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APPLICATION OF THE SONIN-PÓLYA OSCILLATION THEOREM*

BY

E. V. LAITONE AND WEN-FAN LIN

University of California, Berkeley

Summary. The necessary conditions for the Sonin-Pólya theorem, which predicts the variation of the successive relative maxima of the oscillating solutions of a second-order ordinary differential equation, are evaluated for the confluent hypergeometric functions to prove that the criteria for their decreasing sequence, as given in [1], [2] and [3], are incorrect. The Sonin–Pólya theorem is then applied to the differential equation for the linearized shallow-water edge wave that is produced by a beach of constant slope. The final application is to Allen's differential equation for the angle of attack oscillations of a re-entry ballistic missile, and new stability criteria are obtained for both ascending and descending missiles that are coasting through the atmosphere.

1. Introduction. The linearized differential equation representing the shallow-water edge waves that are formed perpendicular to a sloping beach when a periodic wave is propagated parallel to the beach has an exact solution in terms of the confluent hypergeometric function defined by Kummer's differential equation

$$xy_{xx} + (b - x)y_x - ay = 0. \tag{1.1}$$

This has the two solutions

$$y(x) = C_1 \, {}_1F_1(a, b, x) + C_2 \, U(a, b, x), \tag{1.2}$$

(see, e.g., Abramowitz and Stegun [1], Bateman [2], or Slater [3]). The first kind of confluent hypergeometric function is defined by Kummer's series as

$${}_1F_1(a, b, x) = 1 + (a)(b)^{-1}x + a(a + 1)(b)^{-1}(b + 1)^{-1}x^2/2! + \cdots, \tag{1.3}$$

while the second kind represented by $U(a, b, x)$ is the logarithmic solution containing ${}_1F_1 \ln x$.

The exact solution for the linearized shallow water edge wave corresponds to $b = 1$, so Eq. (1.3) reduces to

$${}_1F_1(a, 1, x) = 1 + ax + a(a + 1)x^2/2!^2 + a(a + 1)(a + 2)x^3/3!^2 + \cdots. \tag{1.4}$$

This series clearly shows that it is impossible to have an oscillatory solution for $x > 0$ if $a > 0$, and these conclusions are shown to be in agreement with the physical behavior of shallow-water edge waves. However, [1], [2] and [3] have incorrectly applied the Sonin–Pólya theorem so as to predict a decreasing sequence for $|{}_1F_1|$ when $a > 0$ and $x > (b - 1/2)$. In the next section we state, and then correctly apply the Sonin–Pólya theorem to Eq. (1.1), so as to prove that ${}_1F_1$ can oscillate only if $a < 0$.

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2. The Sonin-Pólya oscillation theorem. As given by Szegö (4), this theorem states that if $y(x)$ satisfies the differential equation

$$(Fy_x)_x + G(x)y(x) = 0, \tag{2.1}$$

where $F(x) > 0$ and $G(x) > 0$ and both F_x and G_x are continuous, then the relative maxima of $|y|$ form an increasing or decreasing sequence as x increases, according as $F(x)G(x)$ is decreasing or increasing. Writing Eq. (1.1) in the self-adjoint form

$$(x^b e^{-x} y_x)_x - ax^{b-1}e^{-x}y = 0, \tag{2.2}$$

we see that

$$F = x^b e^{-x} > 0, \quad x > 0; \quad G = -ax^{b-1}e^{-x} > 0, \quad a < 0 \tag{2.3}$$

and

$$(FG)_x = -ae^{-2x}x^{2b-1}(2b - 1 - 2x) > 0, \qquad x < (b - 1/2). \tag{2.4}$$

The fact that we must have $a < 0$ in order to maintain $G > 0$ and therefore have an oscillatory solution had been overlooked in Bateman [2] and Slater [3], and this error is repeated in Abramowitz and Stegun [1]. If $a \geq 0$, then $G \leq 0$ for all $x > 0$ and the solution cannot be oscillatory. The correct statements concerning the oscillations of the confluent hypergeometric functions are therefore only those for $a < 0$, namely, that the successive maxima of $|y|$ are then decreasing if $0 < x < (b - 1/2)$, and increasing if $x > (b - 1/2) > 0$. For $a \geq 0$ the confluent hypergeometric function does not oscillate for $x > 0$. This is immediately apparent from the Kummer's series for ${}_1F_1$, as given by Eq. (1.3), and by the figures presented in [1, p. 514]. It should also be noted that in Abramowitz and Stegun [1, p. 511] there is a misprint in the statement concerning the required variation of FG in the Sonin–Pólya theorem; the words "increasing" and "decreasing" should be interchanged.

