NEAR-BOUNDARY EXPANSION OF GREEN'S FUNCTION ASSOCIATED WITH CLAMPED PLATES*

By

CHIEN-HENG WU

(*University of Illinois at Chicago Circle*)

Abstract. The Green's function $G(P, P')$ associated with a clamped plate of arbitrary shape is considered, when P' is at a distance $O(\epsilon)$ from a regular point O of the boundary. First an outer expansion of G is described, valid when P is not near P'. Then an inner expansion of G is constructed when both P and P' are near 0. The leading term of the inner expansion is just the Green's function G_s for the halfplane bounded by the tangent to the boundary at O, and $\epsilon^{-2}G$ differs from $\epsilon^{-2}G_s$ by $O(\epsilon)$. The first two terms of the inner expansion agree with the first two terms of the expansion of G_c , the Green's function for the interior of the osculating circle of the boundary at 0, if the boundary is convex at O. If it is concave, G_c is the Green's function for the exterior of the osculating circle. Moreover, $\epsilon^{-2}G$ differs from $\epsilon^{-2}G_c$ by $O(\epsilon^2)$. A two-term inner expansion is explicitly given.

1. Introduction. Let (x, y) be rectangular Cartesian coordinates and let D be a domain of the (x, y)-plane. Suppose that O is a regular point of the boundary ∂D of D. We locate the origin of the coordinates at O in such a way that the x-axis is tangent to ∂D and the y-axis points toward the interior of D. For x sufficiently small ∂D has the Taylor expansion

$$y = b(x) = \sum_{n=2}^{\infty} \frac{b_n}{n!} x^n. \tag{1.1}$$

We shall use the radius of curvature of ∂D at O as a length scale so that $b_2 = \pm 1$.

Suppose that $G(x, y, x', y')$ is a function defined on D, satisfying the equations

$$\nabla^4 G = \delta(x - x')\delta(y - y'), \qquad (x, y) \in D, \tag{1.2}$$

$$G = \partial_n G = 0, \qquad (x, y) \in \partial D, \tag{1.3}$$

where ∂_n is taken along the outward normal $\mathbf{n}$ to ∂D.

If D is the semi-infinite plane $y > 0$, the associated Green's function G_s obtained by Michell [1] is

$$G_s = \tfrac{1}{2}[(x - x')^2 + (y - y')^2] \ln \frac{(x - x')^2 + (y - y')^2}{(x - x')^2 + (y + y')^2} + 2yy'. \tag{1.4}$$

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If D is the interior of the unit circle $x^2 + (y-1)^2 = 1$, then the Green's function G_c^+ obtained by Michell [1] is

$$G_c^+ = \tfrac{1}{2}\{1 - [x'^2 + (y'-1)^2]\}\{1 - [x^2 + (y-1)^2]\} + \tfrac{1}{2}[(x-x')^2 + (y-y')^2] \ln [(x-x')^2 + (y-y')^2]z^{-2} \tag{1.5}$$

where

$$z^2 = (x-x')^2 + (y+y')^2 - 2y(x'^2 + y'^2) - 2y'(x^2 + y^2) + (x'^2 + y'^2)(x^2 + y^2). \tag{1.6}$$

The Green's function G_c^- for the exterior of the unit circle $x^2 + (y+1)^2 = 1$ was obtained by Symonds [2]. More general but related problems were investigated by Dundurs and Lee [3] and Amon and Dundurs [4].

For a general domain D, the Green's function G cannot be obtained explicitly in a closed form. Let $G(P, P')$ be the Green's function. Then $G(P, P')$ may be interpreted as the deflection at P of a plate subjected to a unit load applied at P'. Our objective is to determine $G(P, P')$, when the unit load is placed near the boundary point O. Let ϵ denote the distance OP'. We shall determine G asymptotically in terms of the parameter ϵ as $\epsilon \to 0$. Inner and outer expansions will be constructed and matched. Moreover, the first three terms of the inner expansion can be obtained explicitly. Only the first two terms, however, are calculated. The result is

$$\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon\xi', \epsilon\eta') \sim \tfrac{1}{2}[(\xi-\xi')^2 + (\eta-\eta')^2] \ln \frac{[(\xi-\xi')^2 + (\eta-\eta')^2]}{[(\xi-\xi')^2 + (\eta+\eta')^2]} + 2\eta\eta' - \epsilon 4b_2 \frac{\eta^2\eta'(\xi'^2 + \eta'^2) + \eta\eta'^2(\xi^2 + \eta^2)}{(\xi-\xi')^2 + (\eta+\eta')^2} + O(\epsilon^2) \tag{1.7}$$

