QUARTERLY OF APPLIED MATHEMATICS
JULY, 1976

NUMERICAL STUDY OF PERIODIC HAMILTONIAN SYSTEMS BY MEANS OF ASSOCIATED POINT MAPPINGS

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Introduction. This paper is concerned with the study of systems described by second-order differential equations derived from a Hamiltonian with periodically varying coefficierts

$$\frac{dx}{dt} = \frac{\partial H(x, y, t)}{\partial y}; \qquad \frac{dy}{dt} = -\frac{\partial H(x, y, t)}{\partial x}. \tag{1}$$

Among the problems which lead to the study of equations such as (1) can be found, for example, the study of the oscillations of a pendulum under parametric excitation [1], that of the oscillations of a satellite under the effect of periodic variations in its moments of inertia [2] and that of the motion of particles in alternating gradient accelerators [3]. In conservative dynamic systems with two degrees of freedom, studying movements around a periodic solution can amount to studying solutions of (1) around the origin [4]. Since Poincaré [5] it has been known that a point mapping linking the state (x, y) of the system between two instants separated by a time interval can be associated with (1). Seeking this transformation $\mathfrak{T}$ can be justified by the fact that is it particularly suited to numerical treatment. The search for periodic solutions (sub-harmonic) amounts to determining the fixed points of the transformations $\mathfrak{T}^n$, n integer (positive) which in numerical terms is the solution of a two-variable, nonlinear system of algebraic equations. In actual practice, it is not possible, except for a few particular cases, to express the relationships which define $\mathfrak{T}$ by means of classical functional analysis. Thus, it is the aim of this paper to present a method which makes it possible to determine a transformation $\mathfrak{T}^*$ enabling one to carry out an approximate (in a sense which will be defined later and justified on the examples given) analysis of the solutions of the differential system (1). This method can be situated mid-way between that of direct simulation by numerical integration, which poses the problem of sensitivity to discretization, particularly crucial in the case of conservative equations, and the many analytical methods such as those based on successive approximations to determine asymptotic expansions of the solution when the nonlinear terms depend on a small parameter [1].

Before outlining the plan of the paper, it is necessary to give the essential characteristics of the transformation $\mathfrak{T}$ associated with (1). If the Hamiltonian H is analytical with respect to the variables x, y, then the associated transformation $\mathfrak{T}$ is itself analytical and one-to-one and in addition it is area-preserving; the Jacobian, the determinant of the partial derivatives matrix, is 1.

* Received December 11, 1974.

In the first part, an algorithm is given enabling one to determine the coefficients (exact) of the first terms of the series expansion of the expressions defining the mapping $\mathfrak{I}$. This is based on an idea found in Birkhoff [6] and also in Lewis [7]. The idea consists of associating a differential system with the transformation. This idea has been used numerically here in order to establish the transformation $\mathfrak{I}_C$, obtained from $\mathfrak{I}$ by truncation.

It is possible to get some information about the nature and stability of the trivial solution (0, 0) of (1) from this transformation $\mathfrak{I}_C$; however, it fails to give an adequate idea of the behavior of solutions in a wide neighbourhood of the origin for the following fundamental reason: the truncation $\mathfrak{I}_C$ is not generally area-preserving. As a result, the singular center type points for $\mathfrak{I}$ can be transformed into stable or unstable focuses by $\mathfrak{I}_C$. In the second part of the paper, it is shown how to construct, from $\mathfrak{I}_C$ a pointwise transformation $\mathfrak{I}^*$ which:

1. has the same first terms as $\mathfrak{I}_C$ (therefore as $\mathfrak{I}$);
2. is area-preserving ($J = 1$).

Finally, the third part is devoted to numerical tests carried out on three different examples in order to judge the quality of such a simulation by comparing the results provided by $\mathfrak{I}^*$ with those given by $\mathfrak{I}$ (in a case where it is possible to determine it), by numerical integration, and also with those obtained analytically (asymptotic expansions method).

1. Determination of $\mathfrak{I}_C$ (First terms of the series expansions of $\mathfrak{I}$)

1.1. *Point mapping and formal differential system.* Consider the following pointwise transformation T:

$$x_1 = \rho x + \sum_{m,n} \varphi_{mn} x^m y^n$$
$$y_1 = \rho^{-1} y + \sum_{m,n} \psi_{mn} x^m y^n \qquad m + n \geq 2 \tag{2}$$

where the coefficients ρ, φ_{mn} , ψ_{mn} are real, or complex (in this case ψ_{mn} is complex conjugate of φ_{mn}).

