THE DRAG AND SPHERICITY INDEX OF A SPINDLE*

By D. M. STASIW, F. B. COOK, M. C. DETRAGLIA AND L. C. CERNY

(*Masonic Medical Research Laboratory and Utica College of Syracuse University, Utica*)

Introduction. Several years ago, Payne and Pell [1, 2, 3] published a series of articles pertaining to the Stokes flow of a viscous, incompressible fluid about a body in which the flow is two-dimensional or has radial symmetry. Some of the shapes of bodies that were considered were a lens, a torus, a sphere, and oblate and prolate spheroids. Using this procedure, the differential equation to be satisfied in the flow region is found to be

$$L_{-1}{}^{2}\psi_1 = 0, \tag{1}$$

where

$$L_{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{2}$$

If the trace of the boundary of the body is called C, then the condition of vanishing velocity on C can be stated in the form

$$\psi_1 = 0, \qquad \partial\psi_1/\partial n = 0, \text{ on } C. \tag{3}$$

Here n is the unit normal to C exterior to the body.

Recently, our interest has been in the general area of the flow of diseased states of blood. The above papers have been extremely useful in a problem of current medical concern, sickle-cell anemia. The red cells in this diseased state approximate lens, sickles, hemispheres and spindles. It is the purpose of this communication to show the calculation of the flow about a spindle.

Flow about a spindle. A convenient set of coordinates to represent a spindle are the bipolar coordinates (φ, η) [4]. In terms of these coordinates

$$x = \frac{b \sinh \eta}{\cosh \eta - \cos \varphi}, \tag{4}$$

$$r = \frac{b \sin \varphi}{\cosh \eta - \cos \varphi}, \tag{5}$$

$$x^2 + r^2 - 2br \cot \varphi = b^2, \tag{6}$$

with $\varphi > \pi/2$; $r \geq 0$. The calculation of the drag, P, can be made by using

$$P = 8\pi\mu \lim_{\rho\to\infty} (\rho\psi_1/r^2). \tag{7}$$

* Received August 24, 1973. This work was partially supported by a grant from the Public Health Service No. HL 13228.

In this equation $\rho = (x^2 + r^2)^{1/2}$, μ is the coefficient of viscosity of the suspending fluid and ψ_1 is a solution to

$$L_{-1}{}^2\psi_1 = 0. \tag{1}$$

Payne and Pell [3] suggest that the drag can best be evaluated from

$$P = 8\pi b\mu U \int_0^\infty \frac{F(\alpha)}{\cosh \alpha\pi}\, d\alpha, \tag{8}$$

where

$$F(\alpha) = \int_{t_0}^1 K_\alpha(-\tau)K_\alpha{}^{(2)}(\tau)\, d\tau \Big/ \int_{t_0}^1 K_\alpha(\tau)K_\alpha{}^{(2)}(\tau)\, d\tau. \tag{9}$$

In Eq. (9) $K_\alpha(\tau)$ is known as the conal function (5) and defined as

$$K_\alpha(\tau) = P_{i\alpha-1/2}(\tau) \tag{10}$$

and

$$K_\alpha{}^{(n)}(\tau) \equiv d^nK_\alpha(\tau)/d\tau^n, \tag{11}$$

where $P(\tau)$ is the Legendre function, and $\tau = \cos \varphi$.

To date, the drag for a spindle has not been determined, although Payne and Pell suggest that certain tables should facilitate the computation [6, 7]. However, in examining these tables, it was found that they were not adequate with regard to the choice of the angle φ or with ease of determining the derivatives of the conal functions. To overcome these difficulties the approach that follows was finally used.

The functions $B(\alpha)$ and $A(\alpha)$ are evaluated from the boundary conditions of the problem and can be written as

$$A(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi}\, [K_\alpha(-t_0)K_\alpha{}^{(1)}(t_0) - K_\alpha{}^{(1)}(-t_0)K_\alpha(t_0)], \tag{13}$$

$$B(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi}\left[t_0K_\alpha{}^{(1)}(t_0)K_\alpha{}^{(1)}(-t_0) - K_\alpha(t_0)\frac{d}{dt_0}\,(t_0K_\alpha{}^{(1)}(t_0)\right] \tag{14}$$

where $t_0 = \tau_0 = \cos \varphi_0$ and

$$\Omega = t_0[K_\alpha{}^{(1)}(t_0)]^2 - K_\alpha(t_0)\frac{d}{dt_0}\,(t_0K_\alpha{}^{(1)}(t_0)) \tag{15}$$

The value of t_0 defines the shape of the spindle.

