QUARTERLY OF APPLIED MATHEMATICS
OCTOBER, 1974

MULTIPLE FOURIER ANALYSIS IN RECTIFIER PROBLEMS II*

BY

ROBERT L. STERNBERG AND MICHAEL R. SHEETS (*University of Rhode Island*)

HELEN M. STERNBERG (*University of Connecticut*)

ABRAHAM SHIGEMATSU (*Naval Underwater Systems Center*)

AND ALICE L. STERNBERG (*The Williams School*)

Abstract. The nonlinear problem of the multiple Fourier analysis of the output from a cut-off power-law rectifier responding to a two-frequency input, reviewed in general in Part I of this study [1], is further scrutinized here for the special case of a zero-power-law device; i.e., a bang-bang device or a total limiter. Solutions for the modulation product amplitudes or multiple Fourier coefficients as in Part I appear as Bennett functions, and line graphs of the first fifteen basic functions for the problem are given. The new functions $A_{mn}^{(0)}(h, k)$ studied, being based on a discontinuous device, then, together with the functions $A_{mn}^{(1)}(h, k)$ studied in Part I, provide approximate solutions to the two-frequency modulation product problem for an arbitrary piecewise continuous nonlinear modulator, and the solution for this general problem is outlined. Finally, numerical tables of the zeroth-kind functions $A_{mn}^{(0)}(h, k)$ graphed have been prepared and are available separately in the United States and Great Britain. As before, the entire theory is based on the original multiple Fourier methods introduced by Bennett in 1933 and 1947.

1. Introduction and formulation of the problem. In Part I of this paper by Sternberg et al. [1], the much-studied problem in theoretical electronics of the multiple Fourier analysis of the output from a cut-off power-law rectifier responding to a several-frequency input was studied in general, and the multiple Fourier coefficients or Bennett functions

$$A_{mn}^{(\nu)}(h, k) = \frac{2}{\pi^2} \iint_{\mathcal{R}} (\cos u + k \cos v - h)^\nu \cos mu\, du \cos nv\, dv,$$

$$\mathcal{R} : \cos u + k \cos v \geq h, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq \pi, \qquad (1.1)$$

where the indices m, n take all integral values $m, n \geq 0$, and where $\nu \geq 0$, were examined in some detail for the case in which $\nu = 1$. Here we study the zeroth-kind functions (1.1), that is to say the case $\nu = 0$, an apparently simple but actually quite complicated problem.

* Received May 8, 1973. The authors are indebted to the Computation Centers of the University of Rhode Island, the University of Connecticut, and the Naval Underwater Systems Center in New London, Connecticut for the computational and graphical work that went into the project. These computational services were provided the authors by the several institutions named over a continuing period of time without charge of any kind, and the authors wish to express their sincere thanks for this help.

For a cut-off power-law rectifier with output versus input characteristic $Y = Y^{\nu}(X; X_0)$ of the form $Y^{\nu}(X; X_0) = (X - X_0)^{\nu}$ if $X > X_0$ and zero otherwise, the functions (1.1) are the coefficients $A_{\pm mn}{}^{(\nu)}(h, k) = A_{mn}{}^{(\nu)}(h, k)$ in the double Fourier series expansion

$$y(t) = \tfrac{1}{2}P^{\nu}A_{00}{}^{(\nu)}(h, k) + P^{\nu} \sum_{m,n=0}^{\infty}{}^{*} A_{\pm mn}{}^{(\nu)}(h, k) \cos (\omega_{\pm mn}t + \phi_{\pm mn}) \tag{1.2}$$

of the output $y(t) = Y^{\nu}[x(t); X_0]$ when the input $X = x(t)$ has the form

$$x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q), \qquad P \geq Q > 0. \tag{1.3}$$

In (1.1) we have $h = X_0/P$ and $k = Q/P > 0$, while in (1.2) the modulation product frequencies and phase angles, $\omega_{\pm mn}$ and $\phi_{\pm mn}$, are given by the relations $\omega_{\pm mn} = mp \pm nq$ and $\phi_{\pm mn} = m\theta_p \pm n\theta_q$. Finally, in (1.2) the asterisk on the summation sign indicates that we sum only on all distinct arrangements of plus and minus signs, equivalent arrangements being taken only with the plus signs and with the zero-order term, particularly, having been removed from the sum.

The nonlinear device in the problem is a biased linear rectifier when $\nu = 1$ and may be described as a biased zero-one bang-bang device or total limiter when $\nu = 0$, as in Bennett [2, 3]. If we use appropriate linear combinations of functional values $A_{mn}{}^{(1)}(h_i, k)$ summed up with suitable coefficients g_i for suitable choices of the h_i's, the corresponding multiple Fourier coefficients $B_{\pm mn}$ in the double Fourier series expansion of the output of a general nonlinear device with input (1.3) may be approximated to within an arbitrary $\epsilon > 0$ for all m, n when the output versus input characteristic $Y = Y(X)$ is continuous; similarly, the $B_{\pm mn}$ for a general device may be expressed exactly as linear sums of functional values $A_{mn}{}^{(1)}(h_i, k)$ in the form

$$B_{\pm mn} = P \sum_{i=1}^{N(\epsilon)} g_i A_{mn}{}^{(1)}(h_i, k) \tag{1.4}$$

for suitable choices of the numbers g_i and h_i when the device characteristic $Y = Y(X)$ is not only continuous, but also polygonal. Similarly, by a simple extension of the basic technique both of these results can be extended, with the aid of the functions $A_{mn}{}^{(0)}(h, k)$, to the case of an arbitrary nonlinear device with a piecewise continuous characteristic $Y = Y(X)$, approximate linear results being obtained in the one case and exact linear expressions for the $B_{\pm mn}$ in terms of functional values $A_{mn}{}^{(0)}(h_i^*, k)$ and $A_{mn}{}^{(1)}(h_i, k)$ in the form

$$B_{\pm mn} = P \sum_{i=1}^{N^*(\epsilon)} g_i^* A_{mn}{}^{(0)}(h_i^*, k) + P \sum_{j=1}^{N(\epsilon)} g_j A_{mn}{}^{(1)}(h_j, k) \tag{1.5}$$

being obtained in the other case, i.e. when the device characteristic $Y = Y(X)$ is not merely piecewise continuous but piecewise polygonal as well. The method is described in detail by Sternberg and Kaufman [4, 5, 6] for the continuous case and is readily extended to the piecewise continuous case without difficulty. Thus, the functions (1.1) have very broad applicability in all cross-talk problems in communications and control theory.

In addition to the foregoing, the functions $A_{mn}{}^{(1)}(h, k)$ occur also in various related statistical information-processing problems as discussed, for example, by Shipman [7], and the functions $A_{mn}{}^{(0)}(h, k)$ have been independently applied to some unrelated problems in crystallography by Montroll [8]. Similarly, if Feuerstein's results [9] and

Bennett's earlier results [1] are turned around, the Bennett functions $A_{mn}^{(\nu)}(h, k)$ may be viewed generally as new special functions of mathematical physics in terms of which many otherwise difficult generalized Weber–Schafheitlin integrals and Schlömilch series or infinite integrals and sums involving Bessel function products may be directly evaluated.

Finally, since the appearance of Part I of this paper, a very interesting review article by Hsu [10] has appeared, and related problems in transistor circuits were studied in a similar manner by Penfield [11[.

The interest and utility of Bennett functions thus appear to be gradually growing and the basic case of the functions $A_{mn}^{(0)}(h, k)$ has been found to have special significance of its own. Hence, we study these latter functions in some detail in the following.

Line graphs of the first fifteen functions $A_{mn}^{(0)}(h, k)$ are given in Figs. 1 through 15, and six decimal tables with h and k varying in steps of one-tenth unit and with error generally less than 1×10^{-6} units have been deposited in the Unpublished Mathematical Tables file in the editorial offices of the journal *Mathematics of Computation*, in the United States and in the Tables Collections of the Grace Library of the University of Liverpool in Great Britain and with the authors.

2. Basic formulas and expansions for the zeroth-kind functions. In addition to satisfying the numerous relations described in Part I for the Bennett functions $A_{mn}^{(\nu)}(h, k)$ in general, the zeroth-kind functions $A_{mn}^{(0)}(h, k)$ satisfy several special relationships and have a few properties that are more or less unique. As usual, the variable h takes all real values while the variable k ranges over the interval $0 < k \leq 1$, and again, it is convenient to discuss the functions in terms of three cases defined, as in Part I, by the conditions: (0) $h \geq 1 + k$, (a) $|h| \leq 1 + k$, and (∞) $h \leq -1 - k$ where again, as before, case (0) is trivial, the rectifier then being biased so strongly that $A_{mn}^{(0)}(h, k) = 0$ for all m and n.

To begin with, since for $\nu = 0$ the kernel function in the integrals (1.1) defining the functions $A_{mn}^{(\nu)}(h, k)$ reduces to unity, a first integration can always be carried out. This leads, then, for the first several functions $A_{mn}^{(0)}(h, k)$ with $h \geq 0$, graphs of which are shown in Figs. 1 through 15, to the formulas

$$
\begin{aligned}
A_{0n}^{(0)}(h, k) &= \frac{2}{\pi^2} \int_0^a \cos^{-1}(h - k\cos v)\cos nv\, dv, \\
A_{1n}^{(0)}(h, k) &= \frac{2}{\pi^2} \int_0^a [1 - (h - k\cos v)^2]^{1/2}\cos nv\, dv, \\
A_{2n}^{(0)}(h, k) &= \frac{2}{\pi^2} \int_0^a [1 - (h - k\cos v)^2]^{1/2}(h - k\cos v)\cos nv\, dv, \qquad (2.1) \\
A_{3n}^{(0)}(h, k) &= \frac{2}{\pi^2} \int_0^a [1 - (h - k\cos v)^2]^{1/2}[\tfrac{4}{3}(h - k\cos v)^2 - \tfrac{1}{3}]\cos nv\, dv, \\
A_{4n}^{(0)}(h, k) &= \frac{2}{\pi^2} \int_0^a [1 - (h - k\cos v)^2]^{1/2}[2(h - k\cos v)^3 - (h - k\cos v)]\cos nv\, dv,
\end{aligned}
$$

where $a = \pi$ if $|h| + k \leq 1$ and $a = \cos^{-1}[(h - 1)/k]$ if $1 - k \leq h \leq 1 + k$, and where n takes all integral values $n \geq 0$. Except for the restrictions on h, these formulas hold quite generally.