We have limited ourselves to the case where the linear part of the pointwise transformation could be diagonalized; however, this is not essential and the following calculations can be adapted to any case [8]. In [6] Birkhoff showed that the series which define T^k, i.e.

$$x_k = \rho^k x + \sum_{m,n} \varphi_{mn}{}^{(k)} x^m y^n$$
$$y_k = \rho^{-k} y + \sum_{m,n} \psi_{mn}{}^{(k)} x^m y^n, \qquad m + n \geq 2 \tag{3}$$

satisfy the following differential equations:

$$dx_k/dk = U(x_k , y_k); \qquad dy_k/dk = V(x_k , y_k)$$
$$U(x, y) = \left[\frac{\partial x_k}{\partial k}\right]_{k=0}; \qquad V(x, y) = \left[\frac{\partial y_k}{\partial k}\right]_{k=0} \tag{4}$$

with the initial conditions $x_0 = x$, $y_0 = y$ for $k = 0$. Inversely, the series x_k , y_k are determined uniquely by these equations (5) and initial conditions.

Strictly speaking, this association is made from a purely formal point of view. In fact, whatever transformation of the type (2) is taken, the association can only be made by means of certain hypotheses regarding the coefficients ρ (and ρ^{-1}). This problem was solved by Lewis [7]. Finally, it is not essential that the series $U(x, y)$, $V(x, y)$ should converge; this would imply that the system was completely integrable (first analytic integral), which is clearly not the general case [9]. This problem is not of course crucial here, where it is in a sense the opposite step which is being taken: differential system $\rightarrow$ transformation.

It is now clear how to determine the coefficients of the pointwise transformation $\mathfrak{T}$. First, suppose that the periodic coefficients which appear in the differential equation (1) are piecewise constant. On each constant level, (1) has therefore the form of an autonomous system. By identifying it with the system (4) the coefficients φ_{mn}, ψ_{mn} are obtained directly in terms of those of the differential equation (1). Some details of the calculations are given below. While quite simple, they are relatively tedious; now that this work has been done once and for all, however, it can be easily programmed on a numerical computer.

1.2. *Calculation of the coefficients* φ_{mn}, ψ_{mn}. The series (3) which provide the differential system (4) is determined by writing $T^{k+1} = T(T^k)$; i.e.,

$$x_{k+1} = \rho^{k+1}x + \sum_{m,n} \varphi_{mn}{}^{(k+1)}x^m y^n = \rho x_k + \sum_{m,n} \varphi_{mn} x_k{}^m y_k{}^n$$

$$y_{k+1} = \rho^{-(k+1)}y + \sum_{m,n} \psi_{mn}{}^{(k+1)}x^m y^n = \rho^{-1} y_k + \sum_{mn} \psi_{mn} x_k{}^m y_k{}^n. \tag{5}$$

Putting (3) into (5) and identifying in (5) the x and y terms of the same order, we obtain a system of linear difference equations which make it possible to determine the coefficients $\varphi_{mn}{}^{(k)}$ and $\psi_{mn}{}^{(k)}$. For the sake of brevity, the discussion will be limited to the coefficients $\varphi_{20}{}^{(k)}$ and $\varphi_{30}{}^{(k)}$. The results of the calculations for the other terms are given in Appendix I.

$$\begin{aligned} \varphi_{20}{}^{(k+1)} &= \rho\varphi_{20}{}^{(k)} + \rho^{2k}\varphi_{20} \\ \varphi_{30}{}^{(k+1)} &= \rho\varphi_{30}{}^{(k)} + 2\varphi_{20}\rho^k\varphi_{20}{}^{(k)} + \varphi_{11}\rho^k\psi_{20}{}^{(k)} + \rho^{3k}\varphi_{30} \end{aligned} \tag{6}$$

with initial conditions

$$\varphi_{mn}{}^{(1)} = \varphi_{mn}$$

we get:

$$\begin{aligned} \varphi_{20}{}^{(k)} &= \frac{\rho^{2k} - \rho^k}{\rho^2 - \rho}\varphi_{20} \\ \varphi_{30}{}^{(k)} &= \left(\varphi_{30} - \frac{2\varphi_{20}{}^2}{\rho - \rho^2} - \frac{\varphi_{11}\psi_{20}}{\rho^{-1} - \rho^2}\right)\frac{\rho^{3k} - \rho^k}{\rho^3 - \rho} \\ &\qquad + \frac{2\varphi_{20}{}^2}{\rho - \rho^2}\cdot\frac{\rho^{2k} - \rho^k}{\rho^2 - \rho} + \frac{\varphi_{11}\psi_{20}}{\rho^{-1} - \rho^2}\frac{1 - \rho^k}{1 - \rho}. \end{aligned} \tag{7}$$

Eqs. (7) and those given in Appendix I make it possible to write the differential system (4), which here is in the form:

$$\begin{aligned} dx_k/dk &= \log \rho x_k + \sum_{m,n} \frac{d}{dk}\,[\varphi_{mn}{}^{(k)}]_{k=0} x_k{}^m y_k{}^n \\ dy_k/dk &= -\log \rho y_k + \sum_{m,n} \frac{d}{dk}\,[\psi_{mn}{}^{(k)}]_{k=0} x_k{}^m y_k{}^n. \end{aligned} \tag{8}$$

