

NETWORK PROBLEMS, M-FUNCTIONS AND UNIFORMLY MONOTONE NETWORKS*

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Abstract. Four classes of network problems—conductive, resistive, conductive boundary value, and resistive boundary value—are considered in this paper. In each case solution of the network problem is tantamount to determining a zero of a nonlinear system of equations. Under certain monotonicity assumptions, it is shown that the nonlinear Gauss-Seidel iterative procedure is globally convergent when applied to these systems.

1. Introduction. Mathematical models of electrical, hydraulic, and elastic systems, as well as certain considerations in operations research and the numerical solution of partial differential equations, give rise to a wide variety of what may be loosely termed *network problems*. For a survey and analysis of many of these the reader is referred to Berge and Ghouila-Houri [1]. In this paper we shall study four classes of network problems. These are formulated in terms of general statements of the Kirchhoff laws, and a collection of functional relations between the network variables. In electrical, hydraulic, and elastic network problems these relations are known respectively as Ohm's law, Darcy's law, and Hooke's law. Here we shall simply call them *characteristics*.

In Sec. 2 we present some fundamentals of network topology and statements of what we call conductive, resistive, conductive boundary value, and resistive boundary value problems. We have chosen circuit theory terminology because this appears to be the most common area of application of these concepts. The conductive and resistive problems differ only because of the type of characteristic that is involved. The conductive problem is associated with what in circuit theory is known as "voltage controlled" resistors while the resistive problem is associated with "current-controlled" resistors. In the important special case of strictly isotone characteristics, it is theoretically possible to go from one problem to the other by replacing the given characteristics by their inverses. However, in practice this may be difficult or impossible to do. Hence, we have included treatments of both problems.

The main object of this paper is to provide a theoretical justification for use of the nonlinear Gauss-Seidel iterative process to solve numerically certain conductive, resistive, conductive boundary value, and resistive boundary value problems. This is done by establishing its global convergence in these cases. It will turn out that when global convergence occurs, uniqueness (and of course existence) of the solution is also assured. To prove these convergence results it is necessary to reformulate the network problems

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as systems of nonlinear equations. This is done in Sec. 3 by utilizing bases of certain finite-dimensional vector spaces.

Secs. 4 and 5 contain the convergence theorems. These are not obtained *ab ovo*, but proceed from more general results on M -functions in Sec. 4 and uniformly monotone gradient maps in Sec. 5. Isotonicity of the characteristics is an assumption which is common to all of our theorems. However, strict isotonicity and surjectivity of *all* characteristics is *not* required in general. Thus we demonstrate convergence even when some of the characteristics are bounded, or constant on all or portions of their domains.

Many authors have considered existence and/or uniqueness questions for network problems, see for example [1–5, 7–9, 13, 17] and the references contained therein. On the other hand, the computational aspects of network problems do not appear to have been treated very extensively. In 1952 Diaz and Roberts [6] used a monotonicity argument to prove that the Gauss-Seidel process converges when applied to the usual 5-point difference analogue of Laplace's equation. Later Birkhoff and Diaz [2] and Dwyer [9] extended this proof idea and applied it to certain network problems which in our terminology are conductive boundary value problems. Accordingly, in Sec. 4 we are able to obtain a generalization of their work.

A different approach was taken by Minty [1, 13] in developing an algorithm for the solution of conductive and resistive problems when the characteristics are *isotone step functions*. His algorithm is based on the "colored arc lemma" and is finite in the sense that it finds a solution after a finite number of steps. This is in direct contrast to the Gauss-Seidel process which generally requires an infinite number of steps for convergence.

Porsching [15] has analyzed the convergence of the Gauss-Seidel process when it is applied to a class of network problems which are related to the present conductive boundary value problems. In [15] the characteristics are permitted to have a more general form than those considered here, but the isotonicity requirements are stricter.

Computational experience has shown that the Gauss-Seidel—or, as it is known to engineers, the Hardy Cross—method often converges slowly [12]. In this regard the recent paper [10] may be consulted for a practical technique to accelerate convergence.

