

TRIANGULAR, NINE-DEGREES-OF-FREEDOM, C^0 PLATE BENDING ELEMENT OF QUADRATIC ACCURACY*

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Introduction. The triangular plate bending element with the three nodal values w , w_x and w_y at the vertices has a particular appeal—its simplicity. But a C^1 polynomial interpolation scheme defined over the whole element does not exist. To overcome this Clough and Tocher [1] resorted to subdividing the element into three subelements with the transverse displacement w interpolated individually [2] over each subtriangle so as to maintain a C^1 continuity both in the interior of this complex element and on its boundaries. Bazeley *et al.* [3] interpolated w by some rational functions to obtain the desired variation of w and its derivatives along the sides of the element for assuring a C^1 continuity of displacements. They considered also the use of elements which violate the continuity requirements (non-conforming) and for which the variational principle on minimum total potential energy does not hold any more. These elements may, nevertheless, sometimes produce a valid, stable difference scheme and converge to a useful solution. But their excessive flexibility and precarious convergence did not endear them to engineers. Severn, Taylor and Dungar [4, 5] and Allman [6] used a mixed variational principle [7] for generating a compact hybrid [8] finite element.

Stricklin *et al.* [9] (and also others) made a more radical approach to the generation of plate bending elements in general and the nine-degrees-of-freedom triangular element in particular. They started from the basic equations of elasticity rather than from a ready plate theory and did away with the Kirchhoff assumption (the shear) except at the vertices and along the sides of the element (this is the “discrete Kirchhoff assumption” [10]). Since the plate is now considered a three-dimensional solid, the in-plane displacements are introduced independently of the transverse displacements and the continuity requirement for them is only C^0 , as in three-dimensional elasticity. Assuming that the shear energy is at any rate negligible in thin plates, Stricklin neglected it altogether.

Removing the Kirchhoff assumption from the finite-element analysis of plates and starting with the basic three-dimensional elasticity would seem the most natural approach to the generation of bending elements, particularly since the thin plate is obtained as a limiting case from a three-dimensional solid. However, without the *a priori* assumptions of Kirchhoff, and with the shear energy retained, the global stiffness matrix becomes violently ill-conditioned [11] as the thickness t of the structure is reduced. The difficulties, then, in constructing thin-plate bending elements directly from three-dimensional elasticity are of a *numerical* or *computational* nature. The decline in the conditioning of the matrix may cause, in the computational stage of the solution, grave numerical (round-off) errors, or for a computer with insufficient significant digits the matrix may even be numerically singular.

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If the goal is convergence to the Kirchhoff solution then the addition of shear (removing the Kirchhoff assumption) can be considered as an error, which in the energy is $O(t^2)$, t being the thickness of the plate [12]. Apart from this idealization error the finite-element discretization causes an error in the energy which is $O(h^r)$ where h denotes the diameter of the element and where r is positive. Thus if the plate is thin there is no reason for using the exact small t if h is large—the discretization errors might be overwhelming. The principal idea of this paper is, then, to relate the thickness t of the plate to the mesh size h so that the shear and discretization errors are balanced. This eliminates the factor $1/t^2$ from the stiffness matrix and consequently from its condition number, while retaining the full rate of convergence provided by the shape functions.

This can be applied to plates and shell elements of any degree but we will concentrate on the simplest triangular elements with w , w_x and w_y at the nodes. As the shape functions for the in-plane displacements include a complete polynomial of the first degree the discretization error with this element is $O(h^2)$ [13]. A balanced shear and discretization error is obtained then with $t^2 = h^2/c$, where c is a proportionality coefficient. This parameter c governs the stiffness of matrix; its increase causes an increase in the stiffness of the matrix while its decrease causes the element to become more flexible. In this manner a *continuous* control over the element is provided.

We believe that the element presented here is of great practical importance, but above all its derivation provides a remarkable example of the interaction between the computer, the discretization procedure and the theory of plates and shells.