where $b_2 = \pm 1$, depending on whether D is convex or concave at O. It can be checked that

$$G(\epsilon\xi, \epsilon\eta, \epsilon\xi', \epsilon\eta') - G_s(\epsilon\xi, \epsilon\eta, \epsilon\xi', \epsilon\eta') \sim O(\epsilon^3), \tag{1.8}$$

$$G(\epsilon\xi, \epsilon\eta, \epsilon\xi', \epsilon\eta') - G_c^{\pm}(\epsilon\xi, \epsilon\eta, \epsilon\xi', \epsilon\eta') \sim O(\epsilon^4). \tag{1.9}$$

For the purposes of bringing out the underlying idea and illustrating the method of finding (1.7), we consider a simpler problem, namely, the problem of determining $G(x, y, \epsilon) \equiv G(x, y, 0, \epsilon)$ as $\epsilon \to 0$. The function $G(x, y, \epsilon)$ may be defined by

$$G(x, y, \epsilon) = [x^2 + (y-\epsilon)^2] \ln [x^2 + (y-\epsilon)^2]^{1/2} + H(x, y, \epsilon) \tag{1.10}$$

where H is a regular bihormonic function satisfying the boundary condition that G, together with its normal derivative, vanishes on ∂D.

We shall occasionally use polar coordinates (r, θ). We shall also introduce boundary layer variables $(\xi, \eta, \rho, \theta)$ and intermediate variables $(\sigma, \nu, \lambda, \theta)$. The three sets of variables are related by the equations

$$(x, y, r) = \epsilon^{1/2}(\sigma, \nu, \lambda) = \epsilon(\xi, \eta, \rho). \tag{1.11}$$

In terms of the boundary layer variables, ∂D defined by (1.1) has the expansion

$$\eta = \beta(\xi, \epsilon) \equiv \frac{1}{\epsilon} b(\epsilon\xi) = \sum_{m=1}^{\infty} \frac{\beta_m(\xi)}{m!} \epsilon^m, \qquad \beta_m = \frac{b_{m+1}}{m+1} \xi^{m+1}. \tag{1.12}$$

Our method of analysis is that used by Wu and Keller [5] to obtain the corresponding results for Laplace's equation.

2. Outer expansion. The problem satisfied by $G(x, y, \epsilon)$ is defined by (1.10). Since $G(P, P')$ is analytic in P' except at P, $G(x, y, \epsilon)$ is analytic in ϵ. Therefore

$$\epsilon^{-2}G(x, y, \epsilon) \sim \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} G_n(x, y). \tag{2.1}$$

We shall call (2.1) the outer expansion of $G(x, y, \epsilon)$. The function $G_n(x, y)$ is biharmonic and vanishes, together with its normal derivative, on ∂D except at the point O. Near the point O,

$$G_n(x, y) = O(r^{-n}), \qquad r \to 0. \tag{2.2}$$

Because G_n is analytic, we conclude from (2.2) that

$$G_n(x, y) = \sum_{m=-n}^{\infty} J_{nm}(\theta)r^m, \tag{2.3}$$

where

$$J_{nm}(\theta) = A_{sm}'S_m(\theta) + B_{sm}'C_m(\theta) + \bar{A}_{sm}'\bar{S}_m(\theta) + \bar{B}_{sm}'\bar{C}_m(\theta). \tag{2.4}$$

and

$$C_m(\theta) = \cos m\theta - \cos (m - 2)\theta, \qquad m \neq 1, \tag{2.5}$$

$$\bar{C}_m(\theta) = \cos m\theta + \cos (m - 2)\theta, \qquad m \neq 1, \tag{2.6}$$

$$S_m(\theta) = (m - 2) \sin m\theta - m \sin (m - 2)\theta, \qquad m \neq 0, 1, 2 \tag{2.7}$$

$$\bar{S}_m(\theta) = (m - 2) \sin m\theta + m \sin (m - 2)\theta, \qquad m \neq 0, 1, 2 \tag{2.8}$$

$$C_1(\theta) = \cos \theta, \quad S_1(\theta) = \sin \theta, \quad \bar{C}_1(\theta) = \theta \cos \theta, \quad \bar{S}_1(\theta) = \theta \sin \theta, \tag{2.9}$$

$$S_0(\theta) = S_2(\theta) = \sin 2\theta, \quad S_0(\theta) = \bar{S}_0(\theta) = \theta. \tag{2.10}$$

We note that $J_{nm}(\theta)r^m$ are biharmonic functions. Moreover, the functions $C_m(\theta)$ and $S_m(\theta)$ defined by (2.5) and (2.7) vanish, together with their derivatives, for $\theta = 0$ and π.