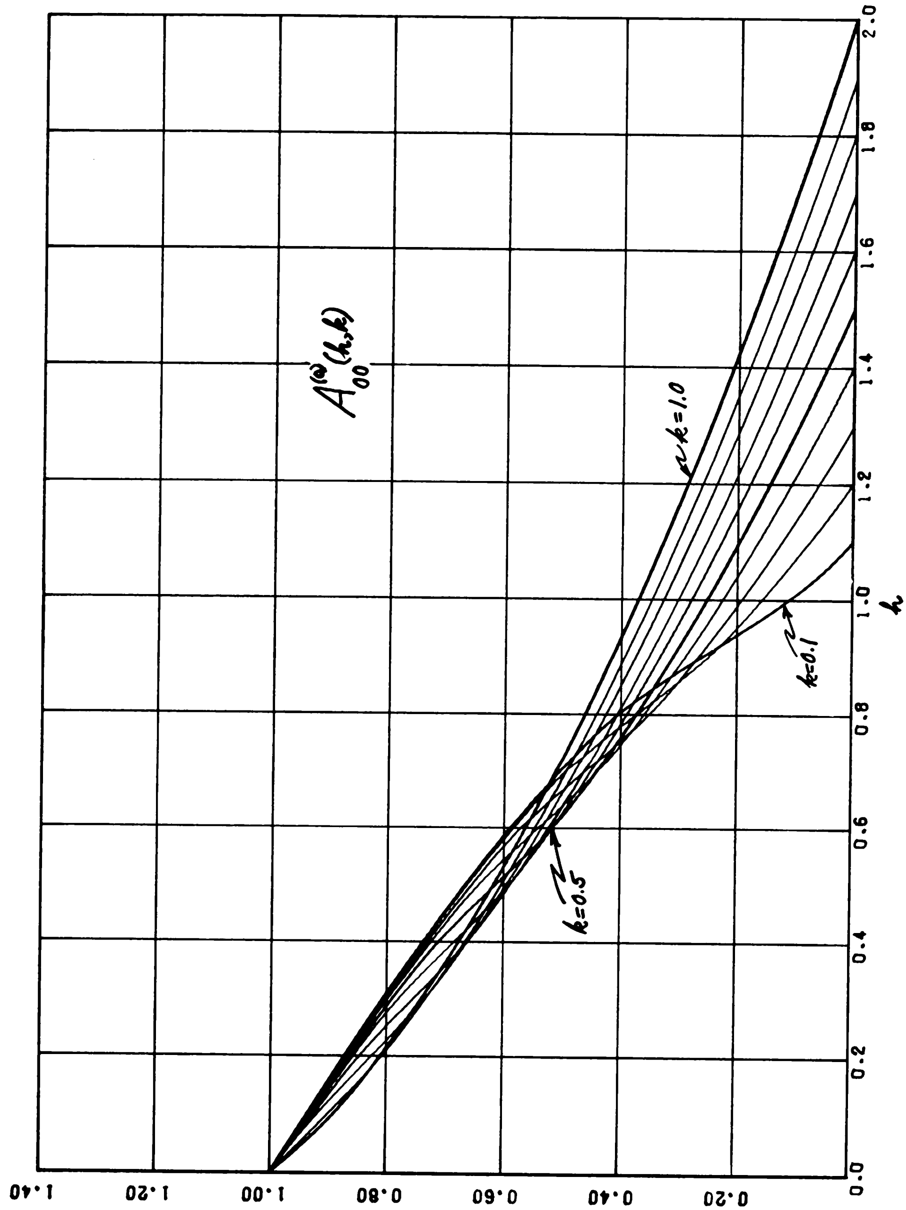

FIG. 1. The function $A_{00}^{(0)}(h, k)$.

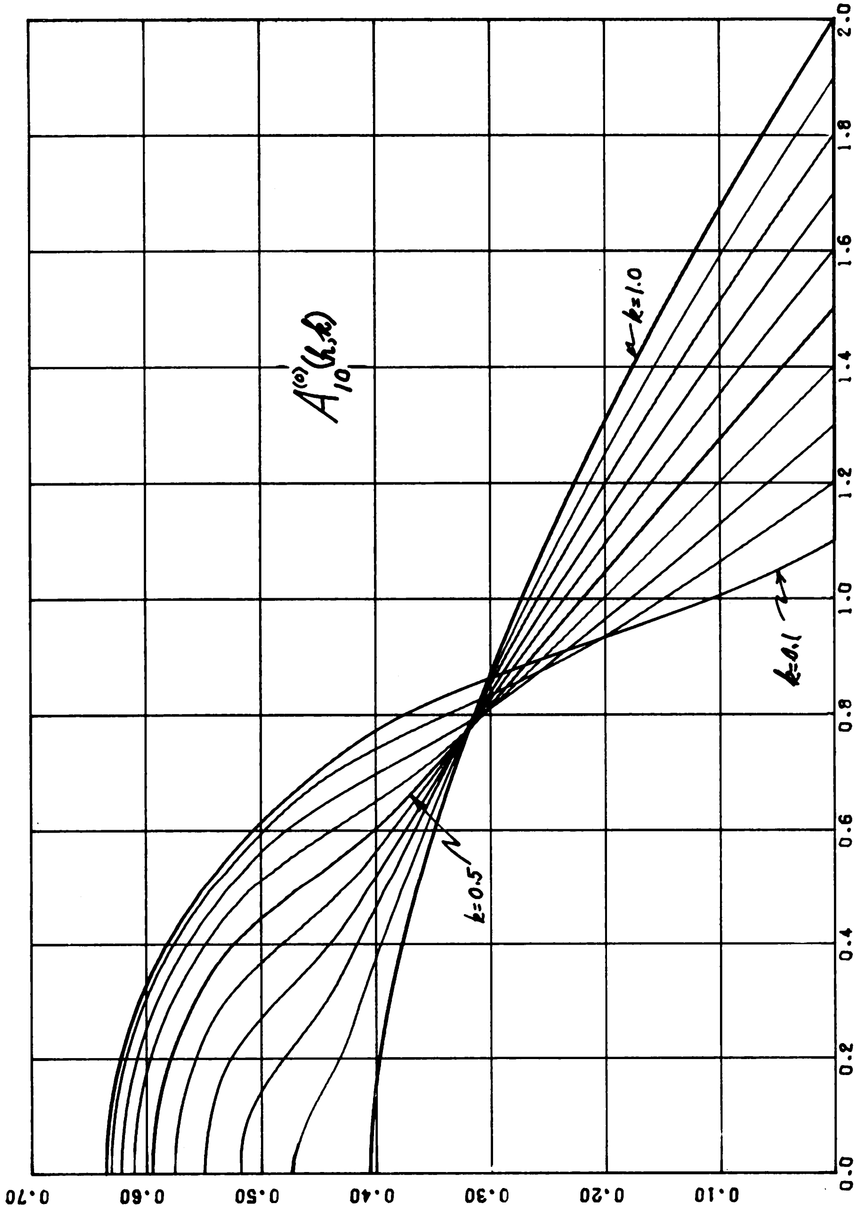

FIG. 2. The function $A_{10}^{(0)}(h, k)$.

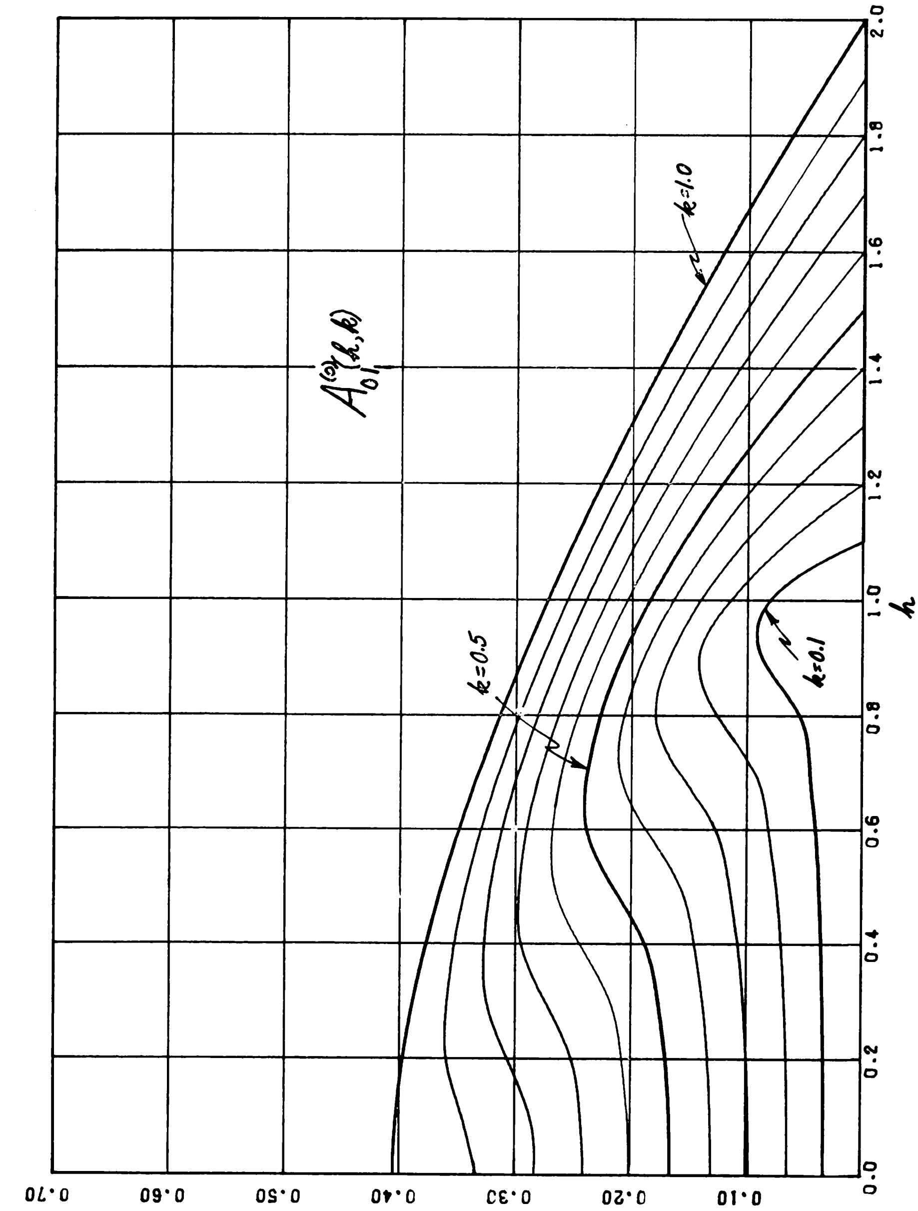

FIG. 3. The function $A_{01}{}^{(0)}(h, k)$.

