

ORTHOGONALITY RELATIONS FOR A CLASS OF LINEAR VIBRATION PROBLEMS*

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1. Introduction. Consider a linear conservative elastic system which has been modeled as an elastic continuum with distributed mass occupying the closure $\bar{\Omega}$ of an open region Ω, and having a lumped mass m attached to a movable point of the boundary $\bar{\Omega} - \Omega$. Denoting the unstressed position of a point in the continuum by $x \in \Omega$, the motion $\eta(x, t)$ usually can be determined by a modal analysis; in particular natural frequencies and normal modes are found by solving an eigenvalue problem of the form

$$\begin{aligned} L_\lambda u(x) &= 0, \qquad x \in \Omega, \\ B_\lambda u(x) &= 0, \qquad x \in \bar{\Omega} - \Omega, \end{aligned} \tag{1.1}$$

where L_λ is a formally self-adjoint differential operator parametrized by λ. However, $m \neq 0$ implies that the boundary condition $B_\lambda u(x) = 0, x \in \bar{\Omega} - \Omega$, is also parameterized by λ; this in turn implies that (1.1) is not a Sturm-Liouville problem and the normal modes u_p usually are not orthogonal in the usual sense. Hence, although it is usually possible to provide a formal solution of the form

$$\eta(x, t) = \sum_{p=1}^{\infty} (b_p \sin \omega_p t + a_p \cos \omega_p t) u_p(x), \qquad x \in \Omega, \qquad t \geq 0, \tag{1.2}$$

it may be less than obvious how the complex numbers b_p , a_p can be found explicitly in terms of given initial data on $\eta(x, 0)$, $(\partial/\partial t)\eta(x, 0)$.

To illustrate this situation consider a uniform elastic bar in longitudinal motion, having a lumped mass m attached to the free end. The initial-boundary-value problem is described by

$$\begin{aligned} \mu \frac{\partial^2}{\partial t^2} \eta(x, t) &= s \frac{\partial^2}{\partial x^2} \eta(x, t), \qquad x \in (0, l), \qquad t > 0, \\ \eta(0, t) = s \frac{\partial}{\partial x} \eta(l, t) &+ m \frac{\partial^2}{\partial t^2} \eta(l, t) = 0, \qquad t > 0, \end{aligned} \tag{1.3}$$

with initial data $\eta(x, 0) = f_0(x)$, $(\partial/\partial t)\eta(x, 0) = 0$, for $x \in (0, l)$. The eigenvalue problem for the natural frequencies ω_p and normal modes $u_p(x)$ is

$$\begin{aligned} su''(x) + \mu\omega^2 u(x) &= 0, \qquad x \in (0, l), \\ u(0) &= 0, \\ su'(l) - \omega^2 m u(l) &= 0, \end{aligned} \tag{1.4}$$

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and we obtain

$$u_p(x) = \sin\left(\omega_p\sqrt{\frac{\mu}{s}}\,x\right),$$
$$m\omega_p = (\mu s)^{1/2}\cot\left(\omega_p l\sqrt{\frac{\mu}{s}}\right), \qquad p = 1, 2, \cdots \tag{1.5}$$

The normal modes $u_p(x)$ span $\mathfrak{L}_2(0, l)$, the real linear vector space of functions which are Lebesgue square-integrable on $(0, l)$. Hence, for any continuous function f_0 satisfying $f_0(0) = 0$, we have

$$\eta(x, t) = \sum_{p=1}^{\infty} a_p u_p(x)\cos\omega_p t, \qquad x \in \Omega, \tag{1.6}$$

where $f_0(x) = \sum_{p=1}^{\infty} a_p u_p(x)$. $x \in \Omega$. Unfortunately, for $m \neq 0$ the u_p are not orthogonal in the $\mathfrak{L}_2$ sense; in general,

$$\int_0^l u_p(x)u_q(x)\,dx \neq 0 \quad \text{for} \quad p \neq q.$$

Hence, it is only for $m = 0$ that we may compute a_p as

$$a_p = \int_0^l u_p(x)f_0(x)\,dx \Big/ \int_0^l u_p^2(x)\,dx.$$

It is the lack of an obvious expression for a_p when $m \neq 0$ that motivates the present work.