In order to illustrate the calculations which lead to the determination of the coefficients φ_{mn}, ψ_{mn}, consider the following differential equation:

$$\ddot{x} + w^2 x = f(x, \dot{x}). \tag{9}$$

It is assumed that this is the form taken by the differential system (1) over the time interval $[t_0, t_1]$. If we transform the complex variables $X = x + j(\dot{x}/w)$, $Y = x - j(\dot{x}/w)$, (9) becomes:

$$\begin{aligned} \frac{dX}{dt} &= -jwX + j\frac{F(X, Y)}{w}, \\ \frac{dY}{dt} &= jwY - j\frac{F(X, Y)}{w}, \qquad F(X, Y) = f\left(\frac{X+Y}{2}, \frac{X-Y}{2j}w\right). \end{aligned} \tag{10}$$

Before going on to the identification of the forms (8) and (10), it is necessary to change the time scale of one or other of the systems so that the pointwise transformation links the states of the systems at the boundaries of $[t_0, t_1]$, i.e.

$$x = x(t_0), \qquad y = y(t_0),$$
$$x_1 = x(t_1), \qquad y = y(t_1).$$

For this, it is enough to write:

$$\begin{aligned} t &= t_0 + k(t_1 - t_0) \qquad (k = 1, t = t_1) \\ &= t_0 + kh. \end{aligned}$$

(10) becomes

$$\begin{aligned} \frac{dX}{dk} &= -jwhX + jh\frac{F(X, Y)}{w} = -jwhX + jwh\sum_{m,n}\alpha_{mn}X^mY^n \\ \frac{dY}{dk} &= jwhY - jh\frac{F(X, Y)}{w} = jwhY - jwh\sum_{m,n}\alpha_{mn}^*X^mY^n \end{aligned} \tag{11}$$

(where * stands for complex conjugate).

Identifying (8) and (11), we have:

$$\rho = \exp(-jwh),$$
$$\left[\frac{d\varphi_{20}^{(k)}}{dk}\right]_{k=0} = -\frac{jwh}{\rho^2 - \rho}\varphi_{20} = jwh\alpha_{20},$$

so that $\varphi_{20} = \alpha_{20}(\rho - \rho^2)$. The results of the calculations for the other coefficients are given in Appendix 1.

1.3. *Expression of* $\mathfrak{J}_C$. If it is assumed that the periodic coefficients which appear in (1) are piecewise constant, the calculation of $\mathfrak{J}_C$ can be carried out as follows:

1. The period, i.e. Θ, is divided into sub-intervals over which each of the coefficients is constant:

$$t_0 \leq t < t_1, \cdots, t_i \leq t < t_{i+1}, \cdots, t_m \leq t < t_0 + \Theta.$$

For each of the intervals the differential equation obtained is autonomous, and the partial, associated pointwise transformation T_i is determined for this interval in the same way as before.

2. If the state variables are assumed to be continuous at times t_i , $i = 1, \cdots, m$, the transformation $\mathfrak{J}_C$ is obtained from the product of the previous partial transformations:

$$\mathfrak{J}_C = T_m T_{m-1} \cdots T_i \cdots T_1 .$$

1.4. *Conclusion.* Obtaining the mapping $\mathfrak{J}_C$ is a result which is interesting in itself, for it enables one to solve certain problems concerned with the stability of the origin fixed point (trivial solution of the differential system (1)). In fact, in the case where the pointwise transformation (2) is such that $\rho = \exp(j\theta)$, i.e. when it is on critical case in the sense of Lyapunov, then the study of the stability and of the nature of the origin fixed point makes it necessary for one to take into account the nonlinear terms.

Two cases need to be considered:

a) $\theta \neq 2k\pi/q$: the stability is fixed by the sign of an expression G_1 which brings in the coefficients of the second- and third-degree terms [10].

b) $\theta = 2k\pi/q$: this case is linked to that of parametric resonance for the differential equation (1). For $q = 3, 4$, the coefficients of $\mathfrak{J}_C$ are still sufficient to determine the nature of the origin fixed point and the existence of analytical lines invariant with respect to $\mathfrak{J}$ passing through this point (node or saddle-type point) [11]. In Sec. 3, this is applied to an example.

Finally, in this first section the fact that we were trying to associate the pointwise transformation (2) to a Hamiltonian system (1) has not been brought out. In other words, the method can be applied to any nonlinear second-order differential equation with periodically varying coefficients.

2. Determination of $\mathfrak{J}^*$. In the first section we have shown how to determine $\mathfrak{J}_C$, i.e. how to determine the first terms (up to the third degree) of the pointwise transformation $\mathfrak{J}$ associated with the differential system (1). It is known that $\mathfrak{J}$ is area-preserving; $\mathfrak{J}_C$ does not generally retain this property. This means that $\mathfrak{J}_C$ cannot give an idea of the trajectories even in a small neighborhood of the origin [13].