The calculation was facilitated by using the following series representation for $K_\alpha(\tau)$ [8]:

$$K_\alpha(\tau) = 1 + \frac{1 + 4\alpha^2}{4(1!)^2}\left(\frac{1-\tau}{2}\right) + \frac{(1 + 4\alpha^2)(3^2 + 4\alpha^2)}{4^2(2!)^2}\left(\frac{1-\tau}{2}\right)^2 + \frac{(1 + 4\alpha^2)(3^2 + 4\alpha^2)(5^2 + 4\alpha^2)}{4^3(3!)^2}\left(\frac{1-\tau}{2}\right)^3 + \cdots \tag{16}$$

for $|\tau - 1| < 2$, where appropriate recursive relationships allowed rapid calculation of the series terms for $K_\alpha(\tau)$ and its derivatives.

In biological cellular flow systems, especially blood, an arbitrary parameter that is used as a reference to changing shapes is the sphericity index (S.I.) [9]. It is defined as

$$\text{S.I.} = 4.84(V^{2/3}/S), \tag{17}$$

where V is the volume and S is the surface area of the particle respectively. If b is taken as unity in Eq. (6), the volume and surface area of a spindle are found to be

$$V = 2\pi\left[a^2 + 2/3 + a(a^2 + 1)\sin^{-1}\left(\frac{1}{a^2 + 1}\right)^{1/2}\right] \tag{18}$$

and

$$S = 4\pi[(\varphi a + 1)(a^2 + 1)^{1/2}] \tag{19}$$

where $a = \cot\varphi$.

These values were used in Eq. (17) to calculate the sphericity index as a function of the changing shape of the spindle.

In Table I, the drag coefficient and sphericity index are listed for several different spindles.

A recent paper by Gluckman, Weinbaum and Pfeffer [10] also presents a solution to the problem of axisymmetric slow viscous flow past a convex body of revolution. Although these authors' presentation is thorough and interesting, the unique use of peripolar or bipolar coordinates as suggested by Pell and Payne [13] seems to encompass all of the above authors' bodies as well as several not included in their publication

TABLE I

Drag coefficients and sphericity indices for spindles

ζ(deg)	drag coefficient $P/b\mu U$	S. I.	D/S. I.
90	18.85	1.000	18.85
95	17.86	0.999	17.88
100	16.95	0.997	17.00
105	16.13	0.992	16.26
110	15.38	0.985	15.61
115	14.71	0.975	15.09
120	14.12	0.962	14.68
125	13.59	0.947	14.35
130	13.13	0.928	14.15
135	12.72	0.905	14.06
140	12.37	0.879	14.07
145	12.07	0.848	14.23
150	11.81	0.811	14.56
155	11.60	0.768	15.10
160	11.43	0.716	15.96
165	11.30	0.654	17.29
170	11.20	0.572	19.58
175	11.15	0.455	24.51

The authors would like to acknowledge the help of Dr. L. E. Payne who introduced us to his work and encouraged us throughout the calculations.

References

[1] L. E. Payne and W. E. Pell, *The Stokes flow problem for a class of axially symmetric bodies*, J. Fluid Mech. **7**, 529–549 (1960)

[2] L. E. Payne, *On axially symmetric flow and the method of generalized electrostatics*, Quart. Appl. Math. **10**, 197–204 (1952)

[3] W. E. Pell and L. E. Payne, *The Stokes flow about a spindle*, Quart. Appl. Math. **18**, 257–262 (1960)

[4] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, University Press, Cambridge, 1965, p. 451

[5] C. Neumann, *Uber die Mehler'schen Kugelfunktionen und deren Anwendung auf electrostatische Probleme*, Math. Ann. **18**, 195–236 (1881)

[6] M. I. Zhurina and L. N. Karmazina, *Tables and formulae for the spherical functions* $P_{-1/2+i\tau}{}^m(Z)$, Pergamon Press, New York, (1966)

[7] M. I. Zhurina and L. N. Karmazina, *Tables of the Legendre functions* $P_{-1/2+i\tau}(X)$, Part I, Macmillan, New York, 1964

[8] A. Erdelyi, *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953, p. 174

[9] P. B. Canham and A. C. Burton, *Distribution of size and shape in populations of normal human red cells*, Cir. Res. **22**, 405–422 (1968)

[10] M. Gluckman, S. Weinbaum and R. Pfeffer, *Axisymmetric slow viscous flow past an arbitrary convex body of revolution*, J. Fluid. Mech. **55**, 677–709 (1972)