

# QUARTERLY

OF

# APPLIED MATHEMATICS

EDITED BY

H. W. BODE  
P. J. DAVIS  
E. T. ONAT  
J. L. SYNGE

G. F. CARRIER  
D. C. DRUCKER  
I. S. SOKOLNIKOFF

HIRSH COHEN  
U. GRENANDER  
P. S. SYMONDS

W. F. FREIBERGER, *Managing Editor*

EDITED WITH THE COLLABORATION OF

M. A. BIOT  
C. FERRARI  
G. E. HAY  
E. REISSNER  
J. J. STOKER

J. M. BURGERS  
P. GERMAIN  
P. LE CORBEILLER  
S. A. SCHELKUNOFF

H. W. EMMONS  
J. A. GOFF  
F. D. MURNAGHAN  
W. R. SEARS

FOUNDER, AND  
MANAGING EDITOR 1943-1965  
W. PRAGER

---

VOLUME XXXIV

APRIL • 1976

NUMBER 1

---



# QUARTERLY OF APPLIED MATHEMATICS

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with applications. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts (two copies) submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to the Editorial Office, Box F, Brown University, Providence, R. I. 02912, either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to scientists or engineers. Authors will receive galley proofs only. The authors' institution will be requested to pay a publication charge of \$25.00 per page which, if honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

The 1976 subscription price for Volume 34 (April 1976-January 1977) is \$25.00. Single issues can be purchased, as far as they are available, at \$8.00 and back volumes at \$25.00 per volume. Subscriptions and orders for back volumes must be addressed to: American Mathematical Society, P.O. Box 1571, Providence, R. I. 02901. All orders must be accompanied by payment. Other subscription correspondence should be addressed to American Mathematical Society, P.O. Box 6248, Providence, R. I. 02940.

---

---

Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia

---

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA



# SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

**Manuscripts:** Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

**Titles:** The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

**Mathematical Work:** As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent  $\frac{1}{2}$  rather than with the sign  $\sqrt{\phantom{x}}$ .

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x / 2 b)}{\cos (\pi a / 2 b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2 b}}{\cos \frac{\pi a}{2 b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

**Cuts:** Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

**Bibliography:** References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (for books—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; for periodicals—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

**Footnotes:** As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

**Abbreviations:** Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.



## CONTENTS

EARL R. BARNES: A variational problem arising in the design of cooling fins	1
Y. V. VENKATESH: Passivity and linear system stability	19
CHANG-YI WANG: Symmetric viscous flow between two rotating porous discs—moderate rotation	29
CHIEN-HENG WU: Near-boundary expansion of Green's function associated with clamped plates	39
T. A. PORSCHING: Network problems, $M$ -functions and uniformly monotone networks	47
K. S. HAVNER AND H. P. PATEL: On convergence of the finite-element method for a class of elastic-plastic solids	59
A. E. DEGANCE AND L. E. JOHNS: The stability of vector renewal equations pertaining to heterogeneous chemical reaction systems	69
ISMAEL HERRERA AND JACOBO BIELAK: Dual variational principles for diffusion equations	85
NOTES:	
MICHAEL D. GREENBERG: Limit cycle city	103
P. T. BLOTTER AND D. H. Y. YEN: Ultraspherical polynomials applied to nonlinear vibrations of continuous media	106
D. TJØSTHEIM AND J. B. THOMAS: Linear time-invariant transformations of some nonstationary random processes	113
P. L. SACHDEV: Some exact solutions of Burgers-type equations	118
BOOK REVIEW SECTION:	
PETER HENRICI: <i>Applied and computational complex analysis</i>	123
R. P. BOAS	
V. A. YAKUBOVICH AND V. M. STARZHINSKII: <i>Linear differential equations with periodic coefficients</i>	123
JACK K. HALE	
E. SKUDRZYK: <i>The foundations of acoustics</i>	123
PHILIP M. MORSE	
E. T. COPSON: <i>Partial differential equations</i>	124
C. M. DAFERMOS	
BOOKS RECEIVED	125



## —BOOKS RECEIVED—

Notice in this section does not preclude later full reviews in the Book Reviews Section.

