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LINEAR TIME-INVARIANT TRANSFORMATIONS OF SOME NONSTATIONARY RANDOM PROCESSES*

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Abstract. We consider the class of nonstationary processes $Y(t)$ which can be represented as $Y(t) = BX(t)$, where $X(t)$ is wide sense stationary and B is a bounded self-adjoint operator with a bounded inverse. An equivalent characterization of this class of processes is given and is used to construct examples of nonstationary processes belonging to this class. A functional analytic treatment is given for describing the effects of linear time-invariant transformations.

1. Introduction. Let $X(t)$, $t \in I$, be a second-order random process. Here the index set I will be assumed to be $(-\infty, \infty)$ or Z, where Z is the set of all integers. We will denote by $L(X)$ the linear manifold generated by all finite linear combinations $\sum_{i=1}^{m} a_i X(t_i)$, $t_i \in I$, a_i complex, and by $H(X)$ the Hilbert space obtained by completing $L(X)$ in the norm defined by $||X||^2 = E\,|X|^2$, $X \in L(X)$.

The process $X(t)$ is said to be *wide sense stationary* (w.s.s.) if for arbitrary t, s, $h \in I$

$$E\{X(t+h)\overline{X(s+h)}\} = E\{X(t)\overline{X(s)}\}.$$

In this paper we will study the class of nonstationary processes $Y(t)$ which can be represented as

$$Y(t) = BX(t) \tag{1}$$

where $X(t)$ is w.s.s. and where B is a bounded self-adjoint operator on $H(Y)$ onto $H(Y)$ such that B^{-1} exists and is bounded. A process $Y(t)$ belonging to this class of processes will be said to be a stochastic process with uniformly bounded shift group (SPUBSG). The representation (1) is clearly not unique for a given process $Y(t)$. We note that a SPUBSG process is a special case of a *deformed stationary process* as defined by Mandrekar [6], in which the deforming operator-valued function of [6] is constant-valued, this constant being the operator B of Eq. (1). Let the shifts U_h, $h \in I$, of $X(t)$ be defined by $U_h X(t) = X(t+h)$. It is interesting to note that $Y(t)$ of Eq. (1) is w.s.s. iff B commutes with the shifts of $X(t)$. This is proved using the technique of [6, p. 282].

Two problems concerning the class of SPUBSG processes will be considered. First, it is important to find explicit examples of SPUBSG processes and to show that this

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class is large enough to contain nonstationary processes occurring in practice. A contribution towards this end is contained in Secs. 2 and 3. The second problem consists of obtaining a representation of linear time-invariant transformations of SPUBSG processes in terms of a frequency response function. We do that in Sec. 4.

2. An equivalent characterization of the SPUBSG property. The following equivalent characterization of SPUBSG processes is needed in Sec. 3. The appellation "SPUBSG" was motivated by Condition (c) of Theorem 1.

THEOREM 1: The following three conditions are equivalent.

(a) $Y(t)$ is SPUBSG.

(b) Let m be an arbitrary positive integer and $a_1, a_2, \cdots, a_m$ be arbitrary complex numbers. Then there exists a constant $M > 0$ such that for arbitrary $t_1, t_2, \cdots, t_m$ and $h \in I$

$$E\left|\sum_{i=1}^{m} a_i Y(t_i + h)\right|^2 \leq ME\left|\sum_{i=1}^{m} a_i Y(t_i)\right|^2. \tag{2}$$

(c) There exists a group T_h, $h \in I$, of linear bounded operators on $H(Y)$ onto $H(Y)$ such that $Y(t + h) = T_h Y(t)$, $t \in I$, and $\sup_{h \in I} ||T_h|| < \infty$.

Proof: (*a*) $\Rightarrow$ (*b*)*:* Let $Y(t)$ be as in Eq. (1). Then, using the boundedness of B and B^{-1} and the definition of norm in $H(Y)$, we have that

$$E\left|\sum_{i=1}^{m} a_i Y(t_i + h)\right|^2 \leq ||B||^2\, ||B^{-1}||^2\, E\left|\sum_{i=1}^{m} a_i Y(t_i)\right|^2$$

from which inequality (2) follows.

(*b*) $\Rightarrow$ (*c*)*:* For $h \in I$, let T_h be defined on the elements $Y(t)$, $t \in I$, by $T_h Y(t) = Y(t + h)$. Using condition (2) it follows (see Getoor [5, p. 176]) that the operators T_h, $h \in I$, can be extended to a group of linear bounded operators on $H(Y)$ onto $H(Y)$ with $\sup_{h \in I} ||T_h|| \leq M$.

(*c*) $\Rightarrow$ (*a*)*:* From the existence of the group T_h, $h \in I$, it follows [10] (see also [7]) that there exists a self-adjoint operator B with a bounded inverse B^{-1} such that $U_h = B^{-1}T_h B$, $h \in I$, is unitary in $H(Y)$. Let $X(t) = U_t B^{-1} Y(0)$. Then $X(t)$ is w.s.s. and $Y(t) = BX(t)$.

