue this question, but shall rather consider a subclass of the materials defined by (3.1) and demonstrate how (3.9) may be used to measure their nonlinear thermal conductivities.

4. Controllable states in materials with temperature-dependent thermal conductivities. We shall from here on restrict our attention to those conductors for which

$$\nabla\theta = -\hat{J}(\theta)\mathbf{h}, \quad (4.1)$$

* That solutions to Laplace's equation characterize the steady flow of heat in bodies with temperature-dependent thermal conductivities is well known (cf. [11, p. 11]). However, we have reached this result without introducing the standard ad hoc transformations of variables. Furthermore, by (3.3), we see that not necessarily all solutions to (3.5) give controllable heat flux fields in conductors defined by (3.1).

and for convenience we shall assume the reciprocal of \tilde{J} can be written as the derivative of some smooth function $\tilde{f}(\theta)$, i.e.,

$$-1/\tilde{J}(\theta) = \tilde{f}'(\theta) \quad (4.2)$$

where the prime denotes differentiation with respect to the primed function's argument.

Since \tilde{J} is independent of $\omega = h^2$, condition (3.3.c) does not restrict controllable states of (4.1)-type materials. Therefore controllable states need only satisfy

$$\operatorname{div} \mathbf{h} = 0, \quad \operatorname{curl} \mathbf{h} = \mathbf{0}. \quad (4.3)$$

Thus we have an exact hydrodynamical analogue, and *every* irrotational flow of an incompressible fluid corresponds to a controllable state of a rigid heat conductor following the law (4.1), the analogous quantities being the fluid velocity and the heat flux fields. Chapter V of Lamb [4] is concerned with the solution of the set of equations (4.3).

If we restrict the heat flux vector field to two dimensions, say the xy -plane, the first of Eqs. (4.3) implies the existence of a "stream function" $\psi = \psi(x, y)$ such that the heat flux components are given by

$$h_x = -(\partial\psi/\partial y), \quad h_y = (\partial\psi/\partial x). \quad (4.4)$$

It follows that ψ is harmonic. The heat flux potential ϕ , where

$$\mathbf{h} = -\nabla\phi, \quad (4.5)$$

and the stream function ψ satisfy Cauchy–Riemann conditions

$$\partial\phi/\partial x = \partial\psi/\partial y, \quad \partial\phi/\partial y = -\partial\psi/\partial x. \quad (4.6)$$

Thus any complex-valued function of the complex variable $x + iy$

$$\phi + i\psi = \tilde{F}(x + iy)$$

gives a possible case of irrotational fluid flow or, by analogy, a controllable state of plane heat flow (cf. [4] and [6]).

In the classical analogy between fluid flow and heat flow the heat conducting materials obey the linear constitutive equation known as Fourier's law,

$$\mathbf{h} = -\kappa\nabla\theta, \quad (4.7)$$

where κ is a constant, and then the heat flux potential of (3.4) is proportional to the temperature field. It follows that the temperature field must satisfy Laplace's equation regardless of the thermal conductivity constant κ .

In the nonlinear case with materials like (4.1), however, the flux potential has no such simple physical interpretation. Furthermore, the temperature field is not harmonic and depends on the material response function through the differential equation (2.3), which for (4.1) reduces to*

$$\nabla \cdot \left(\frac{\nabla\theta}{\tilde{J}(\theta)} \right) = \frac{d}{d\theta} \left(\frac{1}{\tilde{J}(\theta)} \right) \nabla\theta \cdot \nabla\theta + \frac{1}{\tilde{J}(\theta)} \nabla^2\theta = 0. \quad (4.8)$$

* Cf. [11, p. 11] for a discussion of the classical transformation which simplifies this to Laplace's equation by introducing a new variable equivalent to our heat flux potential. Our present intentions, however, are to avoid the use of any such transformation.

Note, however, that the surfaces $\phi = \text{const}$ are isotherms since by (3.4), (4.1) and (4.2) ϕ and θ are functionally related.

5. The use of controllable heat flux fields to measure temperature-dependent conductivities. By (4.1) and (4.2) we have

$$\mathbf{h} = \tilde{f}'(\theta) \nabla \theta \quad (5.1)$$

for (4.1)-type materials. Then by (3.4) we can write

$$\nabla \phi = \tilde{f}'(\theta) \nabla \theta = \nabla f$$

and therefore

$$f = \phi + \text{const.} \quad (5.2)$$

Hence we can derive the constitutive function \tilde{J} from the heat flux potential as

$$\tilde{J}(\theta) = -1 \left/ \frac{d\tilde{\phi}(\theta)}{d\theta} \right., \quad (5.3)$$

and we have, theoretically, a method of determining the nonlinear response function \tilde{J} : subject a rigid heat conductor to any (controllable) potential flux $\mathbf{h} = -\nabla \phi$, determine the function $\phi = \tilde{\phi}(\theta)$, or $f = \tilde{f}(\theta)$, and derive the response function according to (5.3).

We now give an example of the procedure and illustrate that we can get by with many fewer measurements of data by employing this method rather than the direct method of determination described in Sec. 2. Recall that the latter procedure gives a single point

$$\tilde{J}(\theta) = |\mathbf{h}|/|\nabla \theta| \quad (5.4)$$

for each triple of data $(h, \theta, \nabla \theta)$.

According to Tye [10] one of the most widely-used methods to determine thermal conductivity is radial flow in a circular cylinder. We shall employ such a method. Consider a hollow circular cylindrical specimen of (4.1)-type material with inner and outer radii a and b held at uniform temperatures α and β respectively. Away from ends of the cylinder we can argue by symmetry that the cylindrical surface $r = \rho$ is an isotherm and that the heat flux is radial (and outward if $\alpha > \beta$).

For this situation the heat flux potential may be taken to within an unimportant constant in the form

$$\phi = A \ln r \quad (5.5)$$

where A is a constant determined by the boundary conditions. Now if we knew the temperature profile $\theta = \hat{\theta}(r)$ we could express ϕ as a function of θ and derive the response function $\tilde{J}(\theta)$ by (5.3). In standard analyses of this type, when Fourier's law is not assumed, the temperature profile is either assumed to have some simple form, a special form for $\tilde{J}(\theta)$ is presumed, or a mean value of the thermal conductivity is employed. We shall show that none of these assumptions is necessary for a full and exact determination of \tilde{J} through the use of controllable state concepts.

By (3.4) the controllable radial heat flux component corresponding to (5.5) is

$$h = -(A/r) \quad (5.6)$$

so that the constant $A = -hr$. Therefore, by (5.2) and (5.5) we can write

$$f = -hr \ln r. \quad (5.7)$$

Now measuring the heat flux $h = \bar{h}(a)$ at some fixed radius $r = a$, say, and the corresponding temperature θ with a single thermocouple at some fixed position $r = \rho$, say, would determine one value of the function \bar{f} , viz.

$$\bar{f}(\theta) = -\bar{h}(a) a \ln \rho. \quad (5.8)$$

By changing the surface temperatures α or β , but without having to change the instrumentation of the cylinder, one may thus determine the potential function \bar{f} to any desired degree of accuracy. The temperature-dependent thermal conductivity then follows analytically as

$$\bar{J}(\theta) = -1 \left/ \frac{d\bar{f}(\theta)}{d\theta} \right. \quad (5.9)$$

The important thing to note is that \bar{J} is obtained completely from data collected by measuring only pairs (h, θ) rather than triples $(h, \theta, \nabla\theta)$. Furthermore, nowhere is it required to know the temperature profile $\theta = \hat{\theta}(r)$ or the temperature gradient at a point, the determination of either of which is a major problem in experimental technique.

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