Addition of shear. Consider a Cartesian coordinate system $Oxyz$ attached to the plate such that x and y lie in its middle surface and z is normal to it. By u , v and w we denote the displacements of a point in the plate in the x , y and z directions, respectively. We assume that the cross-sections of the plate remain *straight* during bending and we denote by θ and ϕ the inclinations of the cross-sections with respect to x and y , respectively. Then

$$u = -\theta z, \quad v = -\phi z, \quad (1)$$

and the direct strains e_{xx} , e_{yy} and e_{zz} become

$$e_{xx} = -(\partial\theta/\partial x)z, \quad e_{yy} = -(\partial\phi/\partial y)z, \quad e_{zz} = 0. \quad (2)$$

The shear strains e_{xy} , e_{xz} and e_{yz} are given by

$$e_{xy} = -\left(\frac{\partial\theta}{\partial y} + \frac{\partial\phi}{\partial x}\right)z, \quad e_{xz} = -\theta + \frac{\partial w}{\partial x}, \quad e_{yz} = -\phi + \frac{\partial w}{\partial y}. \quad (3)$$

For an isotropic material (and for simplicity with zero Poisson ratio) the elastic energy U_E for the element Δ becomes

$$U_E = \frac{E}{4} \int_{-t/2}^{t/2} \int_{\Delta} [2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + (e_{xy}^2 + e_{yz}^2 + e_{xz}^2)] dx dy \quad (4)$$

where E is the elastic modulus of the material. With the transformations

$$x = h\xi, \quad y = h\zeta, \quad \hat{\theta} = \theta h, \quad \hat{\phi} = \phi h \quad (5)$$

the energy expression in Eq. (5) becomes

$$U_K = \frac{1}{4} E \frac{t^3}{12} \frac{1}{h^2} \left\{ \int_{\Delta} \left[2 \left(\frac{\partial \hat{\theta}}{\partial \xi} \right)^2 + 2 \left(\frac{\partial \hat{\phi}}{\partial \eta} \right)^2 + \left(\frac{\partial \hat{\theta}}{\partial \eta} + \frac{\partial \hat{\phi}}{\partial \xi} \right)^2 \right] d\xi d\eta \right. \\ \left. + 12 \left(\frac{h}{t} \right)^2 \int_{\Delta} \left[\left(\frac{\partial w}{\partial \xi} - \hat{\theta} \right)^2 + \left(\frac{\partial w}{\partial \eta} - \hat{\phi} \right)^2 \right] d\xi d\eta \right\}. \quad (6)$$

The kinetic energy expression U_K is of the form

$$U_K = \frac{\rho}{t} \int_{-t/2}^{t/2} \int_{\Delta} (u^2 + v^2 + w^2) dx dy dz \quad (7)$$

where ρ is the mass per unit area, or

$$U_K = \rho h^2 \left\{ \int_{\Delta} w^2 d\xi d\eta + \frac{1}{12} \left(\frac{t}{h} \right)^2 \int_{\Delta} (\hat{\theta}^2 + \hat{\phi}^2) d\xi d\eta \right\}. \quad (8)$$

In the subsequent numerical examples we will neglect the rotary inertia and calculate the mass matrix from

$$U_K = \rho h^2 \int_{\Delta} w^2 d\xi d\eta. \quad (9)$$

Spectral condition number of global stiffness matrix. The element stiffness and mass matrices k and m derived from Eqs. (6) and (8) are of the general form

$$k = \frac{1}{2} E \frac{t^3}{12} \frac{1}{h^2} \left[k_b + 12 \left(\frac{h}{t} \right)^2 k_s \right] \quad (10)$$

and

$$m = \rho h^2 \left[m_b + \frac{1}{12} \left(\frac{t}{h} \right)^2 m_s \right] \quad (11)$$

where the subscripts b and s refer to bending and shear.