3. The linearized shallow-water edge wave. We use the same notation as in Lamb [5, p. 291] for the surface displacement $\zeta(x, y, t)$ of a shallow-water wave over a bottom of constant slope (s) so the depth is given by $h = sy$, and the horizontal x-axis is selected to coincide with the equilibrium water line so that

$$\zeta_{tt}/g = (h\zeta_x)_x + (h\zeta_y)_y = sy\zeta_{yy} + s\zeta_y + sy\zeta_{xx} . \tag{3.1}$$

The periodic wave along the x-axis then generates an edge-wave $\eta(y)$ so that we may write

$$\zeta(x, y, t) = \eta(y) \cos k(x - ct), \tag{3.2}$$

which thereby reduces Eq. (3.1) to

$$y\eta_{yy} + \eta_y + (k^2c^2/gs - k^2y)\eta = 0. \tag{3.3}$$

In its self-adjoint form Eq. (3.3) becomes

$$(y\eta_y)_y + k^2(c^2/gs - y)\eta = 0 \tag{3.4}$$

which corresponds to Eq. (2.1) with

$$F(y) = y; \qquad G(y) = k^2(c^2/gs - y), \tag{3.5}$$

so that

$$(FG)_y = k^2(c^2/gs - 2y). \tag{3.6}$$

Consequently we have an oscillating solution, corresponding to a wave profile with $\eta(y)$ having nodes, only in the region

$$0 < y < c^2/gs. \tag{3.7}$$

For y values greater than c^2/gs we then would expect an exponential decay in the edge-wave profile $\eta(y)$. However, when the Sonin–Pólya theorem is applied to Eq. (3.6) we find that the relative maxima of $|\eta|$ form a decreasing sequence only for

$$0 < y < c^2/2gs, \tag{3.8}$$

while the relative maxima of $|\eta|$ can increase in the region defined by

$$c^2/2gs < y < c^2/gs. \tag{3.9}$$

For $y > c^2/gs$ the oscillations of $\eta(y)$ cease and the exponential decay of $|\eta|$ commences.

These conclusions, which have been derived from the differential equation representing the linearized shallow-water edge waves, can be confirmed by the exact solution of Eq. (3.1) that was first given by Hunt and Hamzah [6] as

$$\zeta(x, y, t) = \eta(y) \cos k(x - ct); \qquad \eta(y) = e^{-ky}\, {}_1F_1(a, 1, 2ky), \tag{3.10}$$

where the parameter in the confluent hypergeometric function is given by

$$a = (1/2)(1 - kc^2/gs). \tag{3.11}$$

First we note that if $a < 0$ then ${}_1F_1(a, 1, \xi)$ behaves as a simple polynomial of order N, defined by $(-a + 1) > N \geq (-a)$, with all of its roots contained in the interval $0 < \xi < (2 - 4a)$ (e.g., see Slater [3]). Consequently Eqs. (3.10) and (3.11) show that $\eta(y)$ is oscillatory only if

$$y < (c^2/gs) > 1/k. \tag{3.12}$$

The first inequality confirms the relation given by Eq. (3.7) while the second inequality above shows that the beach slope must be sufficiently small, $s < kc^2/g$, before linearized shallow-water theory can provide a satisfactory approximation. With this restriction to sufficiently small beach slopes, shallow-water theory predicts infinitely many edge-wave modes, whereas the exact linearized theory predicts only a finite number of modes for a finite beach slope. Ursell [7] has shown that for a fixed wave number k the exact linearized theory has only one frequency (kc) for beach slopes between $\pi/2$ and $\pi/6$, only two frequencies for slopes between $\pi/6$ and $\pi/10$, and likewise an increasing but still finite number of frequencies as the slope decreases. Consequently, as would be expected, shallow-water theory would be most applicable as the beach slope $s \to 0$. In this limiting case we see from Eq. (3.11) that $-a \gg 1$, so we can introduce the asymptotic form of ${}_1F_1$ [1, p. 508]