The constants A', B', $\bar{A}'$ and $\bar{B}'$ appearing in (2.4) are so far undetermined. We shall show that some of the constants can be determined by matching. The rest can be determined by solving a series of well-defined boundary-value problems.

For the purpose of matching, we need to know the inner expansion of the outer expansion. First we use (2.3) and (2.1) to obtain the expansion of $\epsilon^{-2}G(x, y, \epsilon)$ for r small. It is

$$\epsilon^{-2}G(x, y, \epsilon) \sim \sum_{s=1}^{\infty} \frac{\epsilon^s}{s!} \sum_{m=-s}^{\infty} J_{sm}(\theta)r^m. \tag{2.11}$$

Next we set $(x, y, r) = \epsilon^{1/2}(\sigma, \nu, \lambda)$ in (2.11). Then, upon rearranging the series, we get

$$\epsilon^{-2}G(\epsilon^{1/2}\sigma, \epsilon^{1/2}\nu, \epsilon) \sim \sum_{k=1}^{\infty} \epsilon^{k/2} \sum_{n=1}^{k} \frac{1}{n!} [A_{n,k-2n}' S_{k-2n}(\theta) + B_{n,k-2n}'C_{k-2n}(\theta) + \bar{A}_{n,k-2n}'\bar{S}_{k-2n}(\theta) + \bar{B}_{n,k-2n}'\bar{C}_{k-2n}(\theta)]\lambda^{k-2n}. \tag{2.12}$$

This is the inner expansion of the outer expansion.

3. Inner expansion. We consider the case x and y are $O(\epsilon)$. It follows from (1.10) that we may seek G in the form

$$\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \sim \tfrac{1}{2}[\xi^2 + (\eta - 1)^2] \ln [\xi^2 + (\eta - 1)^2] + \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} h_n(\xi, \eta). \tag{3.1}$$

We shall refer to (3.1) as the inner expansion. The functions h_n are regular and biharmonic. The boundary conditions $G = \partial G/\partial n = 0$ must be satisfied on ∂D defined by (1.12). We have

$$\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \,|_{\eta=\beta(\xi,\epsilon)} \sim 0 \tag{3.2}$$

$$\frac{\partial}{\partial\mu} \epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \,|_{\eta=\beta(\xi,\epsilon)} \sim 0 \tag{3.3}$$

where

$$\frac{\partial(\;)}{\partial\mu} \equiv \left\{1 + \frac{\partial\beta(\xi, \epsilon)^2}{\partial\xi}\right\}^{-1/2} \left\{\frac{\partial(\;)}{\partial\xi} \frac{\partial\beta(\xi, \epsilon)}{\partial\xi} - \frac{\partial(\;)}{\partial\eta}\right\} \Bigg|_{\eta=\beta(\xi,\epsilon)} . \tag{3.4}$$

Substituting (3.1) into (3.2), expanding for small ϵ and equating to zero the coefficient of each power of ϵ, we obtain

$$h_s(\xi, 0) = -f_s(\xi) - \sum_{m=0}^{s-1} \frac{s!}{m!\,(s - m)!} h_{m,s-m}(\xi), \tag{3.5}$$

where

$$f_s(\xi) = \tfrac{1}{2}\partial_\epsilon{}^s\{\xi^2 + [\beta(\xi, \epsilon) - 1]^2\} \ln \{\xi^2 + [\beta(\xi, \epsilon) - 1]^2\} \,|_{\epsilon=0} , \tag{3.6}$$

$$h_{ks}(\xi) = \partial_\epsilon{}^s h_k[\xi, \beta(\xi, \epsilon)] \,|_{\epsilon=0} . \tag{3.7}$$