It has been shown [13, 14, 15, 16] that the spectral condition number $C_2(K)$ of the stiffness matrix K (defined as the ratio between its maximal and minimal eigenvalues) is bounded by

$$\frac{\max (\lambda_n^k)}{\mu_1 \max (\lambda_n^m) p_{\max}} \leq C_2(K) \leq \frac{\max (\lambda_n^k) p_{\max}}{\lambda_1 \min (\lambda_1^m)} \quad (12)$$

where λ_1 and μ_1 are the exact and finite element eigenvalues of the structure, λ_n^k , λ_n^m and λ_1^m are the extremal (n th and 1st) eigenvalues of k and m , p_{\max} denotes the maximum number of elements meeting at a nodal point and where $\max ()$ and $\min ()$ refer to extremal values in the mesh. The maximum eigenvalue λ_n^k of the element stiffness matrix k is, from Eq. (10), of the form

$$\lambda_n^k = \frac{1}{2} E \frac{t^3}{12} \frac{1}{h^2} \left[c_1 + 12 \left(\frac{h}{t} \right)^2 c_2 \right] \quad (13)$$

where c_1 and c_2 are independent of h and t . For the element mass matrix m we have from Eq. (11), after neglecting the rotary inertia,

$$\lambda_n^m = c_3 h^2, \quad \lambda_1^m = c_4 h^2 \quad (14)$$

where c_3 and c_4 are again independent of h and t . Substituting Eqs. (13) and (14) into Eq. (12) we obtain

$$\frac{c_5}{\mu_1} h^{-4} \left[c_1 + 12 \left(\frac{h}{t} \right)^2 c_2 \right] \leq C_2(K) \leq \frac{c_5}{\lambda_1} h^{-4} \left[c_1 + 12 \left(\frac{h}{t} \right)^2 c_2 \right] \quad (15)$$

in which $Et^3/12$ and ρ were set equal to 1. Eq. (15) clearly indicates that if μ_1 is near enough to λ_1 (as with a fine mesh or sufficiently high degree in the polynomial shape functions) then $C_2(K)$ grows like $O(h^{-2}t^{-2})$. The condition number [13, 14, 15, 16] for pure bending C^1 elements grows like $O(h^{-4})$.

Triangular plate bending element. We restrict our attention here to the nine-degrees-of-freedom element for two reasons: first, other elements can be constructed in precisely the same manner; secondly, C^1 interpolation schemes exist for higher-order elements.

We assign to each vertex of the triangle the three nodal values w , $\partial w/\partial x$ and $\partial w/\partial y$, and interpolate w by an incomplete polynomial of the third degree. For an element with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$ the polynomial terms in the shape functions are $1, x, y, x^2, xy, y^2, x^3, x^2y - yx^2, y^3$ and include a complete quadratic. The inclinations θ and ϕ of the cross-sections are chosen so that at the nodal points $\theta = \partial w/\partial x$ and $\phi = \partial w/\partial y$. Also, shear is suppressed *along* the sides of the element and the inclination of the cross-section normal to the sides of the element is made to vary linearly. This assures C^0 continuity for both the transverse and in-plane displacements and amounts, in fact, to an *independent* variation inside the element of w , $\partial w/\partial x$ and $\partial w/\partial y$.

The discretization error estimate for this element is precisely that of three-dimensional elasticity. There [13], if the shape functions for the displacements include a complete polynomial of degree p , the global error in the energy is $O(h^{2p})$. Since in the present element θ and ϕ are interpolated linearly ($p = 1$), the error in the energy with this element is $O(h^2)$. A balance of discretization and shear errors is therefore obtained with

$$t^2 = h^2/c. \quad (16)$$

Introducing Eq. (16) into the element stiffness matrix results in

$$k = \frac{1}{2} E \frac{t^3}{12} \frac{1}{h^2} (k_b + 12ck_s) \quad (17)$$

and the factor $1/t^2$ is removed from it and consequently from $C_2(K)$, which is now $O(h^{-4})$ as with C^1 elements.