*Fourier analysis in probability theory.* By Tatsuo Kawata. Academic Press, 1972. xii + 668 pp. \$32.00.

The first part of this book presents the elements of classical Fourier analysis in the context of their applications to probability theory. The second part covers basic results from the theory of characteristic functions of probability distributions, the convergence of distribution function in terms of characteristic functions, and series of independent random variables.

*General equilibrium.* By Michael Allingham. Halsted Press (a division of John Wiley & Sons), 1975. viii + 113 pp. \$19.95.

This book provides a systematic exposition of the theory of general economic equilibrium. The treatment is self-contained and assumes no prior knowledge of economics or—beyond the most rudimentary ideas—of mathematics.

*The elements of real analysis.* By Robert G. Bartle. Second edition, John Wiley & Sons, 1976. ix + 480 pp. \$16.95.

This is a new and thoroughly revised edition of the textbook first published in 1974, and is based on the author's experience in teaching the subject at the University of Illinois.

*Applied algebra for the computer sciences.* By Arthur Gill. Prentice-Hall, Inc., 1975. xv + 432 pp. \$16.50.

The book covers, in a mathematically rigorous manner, results, techniques and applications of modern algebra that are of particular use to beginning students in the computer sciences. The chapter headings are: Sets; Relations; Functions; Algebraic systems; Propositions; Lattices and Boolean algebras; Combinatorial and sequential networks; Languages and automata; Groups, rings and fields; Codes; Graphs.

*Multivariate statistical methods: within-groups covariation.* Edited by Edwin H. Bryant and William R. Atchley. A book in the series *Benchmark papers in systematic and evolutionary biology*. Dowden, Hutchinson & Ross, Inc.; distributed by Halsted Press, a division of John Wiley & Co. xv + 435 pp. \$26.00.

This volume contains reprints of classical papers (concerning theory and biological applications) on Principal-Components Analysis (K. Pearson, Hotteling, C. R. Rao, Jolicoeur and Mosimann, Gittins); Factor Analysis (Thurstone, Cattell, C. R. Rao, McDonald, Kaiser, Cattell and Muerte, Jennrich and Sampson, McDonald and Burr, Stroud, Reyment, Wallace and Bader, Atchley); Multivariate Regression and Correlation (Bartlett, Hotelling, Glahn, Power, Vogt and Jameson, Lee).

The editors introduce each group of papers with historical comments and include a short bibliography of additional references on theory and applications.



*Multivariate statistical methods: among-groups covariation.* (For editors, publisher and general comments see preceding notice.) xv + 464 pp. \$27.00.

This volume contains papers on Multivariate Analysis of Variance (Hotelling, Roy and Gunanadesikan, Gabriel, Reyment, Sokal and Heryford and Kishpaugh, M. M. Goodman), Discriminant Analysis (Fisher, Mahalanobis, C. R. Rao, Bryan, Porebski, Jolicoeur, Baker and Atchley and MacDaniel, Burnaby, Green), Cluster Analysis (Sokal and Michener, Williams and Lambert, Gower, Rohlf, Williams and Lambert and Lance, M. M. Goodman, Sokal and Rohlf, Farris, Jardine and Sibson); Non-metric Scaling (Kruskal, Rohlf).

Both the above volumes also contain the same introductory paper by Rohlf on the application of multivariate analysis to taxonomy.

*The mathematics of diffusion.* By J. Crank. Second edition. Oxford University Press, 1975. viii + 414 pp. \$39.00.

This book provides a collection of methods for the solution of the differential equations of diffusion—for infinite and semi-infinite media; the plane sheet, cylinder and sphere; for concentration-dependent diffusion; moving boundaries; and other situations. New chapters in this edition include one on non-Fickian (or anomalous) diffusion and one on heterogeneous media. Other chapters, such as that on numerical methods, have been extensively revised.

*The mathematical theory of diffusion and reaction in permeable catalysts.* Vol. I: *The theory of the steady state.* By Rutherford Aris. Oxford University Press, 1975. vii + 444 pp. \$39.50.