Our approach will be to present an abstract mathematical description for a general class of linear conservative elastic systems, and to show that the normal modes for this model do have certain orthogonality properties; we then show that the type of problem just discussed can be viewed in this abstract form and explicit expressions for the coefficients a_p and b_p can be obtained by use of the orthogonality properties of the modes of the abstract system.

I wish to thank my colleague S. Raynor for pointing out that orthogonality relations for this type of problem do not appear to be available in the existing literature on vibration.

2. Abstract linear conservative elastic system. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle_{\mathcal{H}}$ and let $M : \mathcal{H} \to \mathcal{H}$, $K : \mathfrak{D}(K) \to \mathcal{H}$, be positive symmetric linear operators with $\mathfrak{D}(K)$ dense in $\mathcal{H}$. Consider the abstract evolution equation

$$M\ddot{y}(t) + Ky(t) = 0, \qquad t > 0, \tag{2.1}$$

where a mapping $y : [0, \infty) \to \mathfrak{D}(K)$ will be called (here) a solution if it is twice-differentiable and satisfies (2.1). Let us suppose that the eigenvalue problem

$$Kg = \omega^2 Mg, \qquad g \in \mathfrak{D}(K), \tag{2.2}$$

has a set of independent solutions $\{\omega_p^2, g_p\}$. The symmetry of M and K implies that the null space $\mathfrak{N}_p$ of $K - \omega_p^2 M$ is orthogonal to $\mathfrak{N}_q$, relative to both M and K, for $\omega_p^2 \neq \omega_q^2$; that is, for $y_1 \in \mathfrak{N}_p$, $y_2 \in \mathfrak{N}_q$, $\omega_p^2 \neq \omega_q^2$, we have

$$\langle y_1, My_2\rangle_{\mathcal{H}} = 0, \qquad \langle y_1, Ky_2\rangle_{\mathcal{H}} = 0.$$

Hence, it follows that either the set $\{g_p\}$ is also mutually orthogonal in this sense (if $(\dim \mathfrak{N}_p) = 1$ for every p), or can be chosen to be mutually orthogonal in this sense (if $(\dim \mathfrak{N}_p) > 1$ for some p). Therefore, the set $\{g_p\}$ either is, or can be chosen to be, such that

$$\langle g_p , Mg_q \rangle_{\mathcal{H}} = 0 \quad \text{for} \quad p \neq q. \tag{2.3}$$

Moreover, it is clear that $(\cos \omega_p t)g_p$ and $(\sin \omega_p t)g_p$ are solutions of (2.1), and g_p may be thought of as a normal mode. Hence, the orthogonality relation (2.3) may be useful in determining the coefficients a_p, b_p in (1.2), provided the system can be viewed in the abstract form (2.1). How this can be done is demonstrated in the next section.

3. Example. Let us consider the specific problem (1.3) discussed in the introduction and define $\mathcal{H} = \mathcal{L}_2(0, l) \times \mathcal{R}^1$, where $\mathcal{R}^1$ is the real line. Denoting $y \in \mathcal{H}$ by the pair (η, z), $\eta \in \mathcal{L}_2(0, l)$, $z \in \mathcal{R}^1$, we define

$$\langle y_1 , y_2 \rangle_{\mathcal{H}} = \int_0^l \eta_1(x)\eta_2(x)\, dx + z_1 z_2 , \qquad y_1 , y_2 \in \mathcal{H},$$

$$\mathfrak{D}(K) = \{y \in \mathcal{H} \mid \eta, \eta', \eta'' \in \mathcal{L}_2(0, l), \eta(0) = 0, \eta(l) = z\},$$