The parameter c in Eq. (17) is arbitrary and any positive value of it will assure the asymptotic convergence $O(h^2)$. By varying c we can *continuously* control the stiffness of the matrix. The hybrid model also permits a certain control over the stiffness of the matrix by varying the degree of the polynomial interpolation functions for the stresses. However, this control is restricted by numerical stability considerations and is not continuous.

Numerical examples. Several numerical examples will illustrate the effect of c in Eq. (17) on the stiffness of the matrix, suggesting a numerical choice for c .

Fig. 1 refers to a simply supported square plate discretized by right-angular elements. It shows the convergence of the central deflection w_c due to a central point load vs. the number of elements per side $N_{..}$ for different values of c . It is clearly seen from Fig. 1

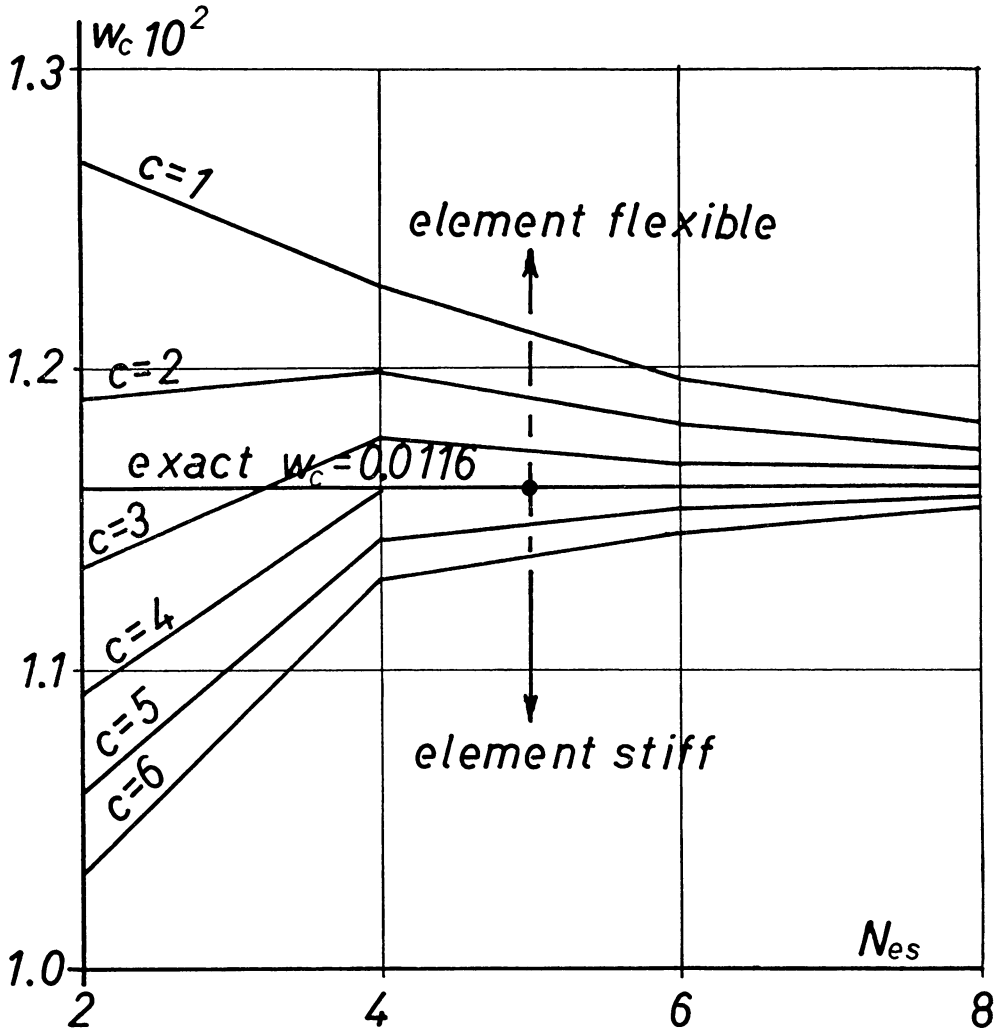


FIG. 1. Simply supported square plate point loaded at the center and discretized by a uniform mesh of right-angular triangular elements. Convergence of the central deflection w_c vs. the number of elements per side N_{es} for different values of c in $t^2 = h^2/c$.