The two volumes of this work describe the mathematical theory of the interaction of diffusion and reaction and its application to problems in chemical engineering and biology. The first volume is a comprehensive survey of the steady state in a catalytic body. The second volume will study uniqueness, stability and transient behavior of reacting systems in which diffusion plays a role.

*Classical electrodynamics.* By J. D. Jackson. Second Edition. John Wiley & Sons, 1975. xxii + 848 pp. \$20.95.

This work provides an advanced treatment of electromagnetic theory for graduate students and professional physicists. A selection of chapter headings is: boundary-value problems of electrostatics, magnetostatics, Maxwell's equations, plane e.m. waves, wave guides, scattering, diffraction, magnetohydrodynamics and plasma physics, special theory of relativity, dynamics of relativistic particles and e.m. fields, collisions between charged particles, radiation by moving charges, bremsstrahlung, multipole fields, radiation damping.

Many chapters have been extensively rewritten for this edition to include recent theoretical and practical applications and 40% of the problems are new.

*Methods of optimization.* By G. R. Walsh. John Wiley & Sons, 1975. x + 200 pp. \$19.95.

This book is an introductory text concerned with the theory and practice of modern methods of maximizing or minimizing a function of many variables, with or without constraints, showing the development of these methods from classical calculus and from mathematical programming techniques.



*Computational methods in linear algebra.* By R. J. Gault, R. F. Hoskins, J. A. Milner, and M. J. Pratt. Halsted Press (a division of John Wiley & Sons), 1975. vii + 204 pp. \$11.95.

This book is designed for engineers and scientists, with the object to provide a reasonably comprehensive survey of numerical methods of linear algebra, with just as much theory as is needed for their proper use. The methods described are all appropriate for digital computers and include: elimination, decomposition and iterative methods for the solution of linear equations, and the Householder, Givens, Hessenberg reduction, and QR-methods for finding eigenvalues and eigenvectors, all with their error analyses.

*Statistical prediction analysis.* By J. Aitchison and I. R. Dunsmore. Cambridge University Press, 1975. xi + 273 pp. \$24.50.

This book regards statistical prediction analysis in a wide sense, to include any form of statistical analysis where consideration of what may happen in the future is essential to the formulation of the problem. The frequency approach and the Bayesian approach are each employed where appropriate. The emphasis is on many examples from practical life, such as: medical prognosis, machine replacement, quality control, process optimization and regulation, biological assay, diagnosis and treatment allocation. The theory and application of statistical tolerance regions are emphasized. Many problems for the reader are presented at the end of each chapter.

*An introduction to applied mathematics.* By J. C. Jaeger and A. M. Starfield. Second Edition. Oxford University Press, 1974. xii + 504 pp. \$27.25.

New material in this edition includes greater emphasis on numerical methods and modern application areas, such as semiconductor circuits, servomechanisms, biological sciences and communication theory. The topics treated encompass: difference equations, ordinary and partial differential equations, electric circuit theory, particle and rigid dynamics, Lagrange's equations, boundary value problems, Fourier series and integrals, matrices, and numerical methods.



## —BOOK REVIEW SECTION—

*Applied and computational complex analysis*, Volume I. By Peter Henrici. John Wiley & Sons, New York, 1974. xi + 686 pp. \$24.95.

This intriguing text is rather different in content, and much different in spirit, from other books on complex analysis, and especially from other books with similar titles. It does not scant the theory in order to get to the applications; and on many applications it gives an unusual amount of detail. Its most striking feature is the principle "not to consider a problem solved unless an algorithm for constructing the solution has been found," a principle which leads the author, in particular, to point out computational difficulties as they arise and give error estimates where needed. It is, for example, instructive to see how analytic continuation by means of power series can actually be carried out, and sobering to see how difficult it is to calculate the winding number of a curve.

This first volume covers basic material but in an unconventional arrangement, also conformal mapping and its applications and (in great detail) the calculation of zeros of polynomials. Two more volumes are promised: one each on topics connected, respectively, with ordinary and partial differential equations.