2. Network problems. In this section we recall some basic graph theoretic notions and give general statements of the network problems to be considered. Throughout this paper we will assume that we are given a finite, connected network \mathfrak{N} with node set V and directed arc set S . Let

$$\omega(U) = \omega^+(U) \cup \omega^-(U)$$

denote the cut (coboundary) associated with the node set $U \subset V$, $\omega^+(U)$ and $\omega^-(U)$ representing respectively the arcs incident from and to U . Let $|S| = m$ and label the arcs of \mathfrak{N} from 1 to m . Then, as is well known, each cut can be uniquely associated with a cut vector

$$\omega(U) = [\omega_1(U), \dots, \omega_m(U)]^T$$

by means of the rule

$$\begin{aligned} \omega_i(U) &= +1 && \text{if } \text{arc } i \in \omega^+(U) \\ &= -1 && \text{if } \text{arc } i \in \omega^-(U) \\ &= 0 && \text{if } \text{arc } i \notin \omega(U). \end{aligned}$$

In an analogous manner each cycle of \mathfrak{N} ,

$$\mathfrak{u} = \mathfrak{u}^+ \cup \mathfrak{u}^-,$$

may be associated with a cycle vector

$$\mu = [\mu_1, \mu_2, \dots, \mu_m]^T,$$

where

$$\begin{aligned} \mu_i &= +1 && \text{if } \text{arc } i \in \mathfrak{u}^+ \\ &= -1 && \text{if } \text{arc } i \in \mathfrak{u}^- \\ &= 0 && \text{if } \text{arc } i \notin \mathfrak{u}. \end{aligned}$$

Here \mathfrak{u}^+ is the set of cycle arcs which are oriented in a given sense and \mathfrak{u}^- is the set of arcs which are oriented in the opposite sense.

A *flow* on \mathfrak{N} is a vector

$$\phi = [\phi_1, \dots, \phi_m]^T$$

which is orthogonal to every cut vector. That is,

$$\omega(U)^T \phi = 0, \quad \forall U \subset V. \quad (2.1)$$

Similarly, a *tension* on \mathfrak{N} is a vector

$$\theta = [\theta_1, \dots, \theta_m]^T$$

such that for any cycle vector μ ,

$$\mu^T \theta = 0. \quad (2.2)$$

In the context of electrical networks Eqs. (2.1) and (2.2) are statements of Kirchhoff's current and voltage laws.

For each arc i of \mathfrak{N} we relate the quantities ϕ_i and θ_i by means of a *conductive characteristic*

$$\phi_i = x_i(\theta_i), \quad (2.3)$$

or a *resistive characteristic*

$$\theta_i = y_i(\phi_i), \quad (2.4)$$

where the functions $x_i, y_i : R^1 \rightarrow R^1$, are given. We shall generally identify properties of the functions $x_i(t)$ or $y_i(t)$ with arc i . Thus, when we speak of a continuous or isotone arc, we mean that the corresponding function is continuous or isotone.

We now state two problems.

Conductive Problem: Find a flow ϕ and a tension θ such that (2.3) is satisfied for $i = 1, \dots, m$.

Resistive Problem: Find a flow ϕ and a tension θ such that (2.4) is satisfied for $i = 1, \dots, m$.

Let $|V| = n$ and let the nodes of \mathfrak{N} be labeled from 1 to n . Then it is not difficult to show that θ is a tension if and only if there exists a set of scalars $\{p_1, \dots, p_n\}$, called a node state, such that for any arc i

$$\theta_i = p_i - p_k, \quad (2.5)$$

where j is the initial node and k the terminal node of arc i . The scalar p_i , which is unique up to an added constant, is called the *state value* at node j , and many practical problems are formulated in terms of these values.

Specifically let V_b be a non-void set of *boundary nodes* of V and let p_i^* be an assigned state value for $j \in V_b$. Then we have the following two problems:

Conductive Boundary Value Problem: Find a tension θ and a vector $\phi \in R^m$ such that (2.3) is satisfied, p_i^* is a state value at node $j \in V_b$ corresponding to θ , and $\omega(U)^T \phi = 0$ if $U \subset V - V_b$.

Resistive Boundary Value Problem: Find a tension θ and a vector $\phi \in R^m$ such that (2.4) is satisfied, p_i^* is a state value at node $j \in V_b$ corresponding to θ , and $\omega(U)^T \phi = 0$ if $U \subset V - V_b$.

Note that the vector ϕ in the conductive and resistive boundary value problems is not necessarily a flow on \mathfrak{N} since the orthogonality condition holds only for a restricted class of cuts. However, these two problems can be reformulated exclusively in terms of a flow and a tension on an augmented network.