Applying the same operation on (3.3), we get

$$\begin{aligned} h_{s,\eta}(\xi) = {} & -f_{,\eta s}(\xi) + \sum_{m=1}^{s} \frac{s!\, b_{m+1}\xi^m}{(s - m)!\, m!} f_{,\xi(s-m)}(\xi) \\ & + \sum_{k=1}^{s} \left\{\sum_{m=1}^{k} \frac{s!\, b_{m+1}\xi^m}{(s - k)\,(k! - m)!\, m!} h_{(s-k),\xi(k-m)}(\xi) - \frac{s!}{k!\,(s - k)!} h_{(s-k),\eta k}(\xi)\right\} \end{aligned} \tag{3.8}$$

where the notation

$$P_{,\xi n}(\xi) = \partial_\epsilon{}^n[\partial_\xi P(\xi, \eta) \,|_{\eta=\beta(\xi,\epsilon)}] \,|_{\epsilon=0} \tag{3.9}$$

$$P_{,\eta n}(\xi) = \partial_\epsilon{}^n[\partial_\eta P(\xi, \eta) \,|_{\eta=\beta(\xi,\epsilon)}] \,|_{\epsilon=0} \tag{3.10}$$

applies to f and h_m . Eqs. (3.5) and (3.8) determine the values of h_s and $h_{s,\eta}$ on the ξ-axis, in terms of the h_m with $m < s$. By determining regular biharmonic functions satisfying (3.5) and (3.8), starting with $s = 0$, the h_s can be found successively.

Setting $s = 0$ in (3.5) and (3.8), we get

$$h_0(\xi, 0) = -\tfrac{1}{2}(\xi^2 + 1) \ln (\xi^2 + 1), \tag{3.11}$$

$$h_{0,\eta}(\xi, 0) = \ln (\xi^2 + 1) + 1. \tag{3.12}$$

An image analysis leads to the choice

$$h_0(\xi, \eta) = -\tfrac{1}{2}[\xi^2 + (\eta - 1)^2] \ln [\xi^2 + (\eta + 1)^2] + 2\eta + A_0(\xi, \eta) \tag{3.13}$$

where A_0 is an arbitrary biharmonic function satisfying the conditions $A_0 = A_{0,\eta} = 0$ on the ξ-axis. However, since the outer expansion is $O(\epsilon^3)$, the term $\epsilon^2 A_0$ cannot be matched with the outer expansion. Thus $A_0 = 0$. Then (3.1) becomes

$$\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \sim \tfrac{1}{2}[\xi^2 + (\eta - 1)^2] \ln \frac{\xi^2 + (\eta - 1)^2}{\xi^2 + (\eta + 1)^2} + 2\eta + O(\epsilon). \tag{3.14}$$

Aside from the term $O(\epsilon)$, this is the Green's function for the half plane $\eta \geq 0$, bounded by the tangent to ∂D at O.

We need the solution of a fundamental problem for the determination of h_n for $n \geq 1$. Let $\omega(\xi, \eta)$ be a biharmonic function defined on the half-plane $\eta > 0$. On the edge $\eta = 0$, ω satisfies the conditions

$$\omega(\xi, 0) = \omega_0(\xi), \qquad \omega_{,\eta}(\xi, 0) = \omega_1(\xi) \tag{3.15}$$

where ω_0 and ω_1 are given functions satisfying the conditions

$$\lim_{\xi \to \infty} \frac{1}{\xi^3}\, \omega_0(\xi), \qquad \frac{1}{\xi}\, \omega_1(\xi) = 0. \tag{3.16}$$

Using G_s for G in (1.4) and making appropriate substitutions, we get

$$\omega(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{2\eta^3 \omega_0(\xi')}{[(\xi - \xi')^2 + \eta^2]^2} + \frac{\eta^2 \omega_1(\xi')}{(\xi - \xi')^2 + \eta^2} \right\} d\xi'. \tag{3.17}$$

This is the solution of a semi-infinite cantilever plate subjected to edge displacement and rotation.