$${}_1F_1(a, 1, \xi) \to \exp\,(\xi/2) J_0[(2\xi - 4a\xi)^{1/2}], \tag{3.13}$$

where J_0 is the zero-order Bessel function of the first kind. Therefore for small beach slopes Eq. (3.10) reduces to

$$\zeta(x, y, t) \to J_0[2kc(y/gs)^{1/2}] \cos k(x - ct). \tag{3.14}$$

Now the edge-wave profile $\eta(y)$ is identical to the linearized shallow-water wave that is climbing a beach of constant slope (see Lamb [5, p. 276]) and has a monotonic decrease in its successive maxima as y increases. In this asymptotic solution the interval predicted by Eq. (3.9) for the increase of $|\eta|$ is eliminated because we essentially have $c^2/gs \to \infty$. It should be noted that for finite values of $-a > 0$ Eqs. (3.8) and (3.9) are more easily derived from the Sonin–Pólya theorem than from the exact solution, Eq. (3.10).

4. Longitudinal oscillations of a ballistic missile. In our second problem we find a situation wherein the direct application of the Sonin–Pólya theorem to the differential equation yields more information than does the exact solution by itself. This occurs because the exact solution involves the confluent hypergeometric function and we find that its behavior, for the range of parameters involved, is not easily predictable. We consider the differential equation developed by Allen [8] for predicting the angle-of-attack (α) oscillations of a coasting missile that is decelerating while it is traveling at hypersonic speeds in a straight-line trajectory that can be either ascending or descending, namely,

$$\alpha_{xx} + 2k_1e^{-x}\alpha_x + (k_2e^{-x} + k_3e^{-2x})\alpha = 0. \tag{4.1}$$

Here k_1, k_2 and k_3 are constants that depend upon the trajectory inclination and the aerodynamic coefficients of the missile, while x represents the non-dimensional altitude above the earth's surface. The exact solution of Eq. (4.1) was found by Vinh and Laitone [9] and may be written as

$$\alpha = \{\exp [(k_1/\xi_0) - 1/2]\xi\}\{c_1\, {}_1F_1(a, 1, \xi) + c_2U(a, 1, \xi)\}, \tag{4.2}$$

where

$$\xi(x) = \xi_0e^{-x} = 2(k_1^2 - k_3)^{1/2}e^{-x}, \tag{4.3}$$

$$a = 1/2 - (k_1 + k_2)\xi_0^{-1} = (1/2)[1 - (k_1 + k_2)(k_1^2 - k_3)^{-1/2}]. \tag{4.4}$$

Allen found that any ballistic missile that had satisfactory static longitudinal stability was dominated by the term k_2 since all of the bodies studied had

$$k_2 \geq 0(10^4) \gg (|k_1| + |k_3|) > 0. \tag{4.5}$$

Consequently for the usual ballistic missile that has static longitudinal stability we have $-a \geq 10^4$, and therefore the asymptotic representation of the confluent hypergeometric functions, as in Eq. (3.13), provides an excellent approximation so that Eq. (4.2) may be replaced by

$$\alpha(x) = [c_1J_0(\zeta^{1/2}) + c_2Y_0(\zeta^{1/2})] \exp (k_1e^{-x}), \tag{4.6}$$

where

$$\zeta = 4(k_1 + k_2)e^{-x} \approx 4k_2e^{-x} \geq 0 \tag{4.7}$$

and J_0, Y_0 are the zero-order Bessel functions of the first and second kind respectively. This Bessel-function solution was first given by Allen [8] as an approximate solution of Eq. (4.1) for the usual missile that satisfied Eq. (4.5), and it is evident that the effect of k_3 has been completely eliminated in this approximate solution.