For the conductive boundary value problem let us construct the network \mathfrak{N}' from \mathfrak{N} by adding one new node, say node $n + 1$, and $|V_b|$ new arcs, say arcs $m + 1, \dots, m + |V_b|$ where each new arc is incident from a different node of V_b to the new node. If arc i connects node $j \in V_b$ to the new node, we define the resistive characteristic

$$\theta_i = p_i^*. \quad (2.6)$$

By introducing the *node cut vectors* $\omega(j)$, and the fact that any cut vector $\omega(U)$ has the representation

$$\omega(U) = \sum_{j \in U} \omega(j), \quad (2.7)$$

it is a straightforward exercise to show that the conductive boundary value problem is equivalent to the

Augmented Conductive Problem: Find a flow and tension on \mathfrak{N}' such that (2.3) is satisfied on arcs $1, \dots, m$ and (2.6) is satisfied on arcs $m + 1, \dots, m + |V_b|$.

In the same way we can show the equivalence of the resistive boundary value problem with the following

Augmented Resistive Problem: Find a flow and tension on \mathfrak{N}' such that (2.4) is satisfied on arcs $1, \dots, m$ and (2.6) is satisfied on arcs $m + 1, \dots, m + |V_b|$.

Finally, we note that an augmented resistive problem is simply a resistive problem on \mathfrak{N}' , but an augmented conductive problem is not a conductive problem on \mathfrak{N}' because the characteristics (2.3) and (2.6) are of *mixed* type. In the next section we examine these problems relative to bases of cuts and cycles.

3. Bases of cuts and cycles. Let Ω and M denote respectively the spans of cut and cycle vectors of \mathfrak{N} . Then it is well known that Ω and M are orthogonal subspaces of R^m and $\dim \Omega = n - 1$, $\dim M = m - n + 1$. Consequently, if $\omega^i, i = 1, \dots, n - 1$, and $\mu^i, i = 1, \dots, m - n + 1$, are bases for Ω and M , then ϕ is a flow on \mathfrak{N} if and only if there are scalars $\gamma_1, \dots, \gamma_{m-n+1}$ such that

$$\phi = \sum_{i=1}^{m-n+1} \gamma_i \mu^i, \quad (3.1)$$

and θ is a tension on \mathfrak{N} if and only if there are scalars $\delta_1, \dots, \delta_{n-1}$ such that

$$\theta = \sum_{i=1}^{n-1} \delta_i \omega^i. \quad (3.2)$$

It follows that relative to the basis $\{\omega^i\}$, solution of a conductive problem is equivalent to determination of scalars $\delta_i, j = 1, \dots, n-1$ which satisfy the *cut equations*,

$$\omega^{iT} x \left(\sum_{j=1}^{n-1} \delta_j \omega^j \right) = 0, \quad i = 1, \dots, n-1, \quad (3.3)$$

where $x(\theta) \equiv [x_1(\theta_1), \dots, x_m(\theta_m)]^T$. Furthermore, relative to the basis $\{\mu^i\}$, solution of a resistive problem is equivalent to determination of scalars $\gamma_i, j = 1, \dots, m-n+1$ which satisfy the *cycle equations*

$$\mu^{iT} y \left(\sum_{j=1}^{m-n+1} \gamma_j \mu^j \right) = 0, \quad i = 1, \dots, m-n+1 \quad (3.4)$$

where $y(\phi) \equiv [y_1(\phi_1), \dots, y_m(\phi_m)]^T$.

Clearly the actual form that either the cut or cycle equations assume depends on the particular basis chosen in the representations (3.3) and (3.4). Some of these forms are better suited for analysis than others. Indeed, in Secs. 4 and 5 we shall utilize particular bases to establish a globally convergent iterative algorithm—the nonlinear Gauss-Seidel method—for the cut and cycle equations.