how the choice of c influences the stiffness of the element. Wishing to have an element which is not too flexible ("soft"), we choose $c = 6$. Fig. 2 compares the performance of the present element, on the same plate, with the *HCT* element of Clough and Tocher, with the Allman hybrid element and with the *QQ3-3* element of Stricklin *et al.* Fig. 3 shows the convergence of the first eigenvalue λ_1 of the simply supported plate, discretized with the present element ($c = 6$), with the *HCT* element and with the *DST* hybrid element used by Dungar *et al.* In Fig. 4 the convergence of the central deflection in a clamped plate point loaded at the center is shown for the present element with $c = 6$, the *HCT* element and the Allman hybrid element. In Fig. 5 we compare the convergence of the central moment M_x in a uniformly loaded clamped plate for the present element with $c = 6$ and the hybrid element of Allman.

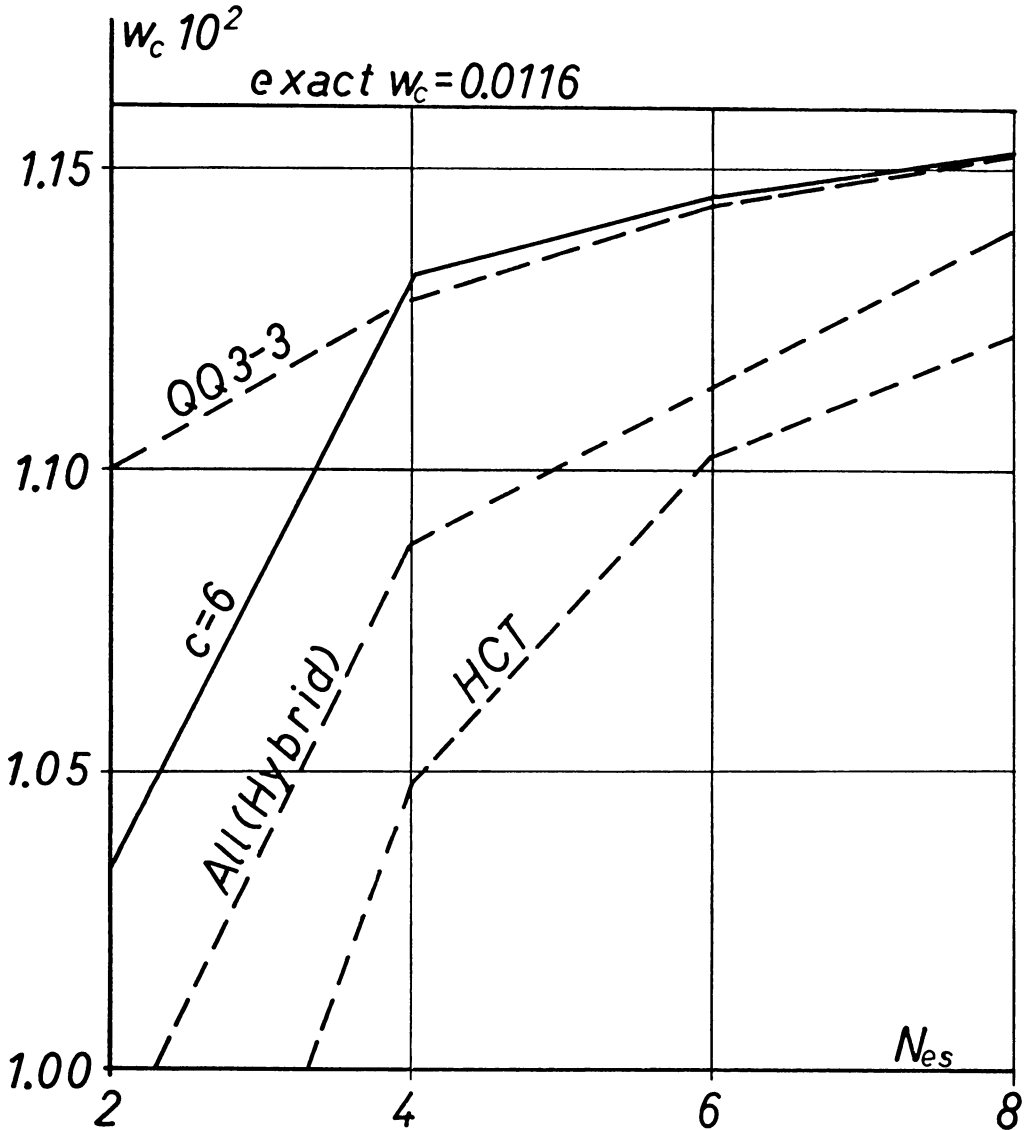


FIG. 2. Simply supported square plate point loaded at the center. Convergence of the central deflection w_c vs. the number of elements per side N_{es} for discretization with the present element ($c = 6$), the HCT element, the hybrid element of Allman and the QQ3-3 element of Stricklin *et al.*