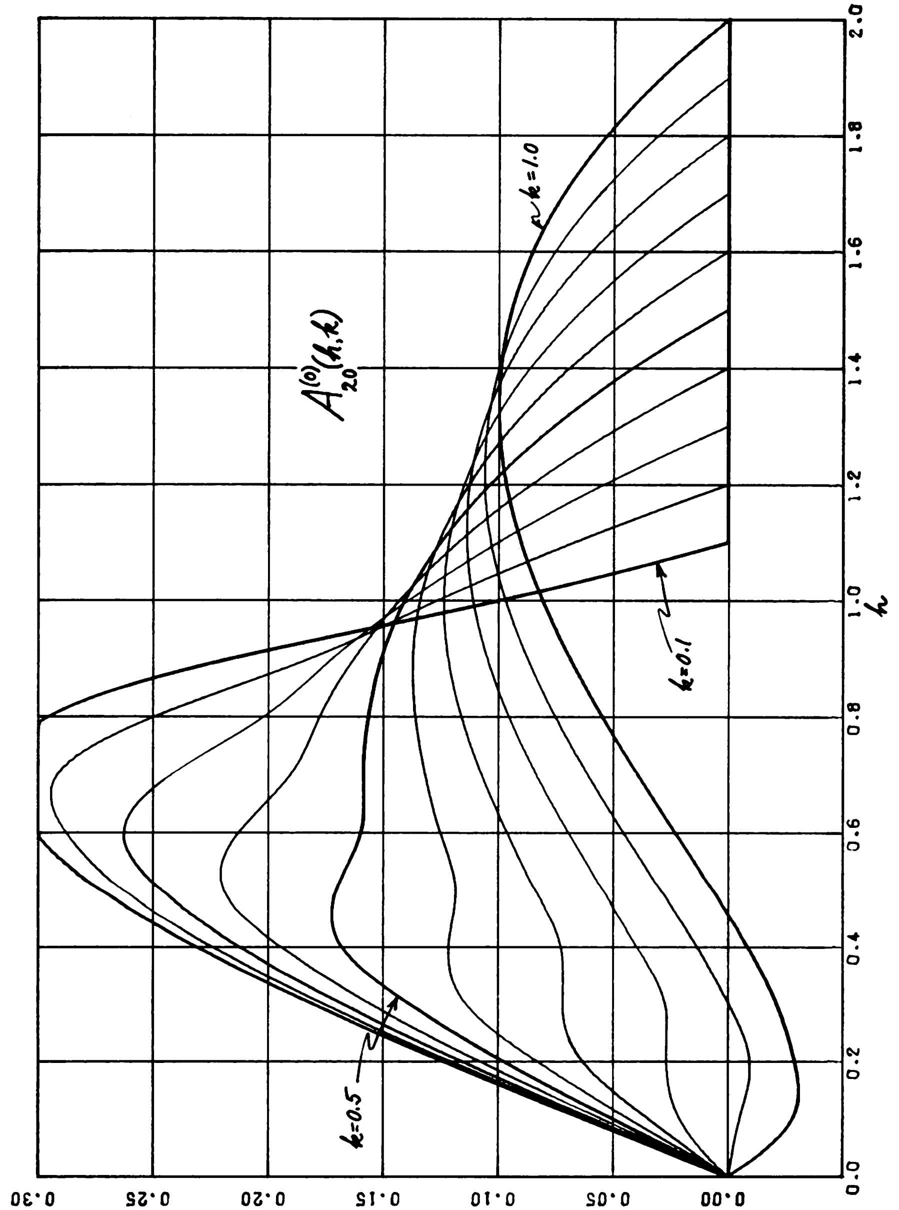

FIG. 4. The function $A_{20}^{(0)}(h, k)$.

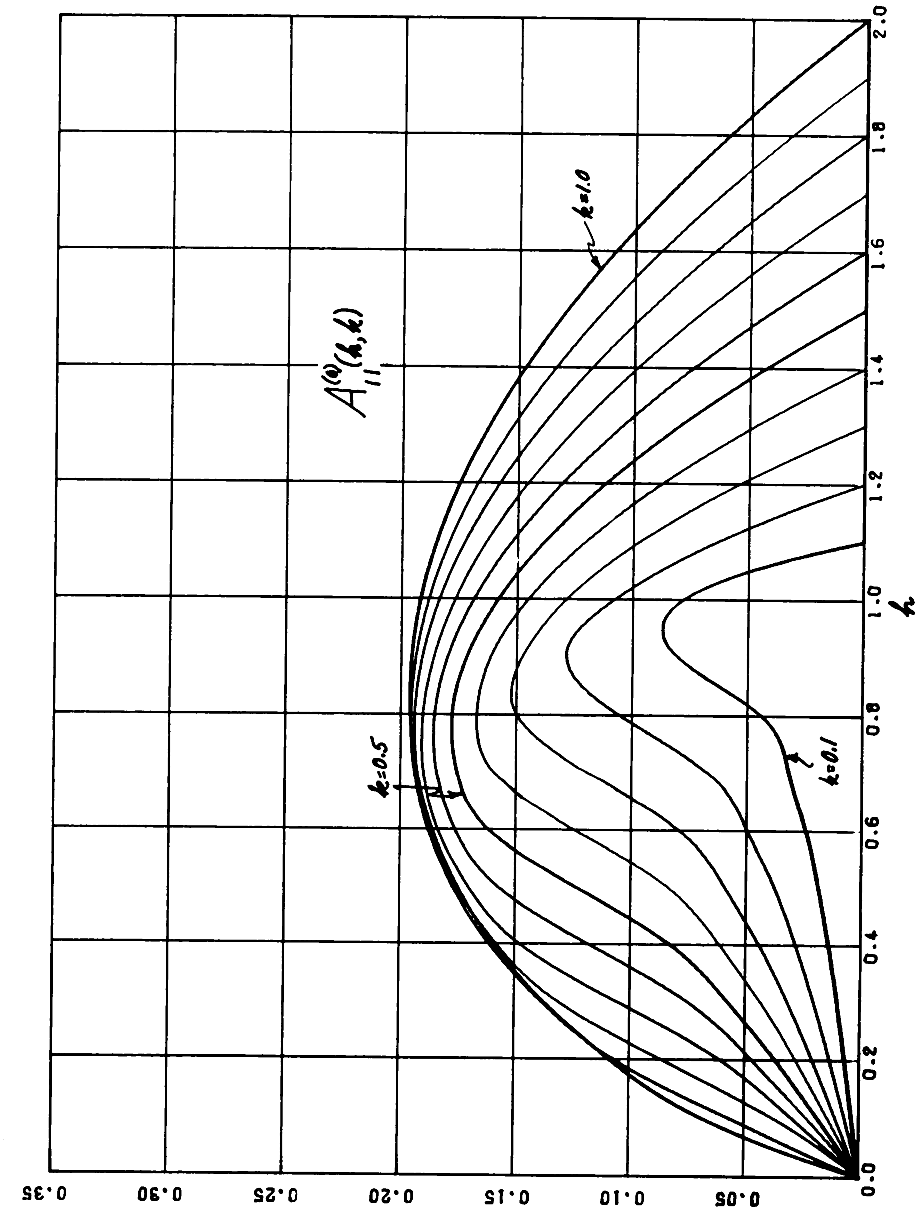

FIG. 5. The function $A_{11}{}^{(0)}(h, k)$.

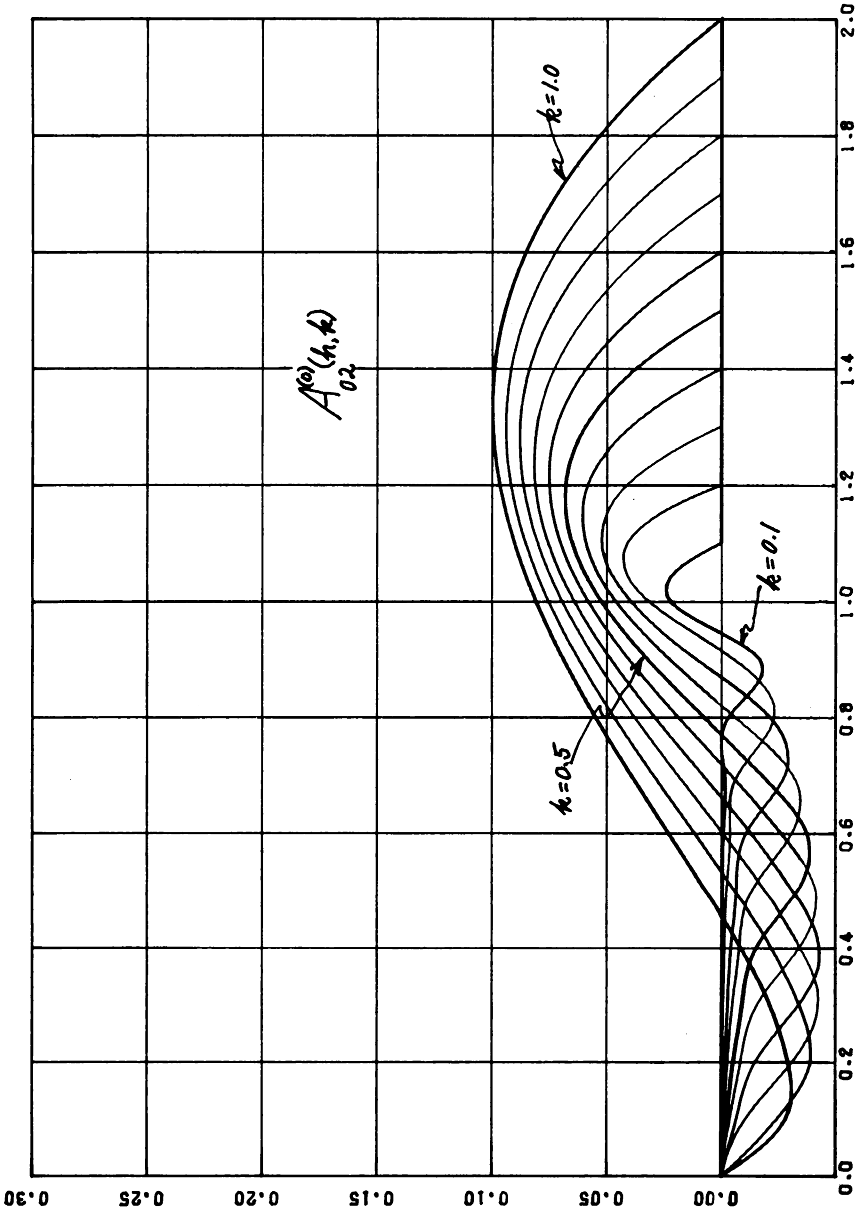

Fig. 6. The function $A_{02}^{(0)}(h, k)$.