The cut and cycle equations are mathematical formulations of conductive, resistive and augmented resistive problems. It remains to deduce a corresponding formulation for an augmented conductive problem. In this case let Ω' be the span of cut vectors of the augmented network \mathfrak{N}' and let $\omega'^i, i = 1, \dots, n$ be a basis for Ω' . We write

$$\omega'^i = [\omega_1'^i, \dots, \omega_{m+|V_b|}^i]^T,$$

and

$$\phi' = [\phi_1', \dots, \phi_{m+|V_b|}]^T.$$

Solution of the augmented conductive problem is then equivalent to determination of scalars $\delta_k, k = 1, \dots, n$ and $\phi_i', j = m+1, \dots, m+|V_b|$ which satisfy the *mixed equations*,

$$\sum_{j=1}^m \omega_j'^l x_j \left(\sum_{k=1}^n \delta_k \omega_k'^l \right) + \sum_{j=m+1}^{m+|V_b|} \omega_j'^l \phi_j' = 0, \quad l = 1, \dots, n \quad (3.5a)$$

$$\sum_{k=1}^n \delta_k \omega_k'^i = p_i^*, \quad i = m+1, \dots, m+|V_b|. \quad (3.5b)$$

In (3.5b), as in (2.6), it is assumed that arc i connects node $j \in V_b$ to node $n+1$.

4. M-functions. Having formulated the cut, cycle and mixed equations, we now investigate conditions under which these equations have unique solutions and for which a globally convergent iterative method exists. In this section we shall do this by relating the equations to a general class of nonlinear mappings called *M-functions*.

Let $e^i \in R^n$ denote the i th unit coordinate vector, $i = 1, \dots, n$. A function $F: R^n \rightarrow R^n, F = [f_1, \dots, f_n]^T$ is *off-diagonally antitone* if $\psi_{ii}(t): R^1 \rightarrow R^1, \psi_{ii}(t) = f_i(x + te^i)$ is

antitone for all $x \in R^n$ and $i \neq j$, $i, j = 1, \dots, n$. It is *inverse isotone* under component-wise partial ordering if $Fx \leq Fy$ implies that $x \leq y$. Finally, F is an *M-function* if it is inverse isotone and off-diagonally antitone. As shown by Rheinboldt [16], if F is a continuous surjective *M-function*, then the system of equations

$$Fx = z \quad (4.1)$$

has a unique solution x^* for any $z \in R^n$, and moreover the (nonlinear) Gauss-Seidel iterates $\{x^k\}$ given by:

$$\text{Solve } f_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \dots, x_n^k) = z_i, \quad \text{for } x_i, \quad (4.2)$$

$$\text{Set } x_i^{k+1} = x_i, \quad i = 1, \dots, n, \quad k = 0, 1, \dots,$$

are well defined and converge to x^* for any starting vector $x^0 \in R^n$.

We shall make use of the following special case of a theorem of Rheinboldt [16, pg. 301] which gives sufficient conditions for F to be a surjective *M-function*.

THEOREM 4.1: Let $F : R^n \rightarrow R^n$ be continuous and off-diagonally antitone. Suppose that for any $x \in R^n$, the functions

$$q_i(t) = \sum_{j=1}^n \psi_{ji}(t), \quad i = 1, \dots, n, \quad (4.3)$$

are isotone. Assume further that for every $i(0)$, $1 \leq i(0) \leq n$, there is a sequence $\{i(1), i(2), \dots, i(p)\} \subset \{1, \dots, n\}$ such that $\psi_{i(k), i(k+1)}(t)$, $k = 0, \dots, p-1$ is strictly antitone and surjective and $q_{i(p)}(t)$ is strictly isotone and surjective. Then F is a continuous surjective *M-function*.

To relate Theorem 4.1 to network equations we require the following definitions. We shall say that an arc is *bijective* if it is strictly isotone and surjective. Two nodes will be said to be *bijectively connected* if they are connected by a path of bijective arcs and two nodes will be termed (bijectively connected) neighbors if they are the extremities of some (bijective) arc.

We now consider the cut equations and prove the following.

THEOREM 4.2: Let \mathfrak{N} have n nodes, m arcs and continuous isotone conductive arcs. If every two nodes of \mathfrak{N} are bijectively connected, then the function

$$F : R^{n-1} \rightarrow R^{n-1}, \quad f_j(\delta) = \omega(j)^T x \left(\sum_{k=1}^{n-1} \delta_k \omega(k) \right), \quad j = 1, \dots, n-1, \quad (4.4)$$

is a continuous, surjective *M-function*.