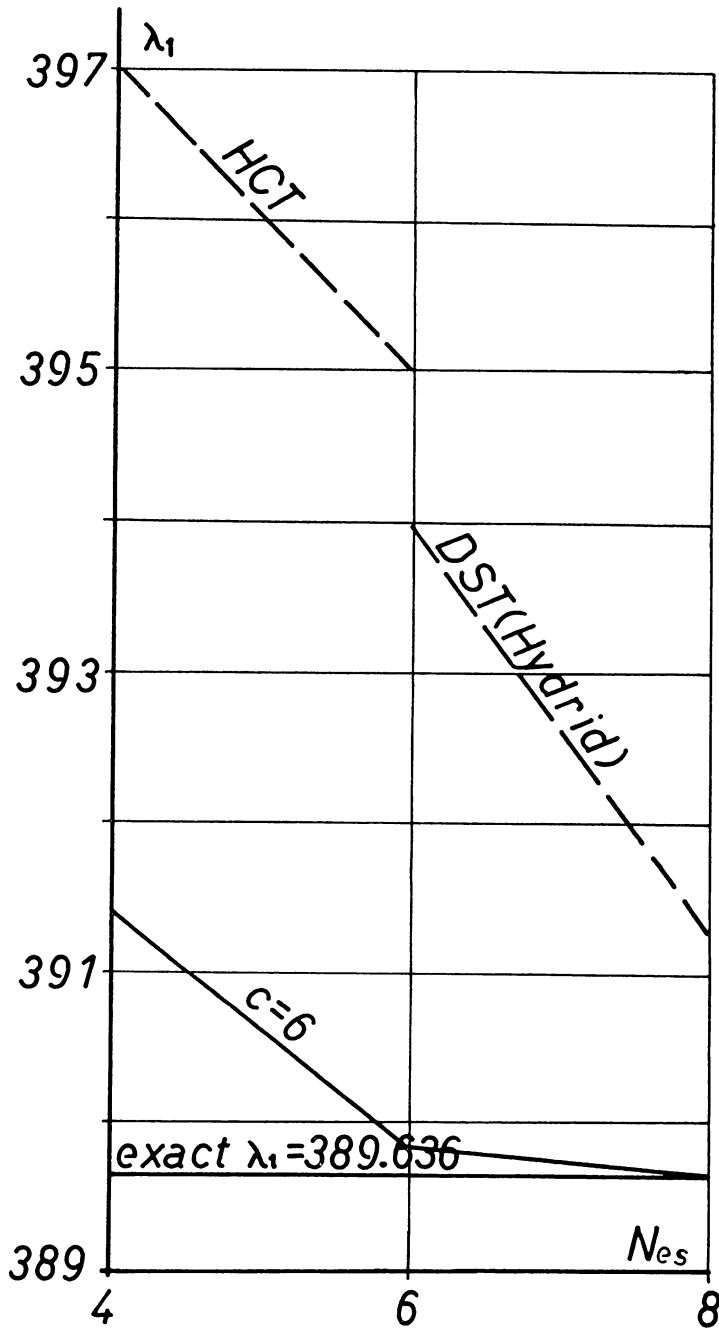


FIG. 3. Convergence of the first eigenvalue λ_1 in a simply supported square plate discretized with the present element ($c = 6$), the HCT element and the DST hybrid element of Dungar, Severn and Taylor.

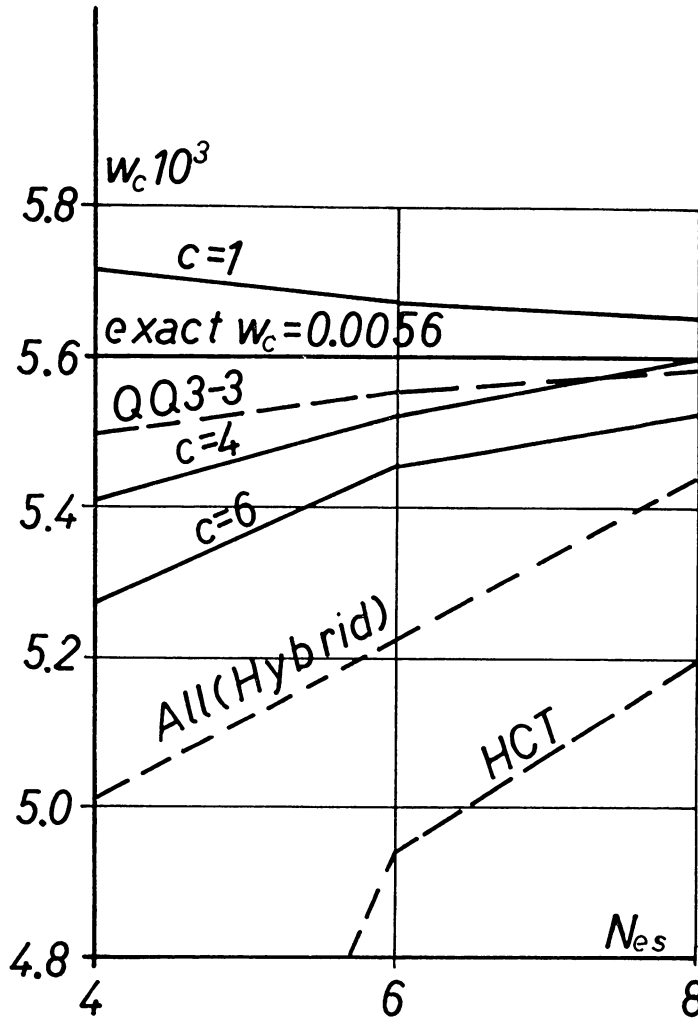


FIG. 4. Clamped square plate point loaded at the center. Convergence of the central deflection for the present element with different values of c in $t^2 = h^2/c$, for HCT element, for the hybrid element of Allman and the QQ3-3 element of Stricklin.