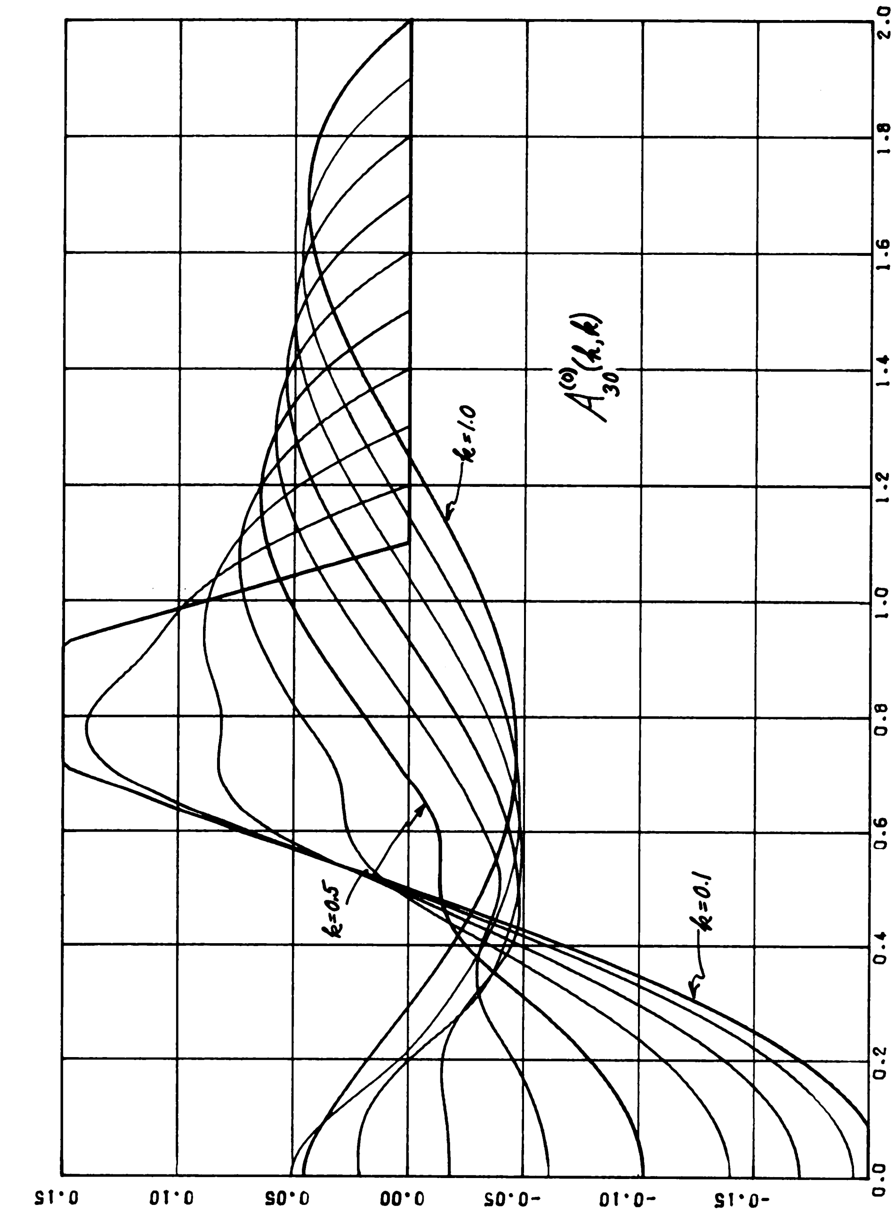

FIG. 7. The function $A_{30}^{(0)}(h, k)$.

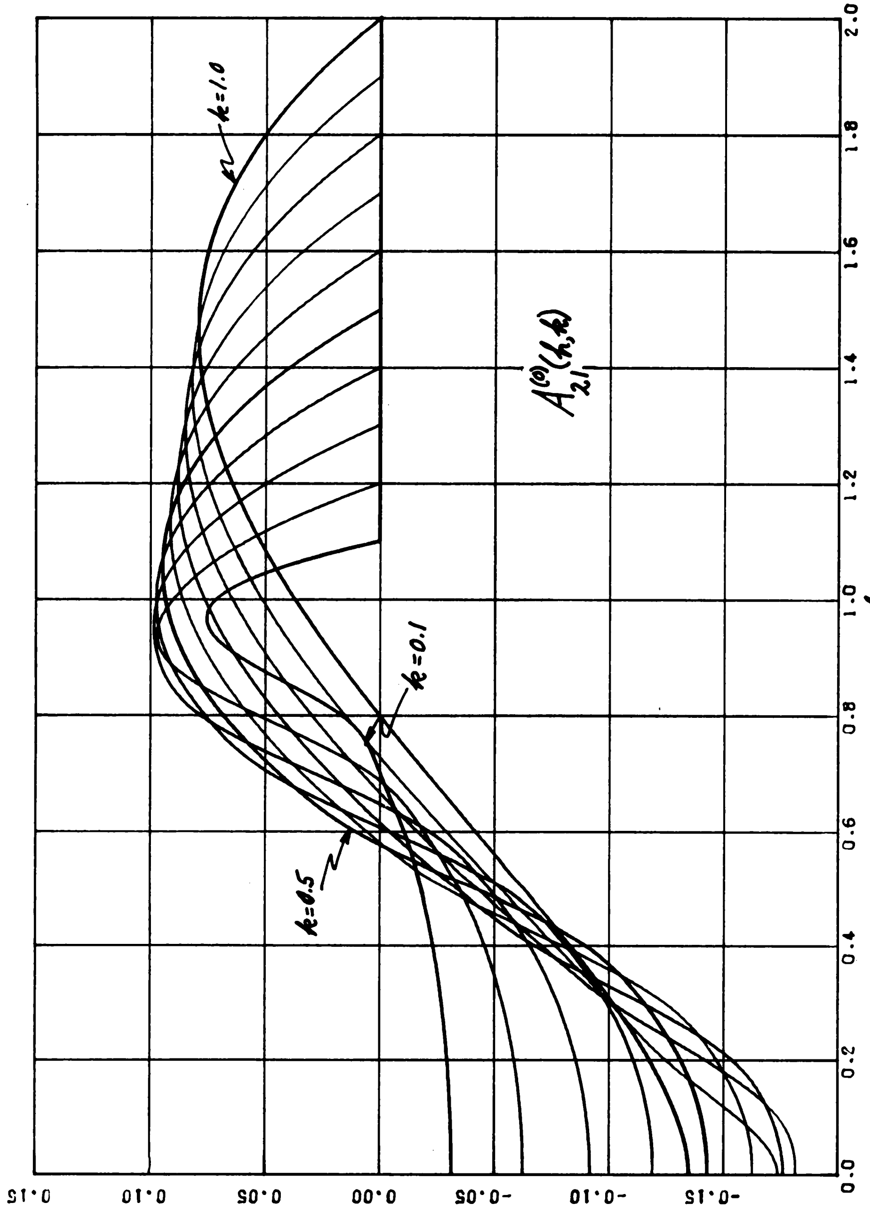

FIG. 8. The function $A_{21}{}^{(0)}(h, k)$.

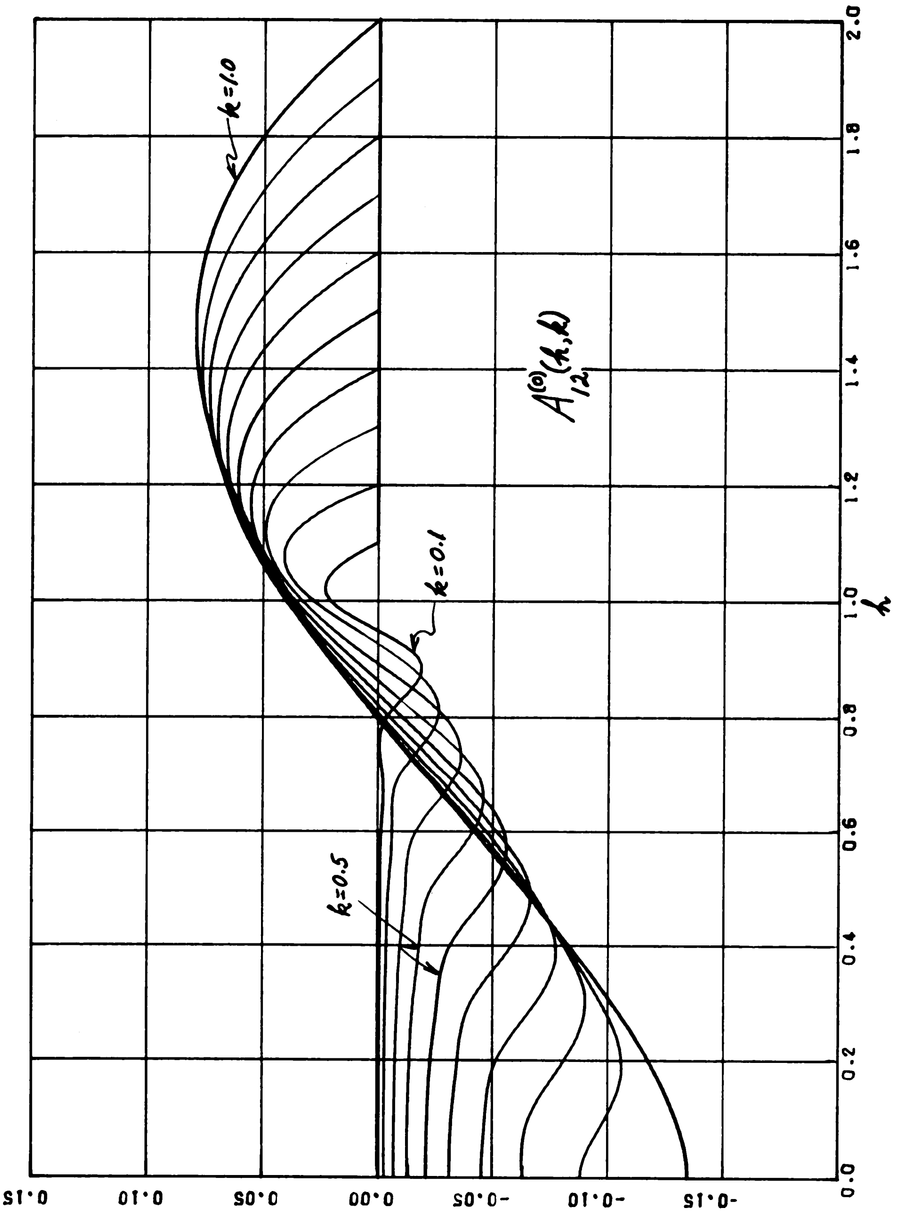

FIG. 9. The function $A_{12}^{(0)}(h, k)$.