For $s = 1$, (3.5) and (3.8) yield

$$h_1(\xi, 0) = -f_1(\xi) - h_{01}(\xi) = 0, \tag{3.18}$$

$$\begin{aligned} h_{1,\eta}(\xi, 0) &= -(f_{,\eta 1} + h_{0,\eta 1}) + (f_{,\xi 1} + h_{0,\xi 1}) b_2 \xi \\ &= -\frac{b_2 4\xi^2}{\xi^2 + 1} \end{aligned} \tag{3.19}$$

where $b_2 = \pm 1$. Using (3.17) we obtain

$$h_1(\xi, \eta) = -4b_2 \frac{\eta^2 + \eta(\xi^2 + \eta^2)}{\xi^2 + (\eta + 1)^2} + A_1(\xi, \eta) \tag{3.20}$$

where A_1 has the same property as A_0 and must again vanish because $\epsilon^3 A_1(\xi, \eta)$ cannot be matched with the outer expansion. Now (3.1) becomes

$$\begin{aligned} \epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \sim \tfrac{1}{2}[\xi^2 + (\eta - 1)^2] \ln \frac{\xi^2 + (\eta - 1)^2}{\xi^2 + (\eta + 1)^2} + 2\eta \\ - \epsilon 4b_2 \frac{\eta^2 + \eta(\xi^2 + \eta^2)}{\xi^2 + (\eta + 1)^2} + O(\epsilon^2). \end{aligned} \tag{3.21}$$

If $b_2 = +1$, this agrees to $O(\epsilon)$ with the expansion G_c^+ given by (1.5). If $b_2 = -1$, it agrees to $O(\epsilon)$ with the expansion of G_c^- for the exterior of the circle $x^2 + (y + 1)^2 = 1$.

For $s > 1$, h_s is just

$$h_s(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{2\eta^3 h_s(\xi', 0)}{[(\xi - \xi')^2 + \eta^2]^2} + \frac{\eta^2 h_{s,\eta}(\xi', 0)}{(\xi - \xi')^2 + \eta^2} \right\} d\xi' + A_s(\xi, \eta) \tag{3.22}$$

where $\fint$ indicates that the integral is to be integrated as a distribution because the integrand may be unbounded at $|\xi'| = \infty$. The function A_s again is regular biharmonic and satisfies the conditions

$$A_s(\xi, 0) = A_{s,\eta}(\xi, 0) = 0. \tag{3.23}$$

It must also satisfy the condition

$$A_s(\xi, \eta) = O(\rho^{s-1}) \quad \text{as} \quad \rho \to \infty \tag{3.24}$$

so that $\epsilon^{s+2}A_s(\xi, \eta)$ can be matched with the outer expansion. It follows from (3.23) and (3.24) that

$$A_s(\xi, \eta) = \sum_{m=2}^{s-1} [C_{sm}C_m(\theta) + S_{sm}S_m(\theta)]\rho^m, \tag{3.25}$$

where C_{sm}, S_{sm} are constants and $S_{s2} = 0$. The sum in (3.25) is taken from $m = 2$ because no biharmonic function of the form $f(\theta)\rho$ can be found to satisfy the required conditions. An immediate consequence is that $A_2 = 0$. This suggests that h_2 can be calculated explicitly. The calculation, however, is not given here.

The constants C_{sm} and S_{sm} are so far undertermined. It is the purpose of matching to determine them. For this purpose we need to know the outer expansion of the inner expansion. To find the behavior of the inner expansion for large ρ, we use (3.22) together with h_0 and h_1 given by (3.13) and (3.20). Then we can prove inductively that

$$h_s(\xi, \eta) = O(\rho^{s-1}), \qquad \rho \to \infty, \qquad s \geq 1, \tag{3.26}$$

and

$$h_s(\xi, \eta) \sim \sum_{m=-\infty}^{s-1} I_{sm}(\theta)\rho^m, \qquad \rho \to \infty, \quad s \geq 1. \tag{3.27}$$

Furthermore

$$\tfrac{1}{2}[\xi^2 + (\eta - 1)^2] \ln [\xi^2 + (\eta - 1)^2] + h_0(\xi, \eta) \sim \sum_{m=-\infty}^{-1} I_{0m}(\theta)\rho^m. \tag{3.28}$$

The fact that h_s is harmonic together with (3.25) implies that

$$I_{sm}(\theta) = (A_{sm} + S_{sm})S_m(\theta) + (B_{sm} + C_{sm})C_m(\theta) + \bar{A}_{sm}\bar{S}_m(\theta) + \bar{B}_{sm}\bar{C}_m(\theta) \tag{3.29}$$

where $S_{s2} = 0$, and $S_{sm} = C_{sm} = 0$ unless $s \geq 2$ and $2 \leq m \leq s - 1$. The constants A_{sm}, B_{sm}, $\bar{A}_{sm}$ and $\bar{B}_{sm}$ are determined by the integral in (3.22).