The first volume begins with formal power series and formal Laurent series, then turns to functions analytic at a point (that is, represented by a power series in a neighborhood of the point). Analytic continuation follows; then residues, Cauchy's formula, and general theorems about zeros. There is a thorough treatment of conformal mapping, including the refinement of Schwarz-Christoffel maps by rounding off the corners.

The final third of the book is in effect a substantial monograph on zeros of polynomials and poles of rational functions, treated algorithmically. No one who is not already as expert as the author should presume to try to calculate a zero without consulting these chapters.

Henrici presents a number of results in simpler or more elegant forms than are usually given, as well as some formulas and methods that have fallen into disuse but now, in the day of high-speed computers, have become quite practical. The algorithmic approach contributes substantial insights. Anyone who takes instruction in complex analysis seriously should consult this book.

R. P. BOAS (*Evanston, Ill.*)

*Linear differential equations with periodic coefficients* (2 vols.). By V. A. Yakubovich and V. M. Starzhinskii. English translation. John Wiley and Sons, 1975. 839 pp. \$49.50.

This is certainly the most comprehensive treatment of linear differential equations with periodic coefficients that is available. The classical as well as the modern aspects of the subject are presented. All results are proved in detail except for the topological characterization of the stable Hamiltonian systems. This choice was wise because the book in its present form is readily accessible and very valuable to researchers in engineering and the physical sciences as well as mathematicians.

JACK K. HALE (*Providence*)

*The foundations of acoustics*. By E. Skudrzyk. Springer-Verlag, Berlin-Heidelberg-New York-Wien, 1972. xxv + 79 pages. \$75.80.

The author has set out to produce a self-contained course of instruction in linear acoustics. He starts with chapters on the relevant mathematics: complex numbers, analytic functions and contour integrals,

Fourier and Laplace transforms, delta and step functions, correlation functions and transmission factors, probability theory and its application to signal processing. Two-fifths of the text, a dozen chapters, is thus occupied before the subject of sound is introduced.

The next four chapters cover in some detail the reflection of a plane wave of sound and its propagation through ducts and horns. Here linear boundary conditions are assumed and the reflection and absorption are related to the acoustic point impedance of the surface—a satisfactory approximation for media of low viscosity. The chapter on ducts and horns considers only those with rigid walls, but deals with various terminations and junctions.

Next comes five chapters on linear sound waves in spherical, cylindrical and spheroidal coordinates, with applications to radiation and scattering processes. Quite a number of special cases are worked out in detail and the results are displayed graphically.

The final quarter of the text takes up the integral equation method of solution, using the Helmholtz-Huygens method. Here the chief applications are to diffraction problems and to the calculation of the radiation from various configurations of point sources, pistons and membranes and of the acoustic impedance of such sources. Again, many solutions are displayed in graphical form. At the end of the book are tables of relevant mathematical formulas and a compendious bibliography.

If one wishes to learn about the calculable cases of linear acoustic theory, this tome is one answer

PHILIP M. MORSE (*Winchester, Mass.*)

*Partial differential equations.* By E. T. Copson. Cambridge University Press, 1975.  
vii + 280 pp. Cloth \$27.50, Paper \$9.95.

The study of partial differential equations has made considerable progress in the last 30 years and, as a result, the methodology itself of the subject has radically changed. Even so, the classical aspects of the qualitative theory should be familiar to the serious student of the field. The content, style and notation of Copson's book have a strong classical flavor. The following topics are covered: equations of first order; the Cauchy-Kowalewsky theorem and the theory of characteristics; Riemann's method and hyperbolic equations of second order; classical potential theory in two variables, subharmonic functions and barriers; elliptic second order equations in two and three variables; the equation of heat conduction. The book also contains a fairly detailed exposition of M. Riesz's theory of integrals of fractional order and its application to hyperbolic equations, a subject not usually discussed in elementary texts of Differential Equations.

The book is dedicated to the memory of the author's father-in-law, Professor Sir Edmund Whittaker, F.R.S.

C. M. DAFERMOS (*Providence*)