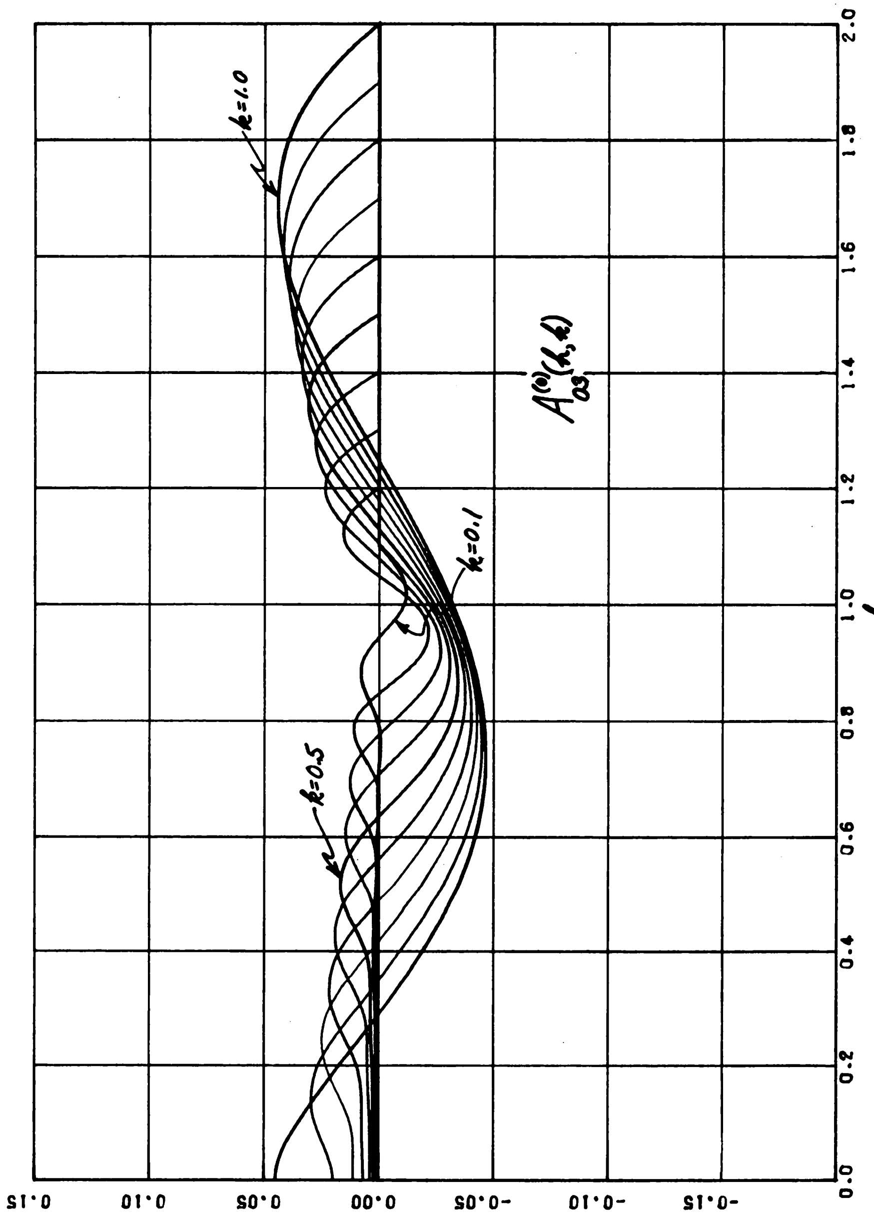

FIG. 10. The function $A_{03}^{(0)}(h, k)$.

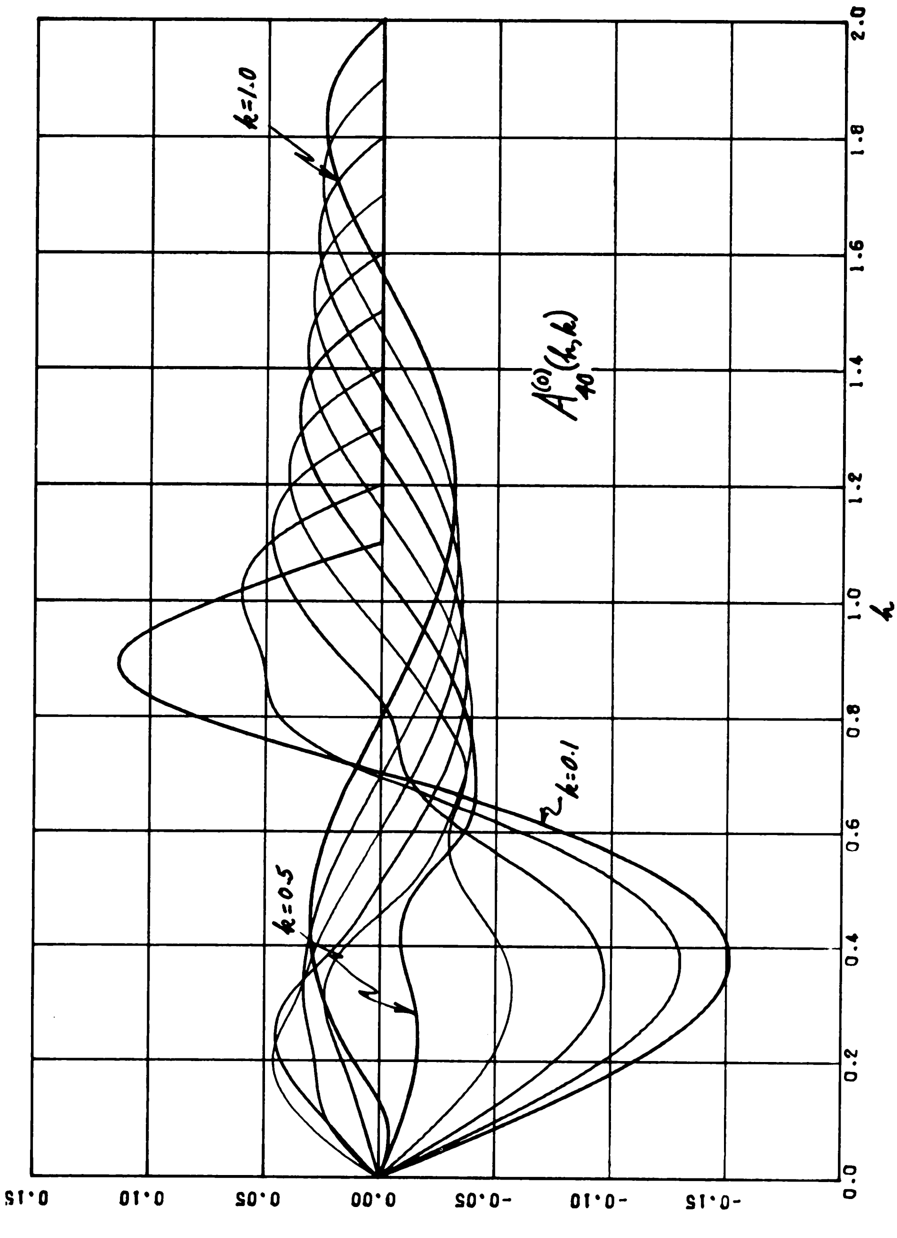

FIG. 11. The function $A_{40}{}^{(0)}(h, k)$.

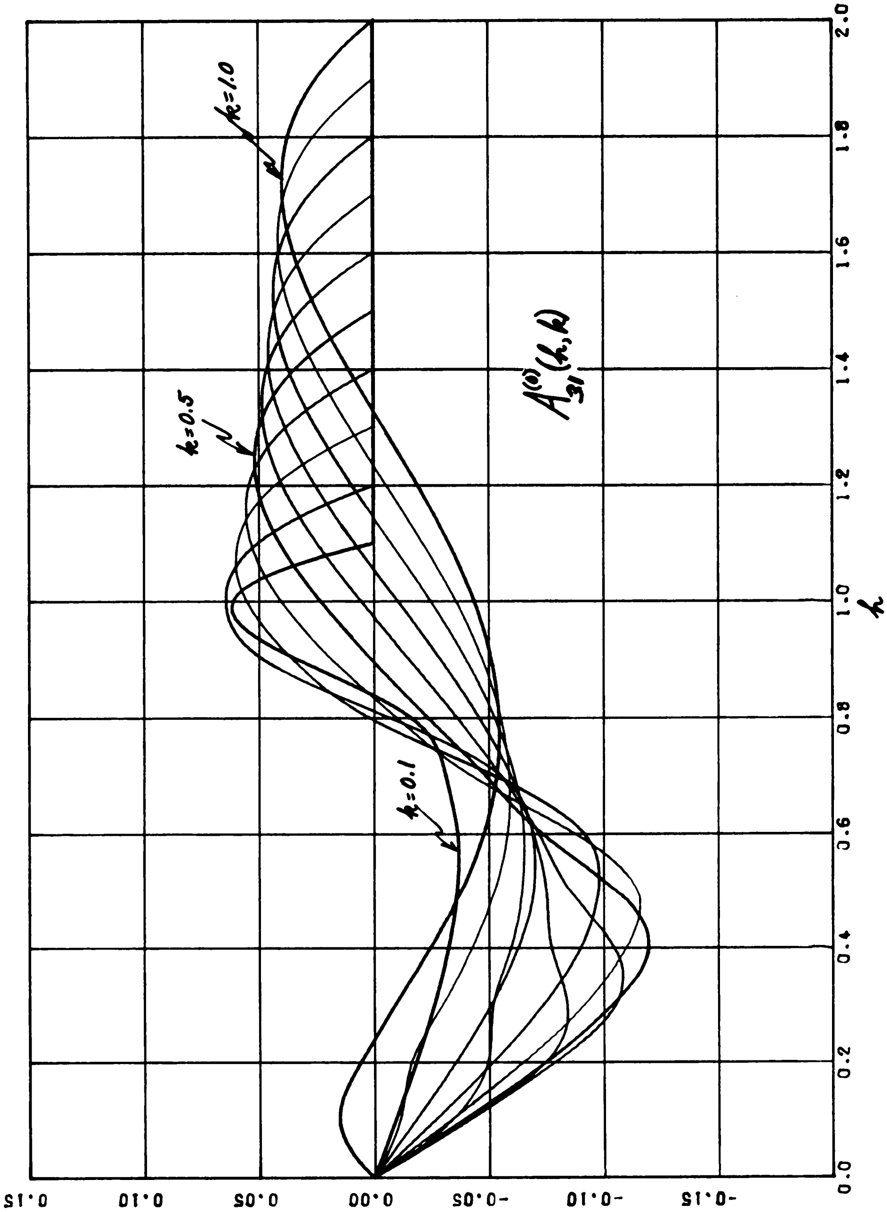

FIG. 12. The function $A_{31}{}^{(0)}(h, k)$.