In this section, without going into the details of the calculations, a method is described enabling one to construct a polynomial pointwise transformation $\mathfrak{J}^*$ with the following properties:

a) $\mathfrak{J}^*$ approximates $\mathfrak{J}$ to the third degree i.e. it has the same terms as $\mathfrak{J}$ (or $\mathfrak{J}_C$) up to the third degree.

b) $\mathfrak{J}$ is area-preserving (Jacobian unity).

Suppose, therefore, that the transformation $\mathfrak{J}_C$ has been determined. In real variables, it is written:

$$\begin{aligned} x_1 &= ax + by + \sum_{m+n=2}^{3} \bar{\varphi}_{mn} x^m y^n \\ y_1 &= cx + dy + \sum_{m+n=2}^{3} \bar{\psi}_{mn} x^m y^n . \end{aligned} \tag{12}$$

For the sake of simplicity, in the calculations which follow the transformation $\mathfrak{J}_C$ is decomposed into the product of two transformations τ_1 and τ_2 :

$$(x, y) \overset{\tau_1}{\ominus\!\!\rightarrow} (x', y') \overset{\tau_2}{\ominus\!\!\rightarrow} (x_1 , y_1).$$

τ_1 is linear, conservative:

$$\tau_1 \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad (ad - bc = 1);$$

τ_2 is in the form:

$$\tau_2 \begin{cases} x_1 = x' + \sum_{m+n=2}^{3} \varphi_{mn} x'^m y'^n \\ y_1 = y' + \sum_{m+n=2}^{3} \psi_{mn} x'^m y'^n \end{cases}$$

where the coefficients φ_{ij} , ψ_{ij} are expressed simply in terms of the coefficients $\bar{\varphi}_{ij}$, $\bar{\psi}_{ij}$. Since τ_1 is already area-preserving, the following calculations refer to τ_2 .

2.1. *Preliminary remarks.* In [12] Engel deals with polynomial transformations with Jacobian constant. We are able to use some of his results in order to carry out the approximation.

a) The following polynomial pointwise transformation:

$$\begin{aligned} u_1 &= u + \beta \sum_{n=2}^{N} a_n(\alpha u + \beta v)^n \\ v_1 &= v - \alpha \sum_{n=2}^{N} a_n(\alpha u + \beta v)^n \end{aligned} \tag{13}$$

where N is any positive integer and α, β, a are constants, is an area-preserving pointwise transformation (Jacobian unity).

b) If the transformation:

$$\begin{aligned} u_1 &= g(u, v) \\ v_1 &= h(u, v) \end{aligned}$$

is area-preserving, then all pointwise transformations of the form:

$$\begin{aligned} u_1 &= g(u, v) + \sum_{n=1}^{N} \alpha_n [h(u, v)]^n \\ v_1 &= h(u, v) \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_1 &= g(u, v) \\ v_1 &= h(u, v) + \sum_{n=1}^{N} \alpha_n [g(u, v)]^n \end{aligned} \tag{15}$$

also have this property.

2.2. *Approximation to the second-degree terms.* Consider again the expression of τ_2 in which only the terms to the second degree are assumed known:

$$\begin{aligned} x_1 &= x + \varphi_{20} x^2 + \varphi_{11} xy + \varphi_{02} y^2 \\ y_1 &= y + \psi_{20} x^2 + \psi_{11} xy + \psi_{02} y^2 . \end{aligned} \tag{16}$$

Since τ_2 comes from the truncation of an area-preserving mapping, some relations can be obtained for the φ_{mn}, ψ_{mn} coefficients, by writing (for the full mapping) $J \equiv 1$, from which we get:

$$\begin{aligned} \psi_{11} + 2\varphi_{20} &= 0 \\ 2\psi_{02} + \varphi_{11} &= 0. \end{aligned} \tag{17}$$

Thus, out of the six coefficients of τ_2 (16), four (at the most) are independent. It is necessary therefore to construct a conservative transformation depending on four parameters. This is easily done in this case from the product of transformations such as (13), (14), (15).

Consider the following mapping:

$$\begin{aligned} x' &= x + \beta[\alpha x + \beta y]^2 = g(x, y) \\ y' &= y - \alpha[\alpha x + \beta y]^2 = h(x, y) \\ x_1 &= g(x, y) + \delta h^2(x, y) \\ y_1 &= h(x, y) + \gamma x_1^2 = h(x, y) + \gamma[g(x, y) + \delta h^2(x, y)]^2 . \end{aligned} \tag{18}$$

This mapping is area-preserving. It depends on the four parameters α, β, γ, δ which can be chosen such that the second-degree terms of (18) are the same as those of (16), by fixing:

$$\begin{aligned} \beta\alpha^2 &= \varphi_{20} , \\ 2\alpha\beta &= \varphi_{11} , \\ \beta^3 + \delta &= \varphi_{02} , \\ -\alpha^3 + \gamma &= \psi_{20} . \end{aligned} \tag{19}$$