Proof: For the function defined by (4.4) we have

$$\psi_{il}(t) = \sum_{j=1}^m \omega_j(j) x_i(c_j + \omega_j(l)t), \quad j, l = 1, \dots, n-1,$$

where

$$c_i \equiv \sum_{k=1}^{n-1} \delta_k \omega_i(k)$$

is independent of t . If $j \neq l$, then it follows from the definition of the node cut vectors $\{\omega(j)\}$ that

$$\psi_{il}(t) = \sum_{i \in \omega(j) - \omega(l)} \omega_j(j) x_i(c_i) + \sum_{i \in \omega(j) \cap \omega(l)} \omega_j(j) x_i(c_i - \omega_j(j)t), \quad (4.5)$$

and therefore F is off-diagonally antitone since the arcs are isotone. Moreover, if any arc $i \in \omega(j) \cap \omega(l)$ is bijective, then $\psi_{il}(t)$ is strictly antitone and surjective. Since every node $j(0)$ is bijectively connected to node n , it follows that there is a node $j(p)$ which is a bijectively connected neighbor of node n and a sequence $\{j(1), \dots, j(p)\} \subset \{1, \dots, n-1\}$ such that $\psi_{j(k), j(k+1)}(t)$, $k = 0, \dots, p-1$ is strictly antitone and surjective.

Now by (2.7),

$$\begin{aligned} q_l(t) &= \sum_{i=1}^{n-1} \psi_{il}(t) = \sum_{i=1}^m \left(\sum_{j=1}^{n-1} \omega_i(j) \right) x_i(c_i + \omega_i(l)t) \\ &= - \sum_{i=1}^m \omega_i(n) x_i(c_i + \omega_i(l)t) \\ &= - \sum_{i \in \omega(n) - \omega(l)} \omega_i(n) x_i(c_i) + \sum_{i \in \omega(n) \cap \omega(l)} \omega_i(l) x_i(c_i + \omega_i(l)t), \quad l = 1, \dots, n-1. \end{aligned}$$

Hence $q_l(t)$ is isotone and if node l is a bijectively connected neighbor of node n , then $q_l(t)$ is strictly isotone and surjective. Therefore, all of the hypotheses of Theorem 4.1 are satisfied and so F is a continuous, surjective M -function. Q.E.D.

We note that if F is the function defined by (4.4), then the equations $F\delta = 0$ are the cut equations relative to a basis of node cut vectors. However, any two bases of Ω are related by a nonsingular matrix transformation. This gives us the following result on the solvability of the cut equations.

COROLLARY 4.3: Under the hypotheses of Theorem 4.2 the cut equations (3.3) have a unique solution. Furthermore, relative to any basis of node cut vectors, the iterates of the Gauss-Seidel process (4.2) converge to the solution for any starting vector.

We now turn to the mixed equations (3.5).

THEOREM 4.4: Let \mathcal{N} have m arcs, n nodes, boundary node set $V_b = \{\nu+1, \dots, n\}$, and continuous isotone conductive arcs. If every node of $V - V_b$ is bijectively connected to a boundary node, then the function

$$F: R^r \rightarrow R^r, \quad f_i(\delta) = \omega(j)^T x \left(\sum_{k=1}^r \delta_k \omega(k) + \sum_{k=\nu+1}^n p_k^* \omega(k) \right), \quad j = 1, \dots, \nu, \quad (4.6)$$

is a continuous surjective M -function.

Proof: As in the case of Theorem 4.2, the proof consists of verifying that the hypotheses of Theorem 4.1 are satisfied. For $j \neq l$, $j, l = 1, \dots, \nu$ the functions $\psi_{il}(t)$ are again given by (4.5) except that now

$$c_i = \sum_{k=1}^r \delta_k \omega_i(k) + \sum_{k=\nu+1}^n p_k^* \omega_i(k).$$

Therefore F is off-diagonally antitone. Furthermore,

$$\begin{aligned} q_l(t) &= \sum_{i=1}^m \left(\sum_{j=1}^r \omega_i(j) \right) x_i(c_i + \omega_i(l)t) \\ &= - \sum_{i=1}^m \omega_i(V_b) x_i(c_i + \omega_i(l)t) \\ &= - \sum_{i \in \omega(V_b) - \omega(l)} \omega_i(V_b) x_i(c_i) + \sum_{i \in \omega(V_b) \cap \omega(l)} \omega_i(l) x_i(c_i + \omega_i(l)t), \quad l = 1, \dots, \nu, \end{aligned}$$

and the isotonicity of $q_l(t)$ follows. Finally, the hypotheses imply that every node of

$V - V_b$ is bijectively connected to a node which is itself a bijectively connected neighbor of some boundary node. Thus the remaining hypotheses of Theorem 4.1 are seen to hold.