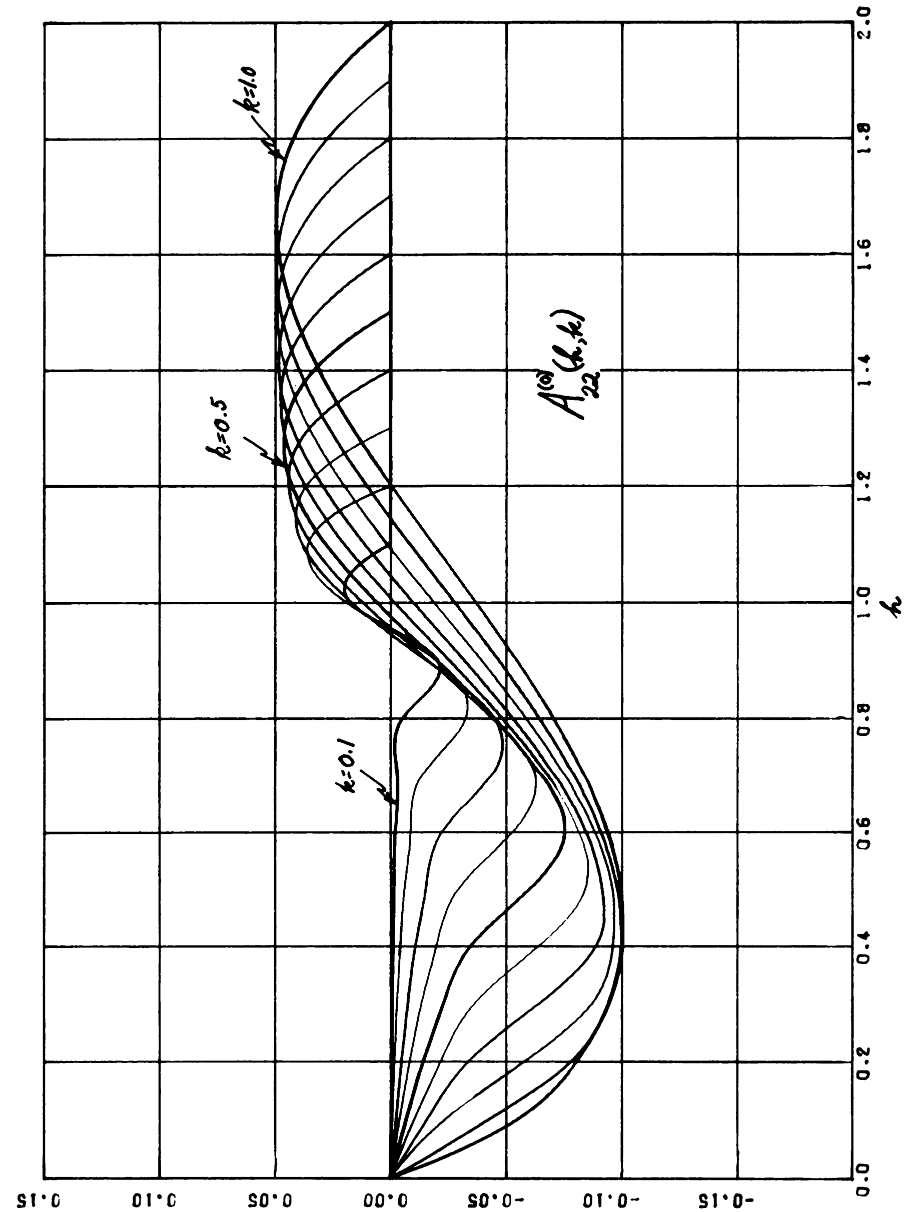

FIG. 13. The function $A_{22}{}^{(0)}(h, k)$.

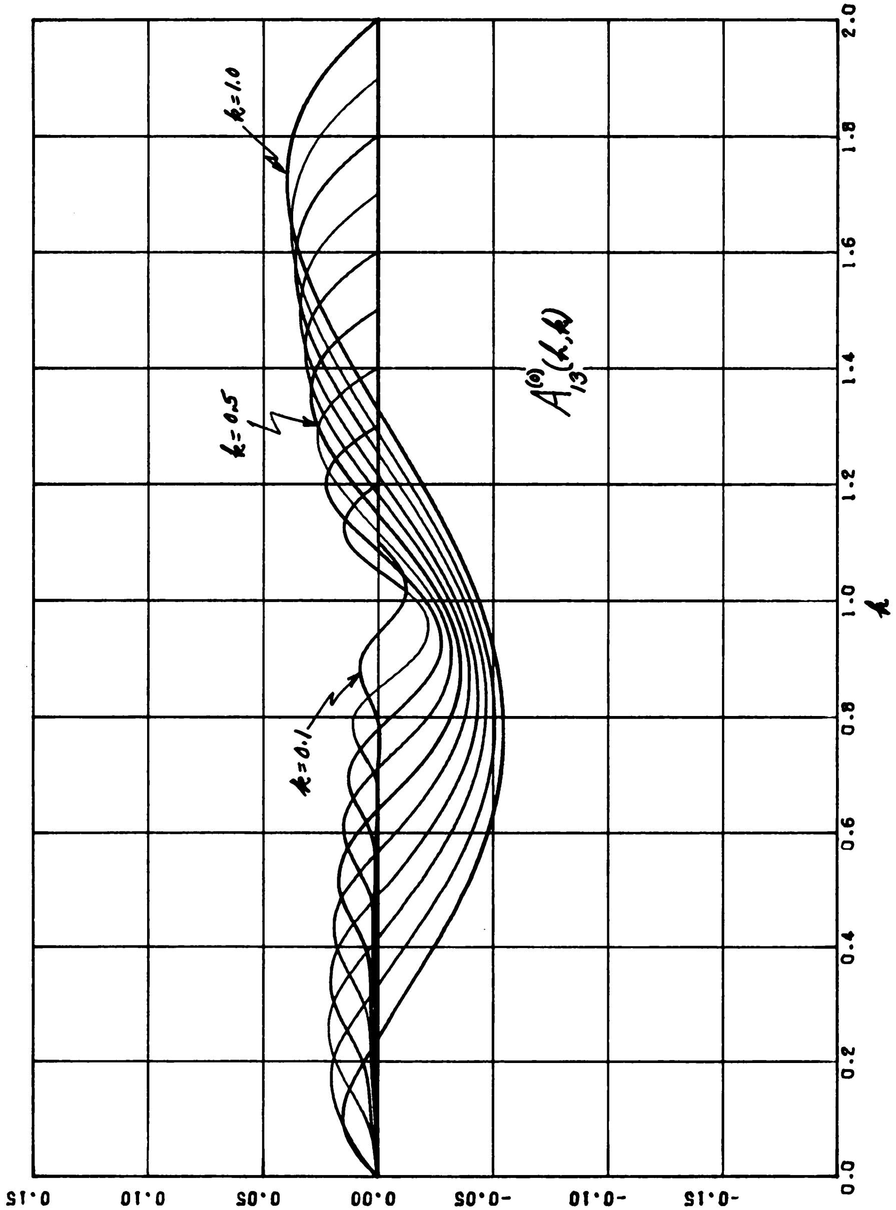

FIG. 14. The function $A_{13}{}^{(0)}(h, k)$.

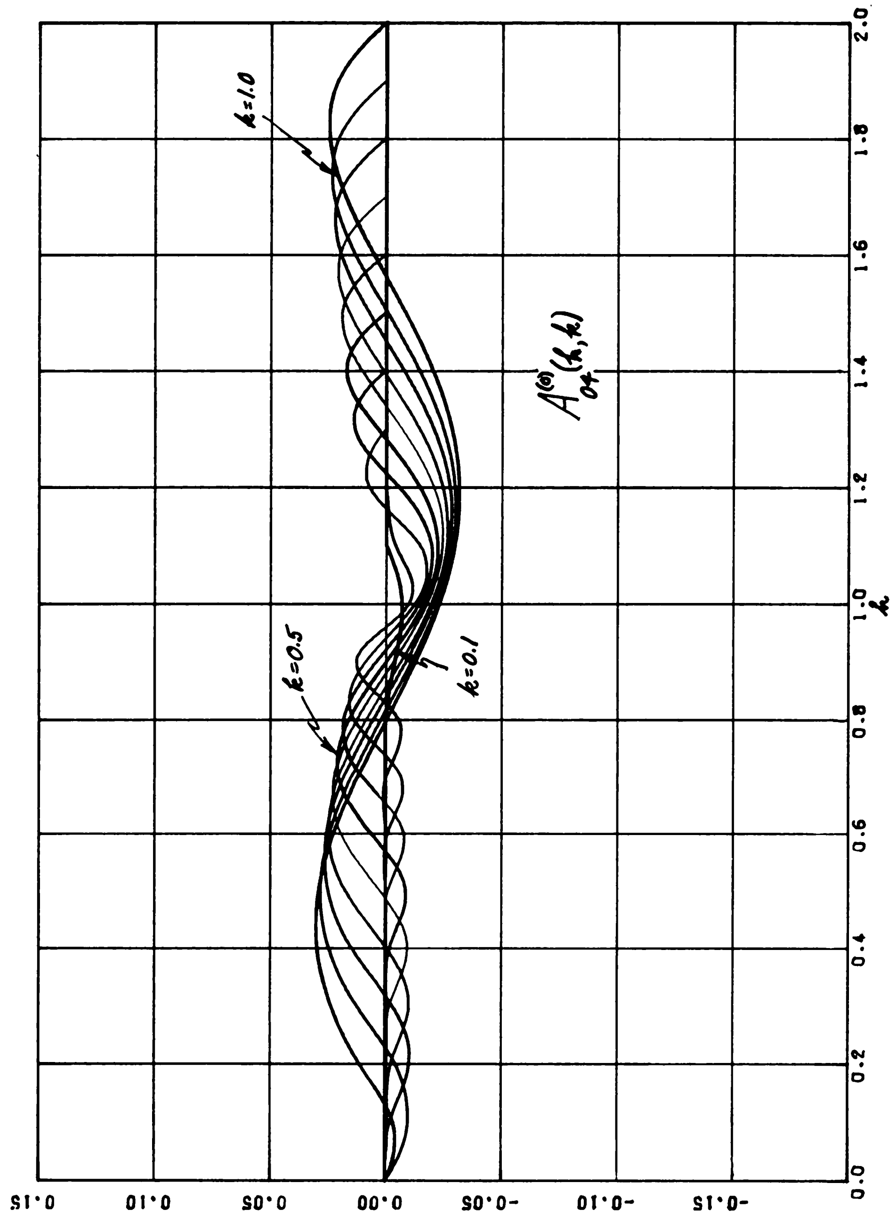

FIG. 15. The function $A_{04}{}^{(0)}(h, k)$.