Now it will be shown that the stability criteria that Allen developed for the α oscillations, by using the approximate solution given by Eq. (4.6), can be made exact and therefore include the effect of k_3 by simply applying the Sonin–Pólya theorem directly to Allen's differential equation. In its self-adjoint form Eq. (4.1) corresponds to Eq. (2.1) with $y = \alpha(x)$ if

$$F(x) = \exp(-2k_1e^{-x}); \qquad G(x) = (k_2e^{-x} + k_3e^{-2x}) \exp(-2k_1e^{-x}), \tag{4.8}$$

so that $F > 0$ for all cases and $G > 0$ as long as $k_2 > |k_3| > 0$. Then the slope of FG is given by

$$\begin{aligned}(FG)_x &= -k_2e^{-x}[1 - (4k_1 - 2k_3/k_2)e^{-x} - (4k_1k_3/k_2)e^{-2x}] \exp(-4k_1e^{-x}) \\ &= -k_2[e^{2x} - (4k_1 - 2k_3/k_2)e^{x} - (4k_1k_3/k_2)] \exp(-3x - 4k_1e^{-x}).\end{aligned} \tag{4.9}$$

Consequently we see that if $k_1 \leq 0$ then the envelope of the $|\alpha|$ oscillations increases with x, since $(FG)_x < 0$ for all $x > 0$ as long as the term in the square brackets remains positive. In other words, the $|\alpha|$ envelope decreases in descending flight, and increases in ascending flight, as shown in Fig. 1, whenever $k_1 \leq 0$ and $k_3 \geq 0$. However, if $k_3 < 0$ when $k_1 < 0$ we then must satisfy the inequality

$$1 - 4k_1(1 + k_3/k_2) > -2k_3/k_2 > 0. \tag{4.10}$$

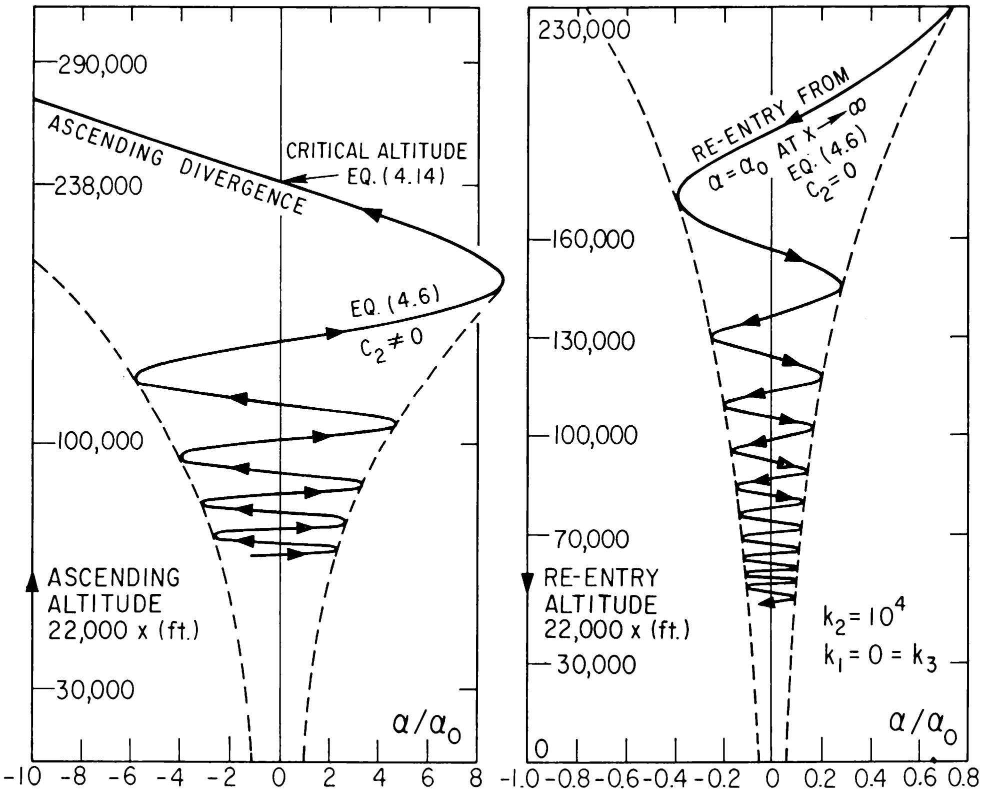

FIG. 1. The oscillations in the angle of attack (α) of a coasting ballistic missile that is decelerating along a straight line trajectory inclined either upward, or downward as in the re-entry case studied by Allen [8, Fig. 1]. In both cases $k_1 = 0 = k_3$ and $k_2 = 10^4$.