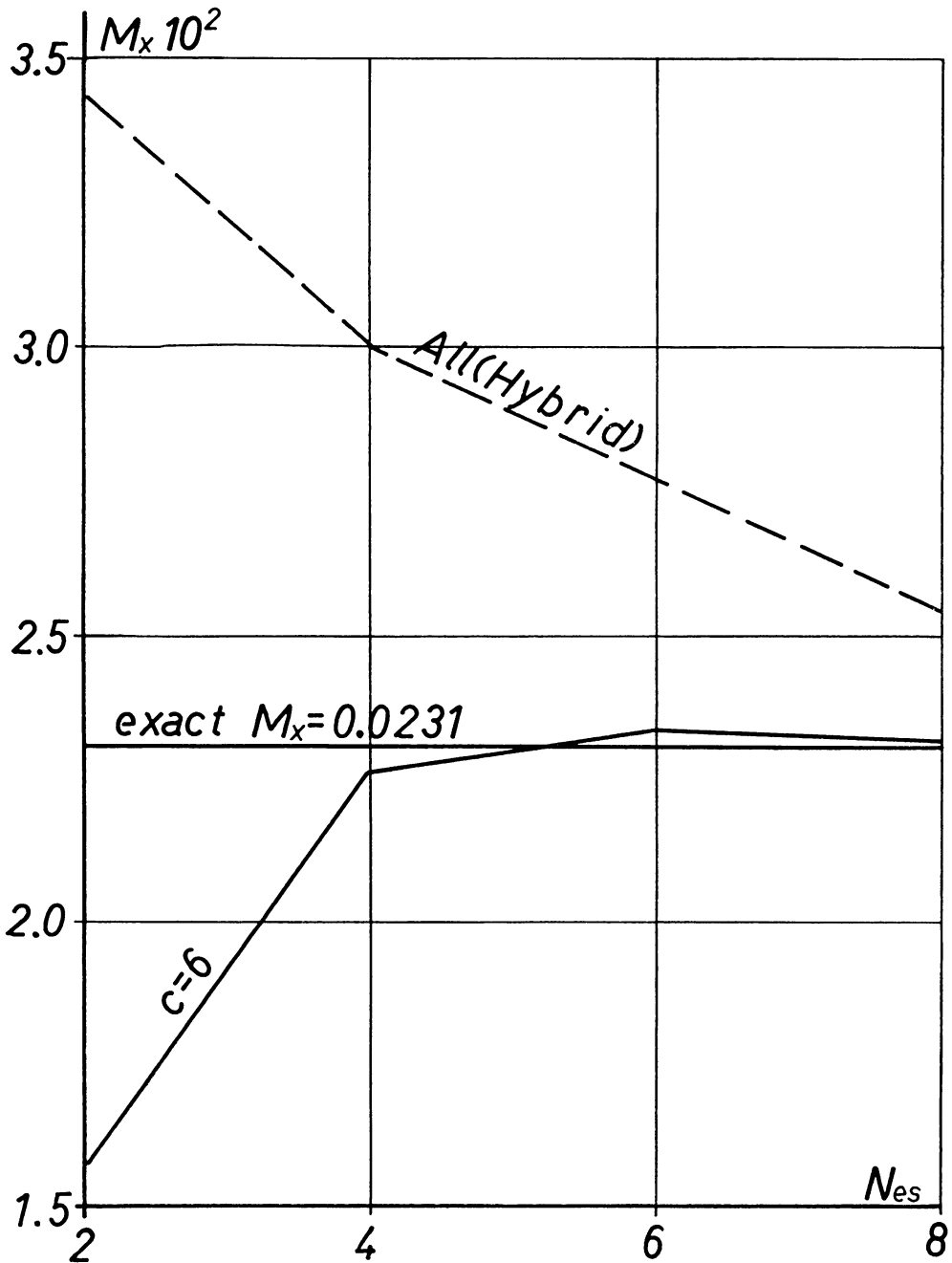


FIG. 5. Clamped square plate uniformly loaded. Convergence of the central moment M_x for the present element ($c = 6$) and the hybrid element of Allman.

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