For $h = 0$ the integrals (2.1) can be further reduced in terms of tabulated complete elliptic integrals. We obtain, thus, the formulas

$$
\begin{aligned}
A_{00}^{(0)}(0, k) &= 1, \\
A_{10}^{(0)}(0, k) &= \frac{4}{\pi^2} E(k), \\
A_{01}^{(0)}(0, k) &= \frac{4}{\pi^2 k} [E(k) - (1 - k^2)K(k)], \\
A_{20}^{(0)}(0, k) &= A_{11}^{(0)}(0, k) = A_{02}^{(0)}(0, k) = 0, \\
A_{30}^{(0)}(0, k) &= \frac{4}{9\pi^2} [(8k^2 - 7)E(k) + 4(1 - k^2)K(k)], \\
A_{21}^{(0)}(0, k) &= \frac{4}{3\pi^2 k} [(1 - 2k^2)E(k) - (1 - k^2)K(k)], \\
A_{12}^{(0)}(0, k) &= \frac{4}{3\pi^2 k^2} [(k^2 - 2)E(k) + 2(1 - k^2)K(k)], \\
A_{03}^{(0)}(0, k) &= \frac{4}{9\pi^2 k^3} [(8 - 7k^2)E(k) - (8 - 3k^2)(1 - k^2)K(k)], \\
A_{40}^{(0)}(0, k) &= A_{31}^{(0)}(0, k) = A_{22}^{(0)}(0, k) = A_{13}^{(0)}(0, k) = A_{04}^{(0)}(0, k) = 0.
\end{aligned}
\tag{2.2}
$$

For $h = 0$ these formulas then yield at once the power-series expansions

$$
\begin{aligned}
A_{10}^{(0)}(0, k) &= \frac{1}{\pi} [2 - \tfrac{1}{2}k^2 - \tfrac{3}{32}k^4 - \cdots], \\
A_{01}^{(0)}(0, k) &= \frac{k}{\pi} [1 + \tfrac{1}{8}k^2 + \tfrac{3}{64}k^4 + \cdots], \\
A_{30}^{(0)}(0, k) &= \frac{1}{\pi} [-\tfrac{2}{3} + \tfrac{3}{2}k^2 - \tfrac{15}{32}k^4 - \cdots], \\
A_{21}^{(0)}(0, k) &= \frac{k}{\pi} [-1 + \tfrac{3}{8}k^2 + \tfrac{5}{64}k^4 + \cdots], \\
A_{12}^{(0)}(0, k) &= \frac{k^2}{\pi} [-\tfrac{1}{4} - \tfrac{1}{16}k^2 - \tfrac{15}{512}k^4 - \cdots], \\
A_{03}^{(0)}(0, k) &= \frac{k^3}{\pi} [\tfrac{1}{24} + \tfrac{3}{128}k^2 + \tfrac{15}{1024}k^4 + \cdots],
\end{aligned}
\tag{2.3}
$$

each of which converges for all k, $0 < k \leq 1$. The series expansions (2.3) are, of course, special cases of double series for the functions $A_{mn}^{(0)}(h, k)$ analogous to the double series expansions given for the functions $A_{mn}^{(1)}(h, k)$ in Part I and convergent for $|h| + k \leq 1$.

For $h = 1$ the integrals (2.1) can also be expressed with great efficiency in terms of polynomial approximations and power-series expansions of the forms

$$
\begin{aligned}
A_{00}{}^{(0)}(1, k) &\cong \frac{k^{1/2}}{\pi^{3/2}}\left[\alpha_0 \frac{\Gamma(3/4)}{\Gamma(5/4)} + \alpha_1 \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \cdots + \alpha_7 \frac{\Gamma(17/4)}{\Gamma(19/4)} k^7\right], \\
A_{01}{}^{(0)}(1, k) &\cong \frac{k^{1/2}}{\pi^{3/2}}\left[\alpha_0 \frac{\Gamma(5/4)}{\Gamma(7/4)} + \alpha_1 \frac{\Gamma(7/4)}{\Gamma(9/4)} k + \cdots + \alpha_7 \frac{\Gamma(19/4)}{\Gamma(21/4)} k^7\right], \\
A_{02}{}^{(0)}(1, k) &\cong \frac{k^{1/2}}{\pi^{3/2}}\left[\frac{\alpha_0}{5} \frac{\Gamma(3/4)}{\Gamma(5/4)} + \frac{3\alpha_1}{7} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \cdots + \frac{15\alpha_7}{19} \frac{\Gamma(17/4)}{\Gamma(19/4)} k^7\right], \\
A_{03}{}^{(0)}(1, k) &\cong \frac{k^{1/2}}{\pi^{3/2}}\left[-\frac{\alpha_0}{7} \frac{\Gamma(5/4)}{\Gamma(7/4)} + \frac{\alpha_1}{9} \frac{\Gamma(7/4)}{\Gamma(9/4)} k + \cdots + \frac{13\alpha_7}{21} \frac{\Gamma(19/4)}{\Gamma(21/4)} k^7\right], \\
A_{04}{}^{(0)}(1, k) &\cong \frac{k^{1/2}}{\pi^{3/2}}\left[-\frac{\alpha_0}{15} \frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{3\alpha_1}{77} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \cdots + \frac{165\alpha_7}{437} \frac{\Gamma(17/4)}{\Gamma(19/4)} k^7\right],
\end{aligned}
\tag{2.4}
$$

and the form

$$
\begin{aligned}
A_{10}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{1}{4} \frac{\Gamma(5/4)}{\Gamma(7/4)} k - \frac{1}{32} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 - \cdots\right], \\
A_{11}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(5/4)}{\Gamma(7/4)} - \frac{1}{4} \frac{\Gamma(7/4)}{\Gamma(9/4)} k - \frac{1}{32} \frac{\Gamma(9/4)}{\Gamma(11/4)} k^2 - \cdots\right], \\
A_{12}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{1}{5} \frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{3}{28} \frac{\Gamma(5/4)}{\Gamma(7/4)} k - \frac{5}{288} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 - \cdots\right], \\
A_{13}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[-\frac{1}{7} \frac{\Gamma(5/4)}{\Gamma(7/4)} - \frac{1}{36} \frac{\Gamma(7/4)}{\Gamma(9/4)} k - \frac{3}{352} \frac{\Gamma(9/4)}{\Gamma(11/4)} k^2 - \cdots\right].
\end{aligned}
\tag{2.5}
$$

Similarly,

$$
\begin{aligned}
A_{20}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{5}{4} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \frac{7}{32} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 + \cdots\right], \\
A_{21}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(5/4)}{\Gamma(7/4)} - \frac{5}{4} \frac{\Gamma(7/4)}{\Gamma(9/4)} k + \frac{7}{32} \frac{\Gamma(9/4)}{\Gamma(11/4)} k^2 + \cdots\right], \\
A_{22}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{1}{5} \frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{15}{28} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \frac{35}{288} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 + \cdots\right],
\end{aligned}
\tag{2.6}
$$

and, finally,

$$
\begin{aligned}
A_{30}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{35}{12} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \frac{63}{32} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 - \cdots\right], \\
A_{31}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(5/4)}{\Gamma(7/4)} - \frac{35}{12} \frac{\Gamma(7/4)}{\Gamma(9/4)} k + \frac{63}{32} \frac{\Gamma(9/4)}{\Gamma(11/4)} k^2 - \cdots\right], \\
A_{40}{}^{(0)}(1, k) &= \frac{2^{1/2}k^{1/2}}{\pi^{3/2}}\left[\frac{\Gamma(3/4)}{\Gamma(5/4)} - \frac{21}{4} \frac{\Gamma(5/4)}{\Gamma(7/4)} k + \frac{231}{32} \frac{\Gamma(7/4)}{\Gamma(9/4)} k^2 - \cdots\right].
\end{aligned}
\tag{2.7}
$$

In (2.4) the coefficients α_0 to α_7 are the quantities

$$\begin{aligned}
\alpha_0 &\quad a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \,, \\
\alpha_1 &= -a_1 - 2a_2 - 3a_3 - 4a_4 - 5a_5 - 6a_6 - 7a_7 \,, \\
\alpha_2 &= a_2 + 3a_3 + 6a_4 + 10a_5 + 15a_6 + 21a_7 \,, \\
\alpha_3 &= -a_3 - 4a_4 - 10a_5 - 20a_6 - 35a_7 \,, \\
\alpha_4 &= a_4 + 5a_5 + 15a_6 + 35a_7 \,, \qquad \alpha_5 = -a_5 - 6a_6 - 21a_7 \,, \\
\alpha_6 &= a_6 + 7a_7 \,, \qquad \alpha_7 = -a_7 \,,
\end{aligned} \tag{2.8}$$

where here a_0 to a_7 are Hastings' coefficients [12] in the polynomial approximation

$$\cos^{-1} x \cong [1 - x]^{1/2}[a_0 + a_1 x + \cdots + a_7 x^7] \tag{2.9}$$

which is accurate to about 2×10^{-8} units uniformly on $0 \leq x \leq 1$; see also Abramowitz and Stegun [13]. The expansions (2.5), (2.6), and (2.7) are based on a comparable expansion of the form

$$\sin(\cos^{-1} x) = [1 - x]^{1/2}[1 + x]^{1/2} = [1 - x]^{1/2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n. \tag{2.10}$$

In each case, x in (2.9) and (2.10) is put equal to $1 - k \cos v$, so that $[1 - x]^{1/2} = k^{1/2} \cos^{1/2} v$, the results are substituted in the integral expressions (2.1), and the resulting finite or infinite series expansions are then integrated with the help of the formula

$$\int_0^{\pi/2} \cos^{\gamma} v \, dv = \frac{\pi^{1/2}}{2} \frac{\Gamma[(\gamma + 1)/2]}{\Gamma[(\gamma/2) + 1]}. \tag{2.11}$$

A factor $k^{1/2}$ can then be factored out and the results, (2.4), (2.5), (2.6), and (2.7), follow.