Q.E.D.

Theorem 4.4 serves as the basis for the following solvability result concerning the mixed equations.

COROLLARY 4.5: Assume that the hypotheses of Theorem 4.4 hold, and let F be defined by (4.6). Then the equations $F\delta = 0$ have a unique solution δ^* which is the limit of the iterates of the Gauss-Seidel process (4.2) for any starting vector. Under the same hypotheses the mixed equations have a unique solution and relative to the basis of node cut vectors $\{\omega'(j)\}_{j=1}^n$ this solution is: $\delta_k = \delta_k^*, k = 1, \dots, \nu$, $\delta_k = p_k^*, k = \nu + 1, \dots, n$, and $\phi_i' = -\omega(j)^T x(\sum_{k=1}^n \delta_k \omega(k))$, $i = m + 1, \dots, m + |V_b|$, where arc i connects node $j \in V_b$ to node $n + 1$.

Proof: Relative to the basis $\{\omega'(j)\}_{j=1}^n$ and boundary node set $V_b = \{\nu + 1, \dots, n\}$ Eqs. (3.5b) reduce to $\delta_k = p_k^*, k = \nu + 1, \dots, n$. But then $F\delta = 0$ represents the first ν equations of (3.5a) and these have the solution δ^* . The remaining $n - \nu$ (i.e. $|V_b|$) equations become $\omega(j)^T x(\sum_{k=1}^n \delta_k \omega(k)) + \phi_i' = 0$, $i = m + 1, \dots, m + |V_b|$, where arc i connects node $j \in V_b$ to node $n + 1$. Finally, the solution relative to any other basis is obtained from this solution by a linear transformation.

Q.E.D.

Birkhoff and Diaz [2] and Dwyer [9] have considered boundary value problems in which the earlier condition

$$\omega(U)^T \phi = 0, \quad U \subset V - V_b$$

is replaced by the condition

$$\omega(j)^T \phi = g_i(p_i), \quad j \in V - V_b, \quad (4.7)$$

where $g_i(t)$ is a continuous antitone function. These seemingly more general problems may be reduced to a conductive boundary value problem as follows. We construct an augmented network \mathfrak{N}_+ from \mathfrak{N} by adding one new boundary node, and $|V - V_b|$ new arcs each of which connects a different node of $V - V_b$ to the new node. If arc i connects node $j \in V - V_b$ to this node, then we define the conductive characteristic for that arc as

$$\phi_i = -g_i(\theta_i).$$

Also, we assign the new boundary node a state value of zero. It is not difficult to see that solution of a boundary value problem which employs (4.7) on \mathfrak{N} is equivalent to solution of a conductive boundary value problem on \mathfrak{N}_+ . Furthermore, it is evident that if the hypotheses of Theorem 4.4 hold on \mathfrak{N} , then they also hold on \mathfrak{N}_+ . Hence an obvious analogue of Corollary 4.5 holds for the mixed equations of \mathfrak{N}_+ , and as such it represents a generalization of results of Birkhoff and Diaz and Dwyer on the convergence of the Gauss-Seidel iterates.

In order to apply the M -function theory to the cycle equations (3.4), we need an additional restriction on \mathfrak{N} , namely, that it be a *planar network*. Let \mathfrak{N} have n nodes and m arcs. It follows that \mathfrak{N} can be embedded in the plane in such a way so as to divide it into $m - n + 1$ simply connected bounded regions and one unbounded region. Furthermore, the cycle vectors corresponding to the cycles created by the arcs which constitute the boundaries of the bounded regions form a basis for M (see, for example, Berge and Ghouila-Houri [1, p. 135]).

To fix these ideas let R_1, \dots, R_{m-n+1} denote the bounded regions and R_{m-n+2} the

unbounded region. We will say that two of these regions are (bijectively connected) neighbors if their boundaries have at least one (bijective) arc in common. The regions $R_{i(0)}$ and $R_{i(p)}$ are *bijectively connected* if there is a sequence $\{i(1), \dots, i(p-1)\} \subset \{1, \dots, m-n+2\}$ such that $R_{i(k)}$ and $R_{i(k+1)}$, $k = 0, \dots, p-1$, are bijectively connected neighbors. Now let \mathbf{y}^i , $j = 1, \dots, m-n+2$, denote the cycle associated with the boundary of region R_i where the given orientation of the arcs of \mathbf{y}^i is in accordance with the positive sense of the boundary of R_i . The corresponding cycle vectors $\{\mu^i\}$ will be termed *contours*.