$$Ky = (-s\eta'', s\eta'(l)), \qquad y \in \mathfrak{D}(K),$$

$$My = (\mu\eta, mz), \qquad y \in \mathcal{H}.$$

We verify that M and K are symmetric:

$$\langle y_1 , My_2 \rangle_{\mathcal{H}} = \int_0^l \eta_1(x)\mu\eta_2(x)\, dx + z_1 m z_2$$

$$= \langle y_2 , My_1 \rangle_{\mathcal{H}} \quad \text{for all} \quad y_1 , y_2 \in \mathcal{H},$$

$$\langle y_1 , Ky_2 \rangle_{\mathcal{H}} = -\int_0^l \eta_1(x)s\eta_2''(x)\, ds + z_1 s\eta_2'(l)$$

$$= -[\eta_1(x)s\eta_2'(x) - \eta_1'(x)s\eta_2(x)]_0^1 - \int_0^1 \eta_1''(x)s\eta_2(x)\, dx + z_1 s\eta_2' l$$

$$= \eta_1'(l)sz_2 - \int_0^1 \eta_1''(x)s\eta_2(x)\, dx$$

$$= \langle y_2 , Ky_1 \rangle_{\mathcal{H}} \quad \text{for all} \quad y_1 , y_2 \in \mathfrak{D}(K).$$

Hence, (2.1) provides a suitable abstract description of (1.3). Using u_p and ω_p as given by (1.5), we have

$$Kg_p = \omega_p^2 Mg_p , \qquad p = 1, 2, \cdots ,$$

where $g_p = (u_p , u_p(l))$.

Hence, each of the null spaces $\mathfrak{N}_p$ has dimension one and $\{g_p\}$ is a mutually orthogonal set relative to M; that is

$$\langle g_p , Mg_q \rangle_{\mathcal{H}} = \int_0^1 \mu u_p(x)u_q(x)\, dx + mu_p(l)u_q(l) = 0 \quad \text{for} \quad p \neq q.$$

This orthogonality relation now furnishes an explicit solution for (1.3):

$$\eta(x, t) = \sum_{p=1}^{\infty} c_p u_p(x) \cos \omega_p t,$$

$$c_p = \left(\int_0^l \mu f_0(x) u_p(x)\, dx + m f_0(l) u_p(l)\right) \Big/ \left(\int_0^l \mu u_p^2(x)\, dx + m u_p^2(l)\right).$$

Remarks on the abstract formulation. Eq. (2.1) is a specialization of a relatively simple abstract formulation previously found quite useful for investigating several other questions regarding linear elastic systems, nonconservative as well as conservative [1, 2]. A question arises as to whether the formulation (2.1) is also appropriate for studying questions of existence and uniqueness of solutions, and for assuring the continuous dependence of $(y(t), \dot{y}(t))$ on the initial data $(y(0), \dot{y}(0))$ in the induced norm of $\mathcal{H} \times \mathcal{H}$. That is, does (2.1) also serve to define a C_0-semigroup on $\mathcal{H} \times \mathcal{H}$ [3]? In general, the answer is negative unless $\mathcal{H}$ is finite-dimensional; i.e., unless the system does not involve distributed parameters.

This apparently disappointing conclusion suggests that perhaps (2.1) should be replaced by a more sophisticated abstract equation, such as that used by Dafermos [4] and Lions [5]. However, for the present purpose this does not seem necessary; in [6] a simple abstract formulation such as (2.1) is referred to as "preliminary", and it is shown therein that such a "preliminary" formulation can be used to derive a family of closely related abstract evolution equations. Each of the latter equations then serves to define a C_0-semigroup on an appropriate state space, replacing $\mathcal{H} \times \mathcal{H}$. Hence, reference [6] implies that the simple formulation (2.1) is also useful as a preliminary step in the investigation of questions which are considerably more delicate than the question of orthogonality; therefore, the relative simplicity of (2.1) seems to justify its present usage.

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