3. Examples and applications of SPUBSG processes. Let $Z(t)$, $t \in Z$, be a white-noise process with $E\{Z(t)\overline{Z(s)}\} = \delta_{ts}$ for t, $s \in Z$ and let

$$X_1(t) = \sum_{s=-\infty}^{\infty} h(t - s)Z(s)$$

where $h(s)$ is complex-valued and such that the sum converges in q.m. Then $X_1(t)$ is w.s.s. and we have the following lemma.

LEMMA 1: Let $g(t)$, $t \in Z$, be complex-valued and such that there exist two positive constants M_1 and M_2 with $M_1 \leq |g(t)|^2 \leq M_2$ for all $t \in Z$. Let $Y_1(t)$ be the process defined by

$$Y_1(t) = \sum_{s=-\infty}^{\infty} h(t - s)g(s)Z(s).$$

Then $Y_1(t)$ is SPUBSG.

Proof: Let $H(Z)$, $H(X_1)$ and $H(Y_1)$ be the Hilbert spaces spanned by $Z(t)$, $X_1(t)$ and $Y_1(t)$ respectively. Then $H(X_1) \subseteq H(Z)$ and $H(Y_1) \subseteq H(Z)$. Let B be the operator defined by $BZ(s) = g(s)Z(s)$. With the assumptions on g, B is linear bounded with a bounded inverse in $H(Z)$. Thus

$$Y_1(t) = \sum_{s=-\infty}^{\infty} h(t-s)BZ(s) = B \sum_{s=-\infty}^{\infty} h(t-s)Z(s) = BX_1(t).$$

It follows that $BH(X_1) = H(Y_1)$ and $B^{-1}H(Y_1) = H(X_1)$. Using the same argument as when proving (a) $\Rightarrow$ (b) of Theorem 1, it follows that $Y_1(t)$ is SPUBSG.

Let $X(t) = \int_{-\pi}^{\pi} \exp(iut)\, d\Phi(u)$ be a w.s.s. process. Assume that the measure F determined by the spectral process $\Phi(u)$ is absolutely continuous with respect to Lebesgue measure. Then $dF(u) = k(u)\, du$, where $k(u)$ is the spectral density for $X(t)$. The following theorem is a generalization of Lemma 1.

THEOREM 2: Let $X(t)$ be w.s.s. with spectral density $k(u)$ and let $Y(t)$ be defined by

$$Y(t) = \sum_{s=-\infty}^{\infty} h(t-s)g(s)X(s)$$

where g is as in Lemma 1 and $h(s)$ is complex-valued and such that the sum converges in q.m. If there exist two positive constants M_3 and M_4 such that $M_3 \leq k(u) \leq M_4$, $u \in [-\pi, \pi]$, then $Y(t)$ is SPUBSG.

Proof: With the assumptions on the spectral density we have that $X(t)$ is purely nondeterministic, and it can be represented [2, 3] in the form

$$X(s) = \int_{-\pi}^{\pi} \exp(ius) f(\exp(iu))\, d\Phi_Z(u) = \int_{-\pi}^{\pi} \exp(ius) f(\exp(iu))\, dE_u\, Z(0) \qquad (3)$$

where $|f(\exp(iu))|^2 = k(u)$; $E\,|d\Phi_Z(u)|^2 = du$; E_u, $u \in [-\pi, \pi]$, is the resolution of identity associated with the unitary shift operator T: $Z(s) \to Z(s+1)$, and $Z(s) = \int_{-\pi}^{\pi} \exp(ius)\, d\Phi_Z(u)$ is a white-noise process such that $H(Z) = H(X)$. From [9. pp, 283–284] it follows that Eq. (3) can be written as $X(s) = f(T)Z(s)$. Using the boundedness conditions on the spectral density $k(u)$ it is not difficult to show that $f(T)$ is linear bounded with a bounded inverse $f^{-1}(T)$ such that $\|f(T)\| \leq M_4$ and $\|f^{-1}(T)\| \leq 1/M_3$. Thus

$$Y(t) = \sum_{s=-\infty}^{\infty} h(t-s)g(s)f(T)Z(s) = f(T)Y_1(t)$$

where

$$Y_1(t) = \sum_{s=-\infty}^{\infty} h(t-s)g(s)Z(s)$$

is SPUBSG according to Lemma 1. Now $H(Y) \subseteq H(Z)$ and $H(Y_1) \subseteq H(Z)$ and, again, using the same argument as in the proof of Theorem 1, we reach the conclusion that $Y(t)$ is SPUBSG.

The theorem can be interpreted as follows. If we pass a w.s.s. process $X(t)$, $t \in Z$, having a spectral density function bounded from above and below, through the linear time-varying filter with impulse response function $h(t-s)g(s)$, then the output process is SPUBSG. The filter is realizable iff $h(s) = 0$ for $s < 0$.