THEOREM 4.6: Let \mathfrak{N} have continuous isotone resistive arcs and contours $\{\mathbf{y}^i\}$. If every two regions are bijectively connected, then the function

$$F: R^{m-n+1} \rightarrow R^{m-n+1}, \quad f_i(\gamma) = \mu^{i^T} y \left(\sum_{k=1}^{m-n+1} \gamma_k \mu^k \right) \quad j = 1, \dots, m-n+1, \quad (4.8)$$

is a continuous surjective M -function.

Proof: We construct \mathfrak{N}^* , the dual network of \mathfrak{N} , by associating node j of \mathfrak{N}^* with R_j , $j = 1, \dots, m-n+2$ and connecting nodes j and k of \mathfrak{N}^* by arc i of \mathfrak{N}^* if the boundaries of R_j and R_k have arc i of \mathfrak{N} in common. Arc i of \mathfrak{N}^* is chosen to be incident from node j to node k if arc i of \mathfrak{N} is a member of \mathbf{y}^{j+} . Otherwise it is incident from node k to node j . Thus the dual \mathfrak{N}^* has $m-n+2$ nodes, m arcs, and $\omega(j)$ of \mathfrak{N}^* is precisely μ^j of \mathfrak{N} . For each arc i of \mathfrak{N}^* we define the *conductive* characteristic $\phi_i = y_i(\theta_i)$, where $\theta_i = y_i(\phi_i)$ is the corresponding *resistive* characteristic for arc i of \mathfrak{N} . This yields a conductive problem on \mathfrak{N}^* . Since \mathfrak{N}^* has continuous isotone conductive arcs and since every two nodes of \mathfrak{N}^* are bijectively connected, the proof now reduces to that of Theorem 4.2 with (4.8) in place of (4.4). Q.E.D.

As in the previous cases, we have the following corollary which we state without proof.

COROLLARY 4.7: If \mathfrak{N} is a planar network with continuous isotone resistive arcs, and if every two regions of \mathfrak{N} are bijectively connected, then the cycle equations (3.4) have a unique solution. Furthermore, relative to the basis of contours μ^i , $j = 1, \dots, m-n+1$, the iterates of the Gauss-Seidel process (4.2) converge to the solution for any starting vector.

The M -function result of Theorem 4.6 required that \mathfrak{N} be planar. This is in contrast to Theorem 4.2 where nonplanar \mathfrak{N} were allowed. If the arcs are resistive and isotone but \mathfrak{N} is nonplanar, then it will not be possible in general to select a basis of cycle vectors $\{\mu^i\}$ such that the function (4.8) is an M -function. This is because the off-diagonal antitonicity of F can not be guaranteed. This in turn is a consequence of MacLane's result [11] that a necessary (and sufficient) condition for \mathfrak{N} to be planar is that it contain a basis of cycles such that every arc appears at most twice in the basis. Thus if \mathfrak{N} is nonplanar, then any basis of cycle vectors will be such that for some index i the i th component will be nonzero in at least three vectors. Necessarily, two of these components will have the same sign. Consequently there will exist indices j and l , $j \neq l$, such that one of the terms appearing in the sum that defines $f_i(\gamma)$ in (4.8) will be of the form $\sigma y_i(\sigma(\gamma_i + \gamma_l) + \sum_{k \neq i, l} \pm \gamma_k)$ where $\sigma = \pm 1$. Hence if $y_i(t)$ grows sufficiently fast, then the function $\psi_{ii}(t)$ can not be antitone.

5. Uniformly monotone networks. To obtain a convergence theorem for the cycle equations when \mathfrak{N} is not necessarily planar, as well as further results for the cut equations, we introduce the idea of a uniformly monotone network.