$\mathfrak{J}^*$ can thus be written:

$$\begin{aligned} x_1 &= x + \varphi_{20}\left[x + \frac{\varphi_{11}}{2\varphi_{20}} y\right]^2 + \left(\varphi_{02} - \frac{\varphi_{11}^2}{4\varphi_{20}}\right)\left[y - \frac{2\varphi_{20}^2}{\varphi_{11}}\left(x + \frac{\varphi_{11}}{2\varphi_{20}} y\right)^2\right]^2 \\ y_1 &= y - \frac{2\varphi_{20}^2}{\varphi_{11}}\left(x + \frac{\varphi_{11}}{2\varphi_{20}} y\right)^2 + \left(\psi_{20} + \frac{2\varphi_{20}^2}{\varphi_{11}}\right)x_1^2. \end{aligned} \tag{20}$$

The case where φ_{11} or φ_{20} is zero can be made to fit in with the above case by carrying out on (18) a transformation of variables (nonsingular, i.e. reversible). This will be discussed in the remarks at the end of the section.

2.3. *Approximation to the third-degree terms.* Consider now the mapping τ_2 with the third-degree coefficients:

$$\begin{aligned} x_1 &= x + \sum_{m+n=2}^{3} \varphi_{mn} x^m y^n \\ y_1 &= y + \sum_{m+n=2}^{3} \psi_{mn} x^m y^n . \end{aligned} \tag{21}$$

The approximation method for determining is very similar to that for the second order, although there are certain minor difficulties which are discussed below. As before, it is easy to verify that at the most 9 out of the 14 coefficients of the nonlinear terms of (21) are independent ($J = 1$). Using the same procedure as in the previous section, construct a polynomial area-preserving transformation:

$$\begin{aligned} x' &= x + \beta[(\alpha x + \beta y)^2 + a(\alpha x + \beta y)^3] \\ y' &= y - \alpha[(\alpha x + \beta y)^2 + a(\alpha x + \beta y)^3] \\ x_1 &= x' + \delta y'^2 + \epsilon y'^3 \\ y_1 &= y' + \gamma x_1^{\,2} + \nu x_1^{\,3} . \end{aligned} \tag{22}$$

The transformation (22) has seven independent parameters, α, β, a, δ, ϵ, γ, ν, so that in general it is not possible to carry out the approximation between (21) and (22) directly. It was shown above that the parameters α, β, a, δ are sufficient to solve the approximation on the second-order terms. It is therefore necessary for us to free two extra parameters which would act on the third-degree terms. This difficulty can be overcome by a transformation of variables in (21) which would give two independent parameters. The choice of this transformation of variable should be made in such a way that it does not alter the conservativity of the mapping $\mathfrak{I}$; the necessary condition is that its Jacobian should be constant, not zero (so that it has an inverse).

The following transformations were chosen:

$$\begin{aligned} \xi' &= x \\ \eta' &= y + t_1 x^2, \end{aligned} \tag{23}$$

$$\begin{aligned} \xi &= \xi' + t_2 \eta'^2 \\ \eta &= \eta', \end{aligned} \tag{24}$$

because of their simplicity (the inverse is easy to calculate) and because the parameters t_1 and t_2 come linearly into the third-degree coefficients, making the calculations considerably easier.

$\mathfrak{I}^*$ is determined in the following way: the coefficients α, β, γ, δ are calculated as shown in Sec. 2.2; the determination of the remaining coefficients then amounts to the solution of a linear system. The details of the calculations are given in Appendix 2. Finally, the approximating transformation is written:

$$\tau_1 \cdot (t_1) \cdot (t_2) \cdot \mathfrak{I}^* \cdot (t_2)^{-1} (t_1)^{-1}.$$

Remarks. 1) In expression (20) it can be seen that φ_{20} or φ_{11} appears in the denominator of the nonlinear terms higher than second degree. It is in fact the ratio $\varphi_{11}/2\varphi_{20}$ and its inverse which comes into these terms. In the case where one of these coefficients becomes very small (in comparison to the other), these terms can take on an exaggerated importance with respect to those of the second and third order, and can thus reduce the area around the origin in which the approximation made using the approximating transformation can be justified. This problem can be solved by a change of variables in the transformation τ_2 . As, before, the Jacobian should be constant and nonzero. For φ_{20} zero, or very low, the following transformation was adopted:

$$\begin{gathered} u = x, \qquad v = y - \mu x, \\ (\varphi_{20} \to \varphi_{20} + \mu \varphi_{11} + \mu^2 \varphi_{02} \; ; \; \varphi_{11} \to \varphi_{11} + 2\mu \varphi_{02}) \end{gathered} \tag{25}$$

aud for φ_{11} zero:

$$u = x - \mu y, \qquad v = y.$$

This transformation makes it possible to give a suitable value to the relationship $R = \varphi_{11}/2\varphi_{20}$. In the examples, this choice was determined empirically.