The polynomial approximations (2.4) are clearly extremely accurate and in use check other results to better than 1×10^{-8} units generally. The series expansions (2.5), (2.6), and (2.7) converge for all k, $0 < k \leq 1$, and yield accuracy of about 1×10^{-6} units even for $k = 1$ when about 12 terms in each series are carried. These expansions are therefore highly efficient.

Finally, it should be remarked that the functions $A_{mn}^{(0)}(-h, k)$ can be computed in terms of the functions $A_{mn}^{(0)}(h, k)$ using the reflection relations

$$\begin{aligned}
A_{00}^{(0)}(-h, k) &= 2 - A_{00}^{(0)}(h, k), \\
A_{mn}^{(0)}(-h, k) &= (-1)^{m+n+1} A_{mn}^{(0)}(h, k),
\end{aligned} \tag{2.12}$$

similarly as in the case of the first-kind Bennett functions $A_{mn}^{(1)}(h, k)$ and that the functions reduce to rational expressions in π for $h = 0$ and $k = 1$.

3. Computation of the zeroth-kind functions. The computation of the zeroth-kind Bennett functions $A_{mn}^{(0)}(h, k)$ tabulated in this work and graphed in Figs. 1 through 15 was based mainly on numerical integration of the functions $A_{mn}^{(0)}(h, k)$ when expressed in the integral forms (2.1) using a modified Simpson's-rule method with stepped-down intervals of integration in critical regions of the domain of integration. In addition, the eleven higher-order functions $A_{20}^{(0)}(h, k)$, $A_{02}^{(0)}(h, k)$, $\cdots$, $A_{04}^{(0)}(h, k)$ were computed separately from values of the four basic functions $A_{00}{}^{(0)}(h, k)$, $A_{10}{}^{(0)}(h, k)$, $A_{01}{}^{(0)}(h, k)$, and $A_{11}{}^{(0)}(h, k)$ using the recurrence relations

$$\begin{aligned} A_{20}{}^{(0)}(h, k) &\equiv hA_{10}{}^{(0)}(h, k) - kA_{11}{}^{(0)}(h, k), \\ A_{02}{}^{(0)}(h, k) &\equiv h'A_{01}{}^{(0)}(h, k) - k'A_{11}{}^{(0)}(h, k), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} A_{30}{}^{(0)}(h, k) &\equiv \tfrac{1}{3}[(4h^2 + \tfrac{8}{3}k^2 - 1)A_{10}{}^{(0)}(h, k) - \tfrac{4}{3}kA_{01}{}^{(0)}(h, k) - \tfrac{20}{3}hkA_{11}{}^{(0)}(h, k)], \\ A_{21}{}^{(0)}(h, k) &\equiv \tfrac{1}{3}[2hA_{11}{}^{(0)}(h, k) - 2kA_{10}{}^{(0)}(h, k) + A_{01}{}^{(0)}(h, k)], \\ A_{12}{}^{(0)}(h, k) &\equiv \tfrac{1}{3}[2h'A_{11}{}^{(0)}(h, k) - 2k'A_{01}{}^{(0)}(h, k) + A_{10}{}^{(0)}(h, k)], \\ A_{03}{}^{(0)}(h, k) &\equiv \tfrac{1}{3}[(4h'^2 + \tfrac{8}{3}k'^2 - 1)A_{01}{}^{(0)}(h, k) - \tfrac{4}{3}k'A_{10}{}^{(0)}(h, k) - \tfrac{20}{3}h'k'A_{11}{}^{(0)}(h, k)], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} A_{40}{}^{(0)}(h, k) &\equiv \tfrac{1}{4}[(2k - \tfrac{52}{3}h^2k - 6k^3)A_{11}{}^{(0)}(h, k) \\ &\qquad + (8h^3 + \tfrac{46}{3}hk^2 - 4h)A_{10}{}^{(0)}(h, k) - \tfrac{14}{3}hkA_{01}{}^{(0)}(h, k)], \\ A_{31}{}^{(0)}(h, k) &\equiv \tfrac{1}{4}[(\tfrac{8}{3}h^2 + 4k^2)A_{11}{}^{(0)}(h, k) - \tfrac{20}{3}hkA_{10}{}^{(0)}(h, k) + \tfrac{4}{3}hA_{01}{}^{(0)}(h, k)], \\ A_{22}{}^{(0)}(h, k) &\equiv \tfrac{1}{4}[(\tfrac{4}{3}hh' - 2k - 2k')A_{11}{}^{(0)}(h, k) + \tfrac{2}{3}hA_{10}{}^{(0)}(h, k) + \tfrac{2}{3}h'A_{01}{}^{(0)}(h, k)], \\ A_{13}{}^{(0)}(h, k) &\equiv \tfrac{1}{4}[(\tfrac{8}{3}h'^2 + 4k'^2)A_{11}{}^{(0)}(h, k) - \tfrac{20}{3}h'k'A_{01}{}^{(0)}(h, k) + \tfrac{4}{3}h'A_{10}{}^{(0)}(h, k)], \\ A_{04}{}^{(0)}(h, k) &\equiv \tfrac{1}{4}[(2k' - \tfrac{52}{3}h'^2k' - 6k'^3)A_{11}{}^{(0)}(h, k) \\ &\qquad + (8h'^3 + \tfrac{46}{3}h'k'^2 - 4h')A_{01}{}^{(0)}(h, k) - \tfrac{14}{3}h'k'A_{10}{}^{(0)}(h, k)], \end{aligned} \tag{3.3}$$

where in these formulas $h' = h/k$ and $k' = 1/k$; for derivations see formula (4.4) in Part I. Each of these computational procedures was carried out to several more decimal places than the six decimals finally tabulated; agreement between the values computed using Simpson's rule and those computed using the recurrence relations (3.1), (3.2), and (3.3), to better than 1×10^{-6} units was obtained in about 95 percent of the table. Round-off errors inherent in the use of the last of formulas (3.2) and (3.3), i.e., those for $A_{03}{}^{(0)}(h, k)$, $A_{13}{}^{(0)}(h, k)$, and $A_{04}{}^{(0)}(h, k)$, for small k values, resulted in a check of the numerical integration to somewhat less than six decimal places in the remaining part of the table.

To provide a further independent check on the computation, therefore, almost all of the functional values for $h = 0$ and for $h = 1$ were recomputed to at least six-decimal accuracy using the formulas (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7), and finally, a few values of the particularly difficult function $A_{04}{}^{(0)}(h, k)$ near $h = 0.9$ and $k = 0.1$ were computed in an entirely separate fashion using the special double series

$$A_{04}{}^{(0)}(h, k) = \frac{2^{3/2}\xi^{1/2}\eta^4}{\pi}\left\{\begin{array}{c} \dfrac{-5}{2048} + \dfrac{1}{8192}\xi - \dfrac{3}{65,536}\xi^2 + \cdots \\ \dfrac{-63}{32,768}\eta^2 + \dfrac{7}{32,768}\eta^2\xi + \cdots \\ \dfrac{-3003}{2,097,152}\eta^4 + \cdots \\ \cdots \end{array}\right\} \tag{3.4}$$

where here $\xi = 1 - h$ and $\eta = k/(1 - h)$ and the expansion is valid near $h = 1$ with $h \neq 1$ and k small in the region $|h| + k \leq 1$.

This then completed the basic computation and check calculations.

Formula (3.4) is based on the expansion of $\cos^{-1}(1 - x)$ in powers of x with $x = 1 - h + k \cos v$. We omit the details.

4. Power computations and concluding remarks. As noted in Part I, when the input (1.3) to the rectifier is non-periodic, i.e., when p/q is irrational, and assuming the average output power $\mathcal{P}_0$ to be one-half of the sum of the squares of the Fourier coefficients, we have, by the Parseval theorem [14, 15] or Bessel equality for double Fourier series, the result that

$$\mathcal{P}_0 = \tfrac{1}{4}A_{00}{}^{(0)2}(h, k) + \tfrac{1}{2} \sum_{m,n=0}^{\infty}{}^{*} A_{\pm mn}{}^{(0)2}(h, k) = \tfrac{1}{2}A_{00}{}^{(0)}(h, k) \tag{4.1}$$

where we have used the fact that for $\nu = 0$ we have $A_{00}{}^{(2\nu)}(h, k) = A_{00}{}^{(\nu)}(h, k)$ to simplify the right-hand side; see also (6.2) in Part I. Finally, note that although (4.1) holds here, the limiting relations discussed in the concluding section of Part I do not necessarily follow, as here $\nu = 0$.

References

[1] R. L. Sternberg, J. S. Shipman and S. R. Zohn, *Quart. Appl. Math.* **16**, 335 (1958)
[2] W. R. Bennett, *Bell System Tech. J.* **12**, 228 (1933)
[3] W. R. Bennett, *Bell System Tech. J.* **26**, 139 (1947)
[4] R. L. Sternberg and H. Kaufman, *J. Math. Phys.* **32**, 233 (1953)
[5] R. L. Sternberg, *J. Math. Phys.* **33**, 68 (1954)
[6] R. L. Sternberg, *J, Math. Phys.* **33**, 199 (1954)
[7] J. S. Shipman, *Quart. Appl. Math.* **13**, 200 (1955)
[8] E. W. Montroll, *Am. Math. Monthly*, Pt. 2, **61**, 46 (1954)
[9] E. Feuerstein, *Quart. Appl. Math.* **15**, 183 (1957)
[10] J. C. Hsu, *Bell System Tech. J.* **41**, 1813 (1962)
[11] Paul Penfield, Jr., *J. Franklin Inst.* **273**, 107 (1962)
[12] Cecil Hastings, Jr., *Approximations for digital computers*, Princeton University Press, Princeton N. J., 1955
[13] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, Washington, D. C., 1964
[14] E. W. Hobson, *The theory of functions of a real variable and theory of Fourier's series*, 2nd ed., **2**, Warren Press, Washington, D. C., 1950
[15] L. Tonelli, *Serie trigonometriche*, Nicola Zanichelli, Bologna, 1928