We indicate a possible application to time-series modeling. Let $Y(t)$, $t \in Z$, be the w.s.s. time series (mixed autoregressive—moving average model [1]) represented by the difference equation

$$Y(t) - \sum_{s=1}^{p} a_s Y(t - s) = Z(t) - \sum_{s=1}^{q} b_s Z(t - s) \tag{4}$$

where $Z(t)$ is a w.s.s. white noise process generating the time series $Y(t)$. The coefficients $a_1, a_2, \cdots, a_p$ and $b_1, b_2, \cdots, b_q$ are complex and the $a_1, a_2, \cdots, a_p$ are such that the equation

$$\varphi(u) = 1 - \sum_{s=1}^{p} a_s u^s = 0$$

has all its roots outside the unit circle. Then $Y(t)$ can be represented as [1]

$$Y(t) = \sum_{s=-\infty}^{t} h(t - s)Z(s) \tag{5}$$

where $h(s)$ is determined by the coefficients in Eq. (4).

Nonstationary time-series models are usually obtained [1] from Eq. (4) by relaxing the condition on the coefficients a_i, $i = 1, 2, \cdots, p$. We will consider the time series obtained by relaxing, instead, the conditions on the generating process $Z(s)$. We note in this connection that, with the notation established in Lemma 1 and Theorem 2,

$$E \left| \sum_{s=-\infty}^{t} h(t - s)Z(s) \right|^2 < \infty \quad \text{iff} \quad E \left| \sum_{s=-\infty}^{t} h(t - s)g(s)X(s) \right|^2 < \infty .$$

From Lemma 1, we can allow the generating process to be nonstationary white noise; that is, $Z(s)$ can be replaced by $g(s)Z(s)$ in Eq. (4). The resulting process $Y(t)$ is given by

$$Y(t) = \sum_{s=-\infty}^{t} h(t - s)g(s)Z(s)$$

where $h(s)$ is as in Eq. (5) and $Y(t)$ is SPUBSG according to Lemma 1. From Theorem 2 we can even allow the generating process to be correlated; that is, $Z(s)$ can be replaced in Eq. (4) by $g(s)X(s)$. The corresponding time series $Y(t)$ is given by

$$Y(t) = \sum_{s=-\infty}^{t} h(t - s)g(s)X(s)$$

and is SPUBSG according to Theorem 2.

4. Linear time-invariant filters for SPUBSG processes. We treat the continuous-time case only. The discrete-time results are similar. Let $Y(t) = BX(t)$ be SPUBSG and continuous in q.m. Denote by A_X the self-adjoint operator defined via Stone's theorem [9] by

$$X(t) = \exp (iA_X t)X(0)$$

and define $A_Y = BA_X B^{-1}$. Let $\{E_u^X\}$ be the resolution of identity associated with A_X. Then

$$A_Y = \int_{-\infty}^{\infty} u \, dE_u^Y$$

where for a Borel set Δ, $E^Y(\Delta) = BE^X(\Delta)B^{-1}$. Then, in the terminology of [4, p. 2104], A_Y is a scalar spectral operator. Let f be a complex-valued Borel-measurable function. We define $f(A_Y)$ by $f(A_Y) = Bf(A_X)B^{-1}$. Then

$$Y(t) = \exp(iA_Yt)Y(0) \tag{6}$$

where we note that the operator A_Y, in contradistinction to the operator B of Eq. (1), is uniquely determined by $Y(t)$. From Eq. (6) we obtain the spectral representation

$$Y(t) = \int_{-\infty}^{\infty} \exp(iut)\, d\Phi_Y(u)$$

with $\Phi_Y(u) = E_u{}^Y Y(0)$. This representation makes it possible to represent linear time-invariant transformations by a frequency response function. Let $Y_1(t)$ be the SPUBSG input process to a linear time-invariant filter having frequency response function $f(u)$, and denote by $Y_2(t)$ the output process. Then

$$Y_2(t) = \int_{-\infty}^{\infty} f(u) \exp(iut)\, d\Phi_{Y_1}(u) = f(A_{Y_1})Y_1(t) \tag{7}$$

in analogy with the w.s.s. case.

A given class of nonstationary processes need not be invariant under linear time-invariant transformations. See, for example, the classes of processes considered by Priestley [8]. The class of SPUBSG processes, however, possesses this invariance property.

THEOREM 3: The process $Y_2(t)$ of Eq. (7) is SPUBSG. Furthermore, if $Y_1(t) = BX_1(t)$, then $Y_2(t) = BX_2(t)$, where $X_2(t)$ is the w.s.s. process given by

$$X_2(t) = f(A_{X_1})X_1(t).$$

Also the shift operators for $Y_2(t)$ are the same as the shift operators for $Y_1(t)$.

The proof is elementary and is left to the reader.

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