2) In the case where the coefficients of the second-order terms of $\mathfrak{J}_C$ are zero, the results given in Appendix 2 for the approximation of the third order cannot be directly applied. The basic reason for this is that the transformations of variables t_1 and t_2 are not valid for the third-degree terms. In this case, the mapping τ_2 has five independent parameters (8-3 conservativeness relations).

Consider the transformation:

$$\begin{aligned} x' &= x + \beta(\alpha x + \beta y)^3 \\ y' &= y - \alpha(\alpha x + \beta y)^3 \\ x_1 &= x' + \epsilon y'^3 \\ y_1 &= y' + \nu x_1^{\,3} \end{aligned} \tag{26}$$

This has four independent parameters. The approximation can be thus carried out using the parameter μ of the transformation of variables (25). There is another possible solution which has the advantage of using the results given in Appendix 2. This consists of artificially introducing second-order terms into the mapping τ_2, obtaining a mapping $\bar{\tau}_2$ in which the previous calculations can be used, and finally eliminating these second-order terms from the approximating transformation. The choice of these terms cannot be made arbitrarily, since they must be such that $\bar{\tau}_2$ can be taken as the truncation of an area-preserving transformation. This is true for:

$$\bar{\tau}_2 \begin{cases} x_1 = x + cy^2 + \sum_{m,n} \varphi_{mn} x^m y^n \\ y_1 = y + \sum_{m,n} \psi_{mn} x^m y^n \end{cases} \qquad m + n = 3.$$

It can easily be shown that by applying the area preserving transformation:

$$\begin{aligned} X &= x - cy^2 \\ Y &= y \end{aligned}$$

to the transformation $\mathfrak{J}^*$ approximating $\bar{\tau}_2$, the term cy^2 is eliminated from the resulting transformation (which remains conservative, since it is the product of conservative transformations). This solution has been employed in example 2 in the following section.

3. Examples. In this section, three applications are given to test the validity of the method for the study of the behaviour of the solutions of a conservative dynamic system around the origin (trivial solution of the recurrence equation system).

3.1. *Example* 1. Consider the differential equation:

$$(d^2x/dt^2) + x = f(t)[\alpha x^2 + \beta x^3] \tag{27}$$

where $f(t)$ is an impulse train of period T, amplitude unity and width θ.

Such an equation is found in the study of the transverse movement of a particle in an alternating gradient accelerator. The method given in Sec. 1 enables one, taking into account the relative simplicity of the signal $f(t)$, to determine the first terms on $\mathfrak{J}_C$, which when expressed in terms of the complex variables $X = x + j(dx/dt)$, $Y = X^*$ (conjugate) can be written

$$X_1 = \rho X + \sum_{m+n=2}^{3} \varphi_{mn} X^m Y^n, \qquad Y_1 = X_1^*$$

where:

$$\rho = \exp(-jT)$$

$$\varphi_{20} = \exp(-jT)(\exp(-j\theta) - 1)\frac{\alpha}{4}$$

$$\varphi_{11} = \exp(-jT)(1 - \exp(j\theta))\frac{\alpha}{2}$$

$$\varphi_{02} = \exp(-jT)(1 - \exp(3j\theta))\frac{\alpha}{12}$$

$$\varphi_{30} = \exp(-jT)\left[\exp(j\theta)(\exp(-j\theta) - 1)^3\frac{\alpha^2}{24} + (\exp(-2j\theta) - 1)\frac{\beta}{16}\right]$$

$$\varphi_{21} = \exp(-jT)\left[(30j\theta - \exp(3j\theta) - 8 - 9\exp(j\theta) + 18\exp(-j\theta))\frac{\alpha^2}{72} - 3j\frac{\theta\beta}{8}\right]$$

$$\varphi_{12} = \exp(-jT)\left[\exp(-j\theta) - 1 - 3\exp(-j\theta) + 5\exp(2j\theta) - 2\exp(3j\theta))\frac{\alpha^2}{24} - 3(\exp(2j\theta) - 1)\frac{\beta}{16}\right]$$

$$\varphi_{03} = \exp(-jT)\left[\exp(4j\theta) - 2\exp(3j\theta) - 1 + 2\exp(j\theta))\frac{\alpha^2}{48} + \frac{\exp(4j\theta) - 1}{32}\beta\right].$$

The results obtained using the osculation method were compared with those obtained by numerical integration (fourth-order Runge-Kutta method with adaptive step [14]), on an IBM 360/75 computer with double precision ($\sim 10^{-16}$). From among the many tests carried out, two have been selected as being representative of certain characteristic phenomena of area-preserving pointwise transformations such as island structures and stochastic behavior (homoclinic or heteroclinic points).