We now use (3.27) and (3.28) to obtain the expansion of $\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon)$ for ρ large. It is

$$\epsilon^{-2}G(\epsilon\xi, \epsilon\eta, \epsilon) \sim \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} \sum_{m=-\infty}^{s-1} I_{sm}(\theta)\rho^m. \tag{3.30}$$

Next we set $(\xi, \eta, \rho) = \epsilon^{-1/2}(\sigma, \nu, \lambda)$ in (3.30). Then upon rearranging the series, we get

$$\begin{aligned} \epsilon^{-2}G(\epsilon^{1/2}\sigma, \epsilon^{1/2}\nu, \epsilon) \sim \sum_{k=1}^{\infty} \epsilon^{k/2} \sum_{n=0}^{k-1} \frac{1}{n!} [A_{n,2n-k} + S_{n,2n-k})S_{2n-k}(\theta) \\ + (B_{n,2n-k} + C_{n,2n-k})C_{2n-k}(\theta) + \bar{A}_{n,2n-k}\bar{S}_{2n-k}(\theta) + \bar{B}_{n,2n-k}\bar{C}_{2n-k}(\theta)]\lambda^{2n-k}. \end{aligned} \tag{3.31}$$

We note that (3.31) satisfies asymptotically the boundary condition on $\nu = \beta(\sigma, \epsilon^{1/2})$ for all values of C_{mn} and S_{mn}, which are still undetermined.

4. Matching. To match the inner and outer expansions, we note that the left sides of (2.12) and (3.31) are the same. Therefore the right sides must be asymptotically equal. This yields

$$\frac{1}{(p-q)!}\{\bar{A}_{p-q,q}' \text{ or } \bar{B}_{p-q,q}'\} = \frac{1}{p!}\{\bar{A}_{pq} \text{ or } \bar{B}_{pq}\}, \qquad p \geq 0, \quad q \leq p-1, \tag{4.1}$$

$$\frac{1}{(p-q)!}\{A_{p-q,q}' \text{ or } B_{p-q,q}'\} = \frac{1}{p!}\{A_{pq} \text{ or } B_{pq}\}, \qquad p \geq 0, \quad q \leq 1, \tag{4.2}$$

$$\frac{1}{(p-q)!}A_{p-q,q}' = \frac{1}{p!}(A_{pq} + S_{pq}), \qquad p > 2, \quad 2 \leq q \leq p-1, \tag{4.3}$$

$$\frac{1}{(p-q)!}B_{p-q,q}' = \frac{1}{p!}(B_{pq} + C_{pq}), \qquad p > 2, \quad 2 \leq q \leq p-1. \tag{4.4}$$

The constants A, B, $\bar{A}$ and $\bar{B}$ are known, and hence (4.1) and (4.2) yield the corresponding A', B', $\bar{A}'$ and $\bar{B}'$. It remains to be shown that the constants A' and B' associated with (4.3) and (4.4) can be determined from the outer expansion. Then C_{pq} and S_{pq} can be found from (4.3) and (4.4) to complete the inner expansion.

The function G_n defined by (2.3) can be written as

$$G_n(x, y) = K_n(x, y) + H_n(x, y) \tag{4.5}$$

where

$$K_n(x, y) = \sum_{m=-n}^{-1} J_{nm}(\theta)r^m, \qquad H_n = \sum_{m=0}^{\infty} J_{nm}(\theta)r^m \tag{4.6}$$

and J_{nm} is defined by (2.4). We note that the coefficients involved in K_n are completely determined by (4.1) and (4.2).

Let $R_n(x, y)$ be a regular biharmonic function defined by

$$\nabla^4 R_n = 0, \qquad (x, y) \in D, \tag{4.7}$$

$$R_n = -K_n, \qquad \partial_n R_n = -\partial_n K_n, \qquad (x, y) \in \partial D, \qquad r \neq 0. \tag{4.8}$$

The functions R_n, $n \geq 1$, are well defined and can be determined by solving the series of boundary-value problems defined by (4.7) and (4.8). The following lemma enables us to determine the unknown coefficients involved in H_n in terms of the coefficients in the expansions of R_n.

LEMMA. $R_n(x, y) = H_n(x, y)$.

Proof. This follows by setting $G_n = K_n + R_n$ and noting that G_n, together with its normal derivative, vanishes on ∂D.

With this lemma, the left-hand sides of (4.3) and (4.4) are completely determined. The coefficients C_{pq} and S_{pq} can now be found.

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