Therefore the conclusions of Allen [8] and Tobak and Allen [10] for $k_1 < 0$ are satisfied by the inequalities in Eq. (4.10), which are much less restrictive than those of Eq. (4.5).

Now for the case of $k_1 > 0$ we have

$$(FG)_x \geq 0 \quad \text{for} \quad x \leq x_* ; \qquad (FG)_x \leq 0 \quad \text{for} \quad x \geq x_* , \tag{4.11}$$

where

$$x_* = \ln [2k_1 - (k_3/k_2) + (4k_1^2 + k_3^2/k_2^2)^{1/2}]. \tag{4.12}$$

Consequently, as long as $k_1 > 0$ and

$$4k_1(1 + k_3/k_2) > (1 + 2k_3/k_2) \tag{4.13}$$

we find that the envelope of the $|\alpha|$ oscillations decreases as x increases up to the critical altitude defined by x_* , and then continually increases with x above this altitude. On the other hand, for a descending missile $|\alpha|$ continually decreases until the critical altitude defined by Eq. (4.12) is reached; from there on the envelope of the $|\alpha|$ oscillations continually increases as the missile descends. When $k_3 = 0$ the critical altitude given by Eq. (4.12) is identical to the one originally derived by Allen [8]. His derivation was for re-entry flight only, since he used Eq. (4.6) with $c_2 = 0$ and $c_1 = \alpha_0$ so as to satisfy the initial conditions when $x \to \infty$ $(\zeta \to 0)$. Consequently Allen was able to use the asymptotic trigonometric approximation for J_0 to find the critical altitude for descending flight.

On the other hand, we must have $c_2 \neq 0$ for ascending flight in order to satisfy the initial conditions when x remains finite, e.g., $\zeta = 4(k_1 + k_2)$ when $x = 0$. Since $\zeta \to 0$ as the missile ascends, and $Y_0(0) \to -\infty$, we can therefore predict another critical altitude that exists for any ascending missile, regardless of the algebraic sign of k_1 , by noting that $Y_0(\zeta^{1/2})$ diverges with no further oscillations whenever $\zeta_c^{1/2} \leq 0.8936$ so that

$$\exp(-x_c/2) \leq (0.8936/2)(k_1 + k_2)^{-1/2}; \qquad x_c \geq \ln 5(k_1 + k_2). \tag{4.14}$$

This critical altitude is shown in Fig. 1 for the ascending missile having $k_1 = 0 = k_3$ so the previous critical altitude $x_* = 0$, since $(FG)_x < 0$ for all $x \geq 0$ as long as $k_2 > 0$. As shown by Allen [8], Fig. 1 would not be greatly altered in appearance if $k_1 < 0$ because the term $k_2 = 10^4$ so dominates Eq. (4.6). However, any $k_1 > 1/4$ produces a severe divergence as $x \to 0$ $(\zeta \to 4k_2 = 4 \times 10^4)$ for the descending missile (see [8, Fig. 1]).

It is interesting to note that when $k_1^2 = k_3 \geq 0$ then Eq. (4.6) becomes the exact solution of Eq. (4.1), while Eq. (4.2) is no longer applicable since $\xi_0 = 0$. This means that Eq. (4.6) can be used to provide an exact oscillatory solution for any value of $(k_1 + k_2) > 0$. However, if $(k_1 + k_2) < 0$ then the solution diverges exponentially with no oscillations, since (J_0 , Y_0) are replaced by the non-oscillating Bessel functions (I_0 , K_0). Allen's solution, as given by Eq. (4.6), is the only useful solution of Eq. (4.1) because $U(a, 1, \xi)$ has not been tabulated, and even though ${}_1F_1(-n, 1, \xi)$ reduces to a finite polynomial of order n, its practical use is restricted in this particular problem because $-a = n > 10^4$.

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