A) $T = 1.588$; $\theta = 0.36$; $\alpha = 10$; $\beta = 0$. The results obtained by numerical integration are shown in Fig. 1. A brief description of this figure is given below.

A fourth-order island structure surrounds the origin which is a center. The regular behavior of the trajectories breaks down at some limit near the curve Γ. Inside it we have an apparently integrable domain, which will be referred to from now on as a "quasi-integrable" region, since actual integrability cannot be generally verified [9]. Outside Γ the iterated points describe a very irregular motion and in this context we refer to it as a "stochastic" motion. This behavior occurs along with the presence of homoclinic and heteroclinic points generated by invariant curves passing through singular points of saddle type [15]. In the present example, this motion gives rise to stochastic instability

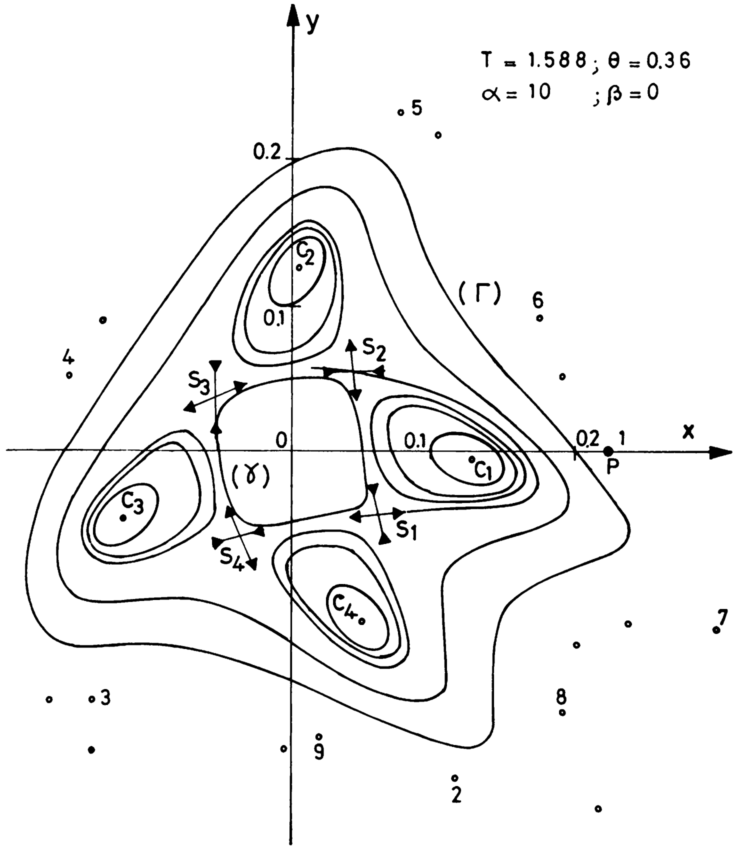

FIG. 1.

[16] so that outside Γ and with an initial condition near it the point describes at first a slowly diverging (with respect to Γ) movement and finally escapes irreversibly from the frame of Fig. 1. The scattered points shown belong to the same trajectory beginning at P (the numbers indicate some successive iterates of $\mathfrak{J}^5$).

As for the comparison between numerical integration and osculating mapping, Table 1 gives some iterated points obtained by both methods, n denoting the number of iterations, starting from the same initial conditions. Obviously, the divergence between the results increases as the distance from the origin becomes larger. However, a point worth noting is that in the quasi-integrable region the error develops in the "phase" rather in the "amplitude". This is reflected by the fact that the phase portrait is almost the same, at least within the degree of precision of Fig. 1, which therefore also represents the results obtained by the approximating transformation.

The position of the cycles C_1, C_2, C_3, C_4 and S_1, S_2, S_3, S_4 of the approximating transformation have been obtained by a numerical program using both the gradient and Newton's method with a precision of about 10^{-9}. A test using numerical integration shows that the exact and approximate cycle positions agree to a precision of 10^{-5}. Either numerical integration or approximating mapping gives invariant curves passing through the saddle S_1, S_2, S_3, S_4 with no detectable homoclinic or heteroclinic points (two of

TABLE 1.