We will say that arc i is *uniformly isotone* if the function $x_i(t)$ or $y_i(t)$ associated with its characteristic is differentiable and $dx_i(t)/dt \geq c$ or $dy_i(t)/dt \geq c$ for some constant $c > 0$ and all $t \in R^1$. Let \mathfrak{N} have n nodes and m arcs. With each basis for M , μ^j , $j = 1, \dots, m - n + 1$, we associate a corresponding *cycle matrix*

$$K = [\mu^1, \mu^2, \dots, \mu^{m-n+1}].$$

This matrix necessarily has at least one $(m - n + 1) \times (m - n + 1)$ submatrix which is nonsingular. If \mathfrak{N} has continuously differentiable, isotone, resistive arcs, and a cycle matrix K such that the arcs corresponding to the rows of some $(m - n + 1) \times (m - n + 1)$ nonsingular submatrix of K are uniformly isotone, then \mathfrak{N} will be said to be *R-uniformly monotone*. Analogously, if \mathfrak{N} has continuously differentiable, isotone, conductive arcs and there is a basis $\{\omega^j\}$ for Ω such that the arcs corresponding to the rows of some $(n - 1) \times (n - 1)$ submatrix of the *cut matrix* $A = [\omega^1, \omega^2, \dots, \omega^{n-1}]$ are uniformly isotone, then \mathfrak{N} will be termed *C-uniformly monotone*.

Uniformly monotone networks are related to the so-called uniformly monotone gradient maps. There are a number of convergence theorems for such mappings. We shall utilize the following one which is proven in Ortega and Rheinboldt [14, p. 516].

THEOREM 5.1: Assume that $F : R^n \rightarrow R^n$ is continuously differentiable, that the Jacobian matrix $F'(x)$ is symmetric for all $x \in R^n$, and that there is a constant $c > 0$ such that for all $x, h \in R^n$

$$h^T F'(x) h \geq c h^T h.$$

Then for any starting vector the Gauss-Seidel iterates (4.2) converge to the unique solution of $Fx = 0$.

The next theorem on *R-uniformly monotone* networks is essentially a corollary of this.

THEOREM 5.2: If \mathfrak{N} is *R-uniformly monotone*, then the cycle equations have a unique solution. Furthermore, there exists a basis for M such that relative to this basis the iterates of the Gauss-Seidel process (4.2) converge to the solution for any starting vector.

Proof: By hypothesis, there is a cycle matrix K with $m - n + 1$ linearly independent rows each of which corresponds to a uniformly isotone arc. With no loss of generality we can assume that these are the first $m - n + 1$ rows of K . Hence we can partition K as

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

where K_1 is nonsingular and arcs $1, \dots, m - n + 1$ are uniformly isotone. Now consider the continuously differentiable map

$$F : R^{m-n+1} \rightarrow R^{m-n+1}, \quad f_i(\gamma) = \mu^{iT} y \left(\sum_{j=1}^{m-n+1} \gamma_j \mu^j \right), \quad i = 1, \dots, m - n + 1.$$

One easily verifies $F'(\gamma) = K^T D(\gamma) K$, where

$$D(\gamma) = \text{diag} \left(\frac{dy_1(\phi_1)}{dt}, \dots, \frac{dy_m(\phi_m)}{dt} \right), \quad \phi_k = \sum_{j=1}^{m-n+1} \gamma_j \mu_k^j, \quad k = 1, \dots, m.$$

The matrix F' is obviously symmetric for all $\gamma \in R^{m-n+1}$. Furthermore, if $h \in R^{m-n+1}$ and we write

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

where D_1 and D_2 are diagonal matrices of orders $m - n + 1$ and $n + 1$ respectively, then

$$h^T F'(\gamma) h = (K_1 h)^T D_1 (K_1 h) + (K_2 h)^T D_2 (K_2 h).$$

It follows from the isotonicity hypotheses that there is a constant $c > 0$ such that

$$h^T F'(\gamma) h \geq c(K_1 h)^T (K_1 h) = c h^T K_1^T K_1 h.$$

However, the matrix $K_1^T K_1$ is positive definite so that if $\lambda > 0$ denotes its smallest eigenvalue, then $h^T F'(\gamma) h \geq (c\lambda) h^T h$. This demonstrates that all of the hypotheses of Theorem 5.1 hold and proves the convergence part of the theorem as well as existence and uniqueness of a solution relative to a certain basis. General existence and uniqueness follow from the invertible relationship between this and any other basis. Q.E.D.

There is an analogous theorem for C -uniformly monotone networks which we state without proof.

THEOREM 5.3: If \mathfrak{N} is C -uniformly monotone, then the cut equations have a unique solution. Furthermore, there exists a basis for Ω such that relative to this basis the iterates of the Gauss-Seidel process (4.2) converge to the solution for any starting vector.

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