$T = 1.588$; $\theta = 0.36$; $\alpha = 10$; $\beta = 0$. *Example A.*

	Approximating mapping		Numerical integration		
n	X	Y	X	Y	
0	0,05	0	0,05	0	
20	−0,00835113	−0,05139432	−0,00834973	−0,05139527	
40	−0,01862271	−0,05294405	−0,01861879	−0,05294774	
60	−0,02587861	−0,05323888	−0,02587089	−0,05324836	
200	−0,05009292	0,00901835	−0,05009050	0,00892006	
400	0,02743005	0,05065437	0,02729490	0,05067465	
600	0,05500973	−0,03379491	0,05503413	−0,03366572	
745	−0,05016173	0,01241045	−0,05016082	0,01204827	
0	0,1	0	0,1	0	
100	0,04383988	−0,09528784	0,04374121	−0,09524857	
200	0,05865594	−0,10588185	0,05846941	−0,10566201	
400	0,06106618	−0,13961848	0,06167375	−0,13966534	
700	0,02937415	−0,09562585	0,02899163	−0,09585284	
0	0,2	0	0,2	0	
100	0,152466	−0,110926	0,15334668	−0,10937630	
200	0,171026	−0,0938892	0,17440640	−0,091387	
400	0,2272165	−0,0720555	0,236263	−0,065829	
700	0,158958	−0,045156	0,140034	0,06575	
0	0,225	0	0,225	0	
10	−0,138654	−0,173515	−0,137934	−0,173137	
20	0,080260	0,233820	0,080038	0,233382	
30	0,301220	−0,114166	0,301447	−0,113954	random
40	0,014320	−0,195143	0,019999	−0,196991	zone
50	−0,114962	0,107698	−0,130526	0,091030	
0	0,0053833	0,1263964	0,0053833	0,1263694	
8			0,0053750	0,1263506	Center
16	id.		0,0053735	0,1263323	(point C_2)
32			0,0053904	0,1263137	
64			0,005434	0,12636651	
0	0,0447153	0,0525959	0,0447153	0,0525959	
4			0,0447139	0,0525977	
8			0,0447125	0,0525997	Saddle
12			0,0447113	0,0526018	(point S_2)
24			0,044707	0,0526135	

these curves were shown in Fig. 1 between S_1 and S_2). This fact is related to the quasi-integrable character of the region inside Γ which contains the island structure corresponding to these saddles.

B) $T = 1, 63$, $\theta = 0, 36$, $\alpha = 10$, $\beta = 0$. In this case the integration and the approximation method also agree (see Table 2). In Fig. 2 one can see that the island structure is further away from the origin. Around the fourth-order center cycles, the behavior of solutions is regular and defines a four-area domain of quasi-integrability. The domain of quasi-integrability around the origin is reduced to the region inside the curve Γ (which

TABLE 2.

$T = 1.63$; $\theta = 0.36$; $\alpha = 10$; $\beta = 0$. *Example B.*

	Approximating mapping		Numerical integration		
n	X	Y	X	Y	
0	0,05	0	0,05	0	
100	0,03727	−0,03919	0,3728	−0,03918	
200	0,05036	−0,00344	0,05035	−0,00342	
400	−0,00726	0,04889	−0,00730	0,04888	
600	−0,04408	−0,02555	−0,04405	−0,02560	
700	−0,00949	−0,04853	−0,00957	−0,04852	
0	0,15	0	0,15	0	
40	0,04454	−0,14284	0,04415	−0,14405	
80	0,04127	−0,15363	0,03822	−0,15158	
120	0,04252	−0,16813	0,03723	−0,16091	Random
160	0,04662	−0,18021	0,03290	−0,16470	zone
200	0,07649	−0,20455	0,02220	−0,15151	
0	0,2	−0,05	0,2	−0,05	
100	0,13841	−0,22519	0,14023	−0,22538	
200	0,13488	−0,19944	0,13655	−0,20198	
300	0,10496	−0,17563	0,10167	−0,17381	
400	0,07614	−0,17172	0,07838	−0,17138	
600	0,13197	−0,22343	0,12287	−0,21994	

is the apparent frontier of this domain) and is separated from the former by a region where the points are scattered. A test of non-integrability for this region has been made by drawing some of the analytical invariant curves passing through the points S_1 and S_4 of the fourth-order saddle cycle. From S_4, the repulsive invariant curve L_1 does not join the attractive curve L_2 (smoothly) but intersects it at heteroclinic points. Some loops of L_1 have been drawn on Fig. 2, using the approximating mapping, and a verification undertaken by numerical integration shows that at least for the first loops the difference between the results appears to be in the "phase" on these curves as for the regular closed curves of the domain of integrability. This stochastic region is also a region of stochastic instability because the point finally escapes. Similar phase portraits can be seen in [17], for example, which deals with a quadratic algebraic transformation.

C) $T = 1{,}5709$, $\theta = 0, 1$, $\alpha = -10$, $\beta = -1$. The value of T is near the value $2\pi/4$, when a parametric resonance may occur. In fact, for the α and β chosen the fixed point origin is a hyperbolic point with eight invariant curves through it, alternately attractive aud repulsive. The detail of Fig. 3 shows the good agreement between numerical integration and approximating transformations.

Table 3 gives the values of the parameters necessary for the construction of the approximating mapping for the three tests given above.

3.2. *Other examples.* The two examples in this section are given in order to show the sort of problem to which the above calculations can be adapted. For brevity each example will be limited to a description and to a qualitative comparison of the results with those given by other authors quoted in the references.

3.2.1. *Movement of a satellite.* The problem here is that of the analysis of the behavior

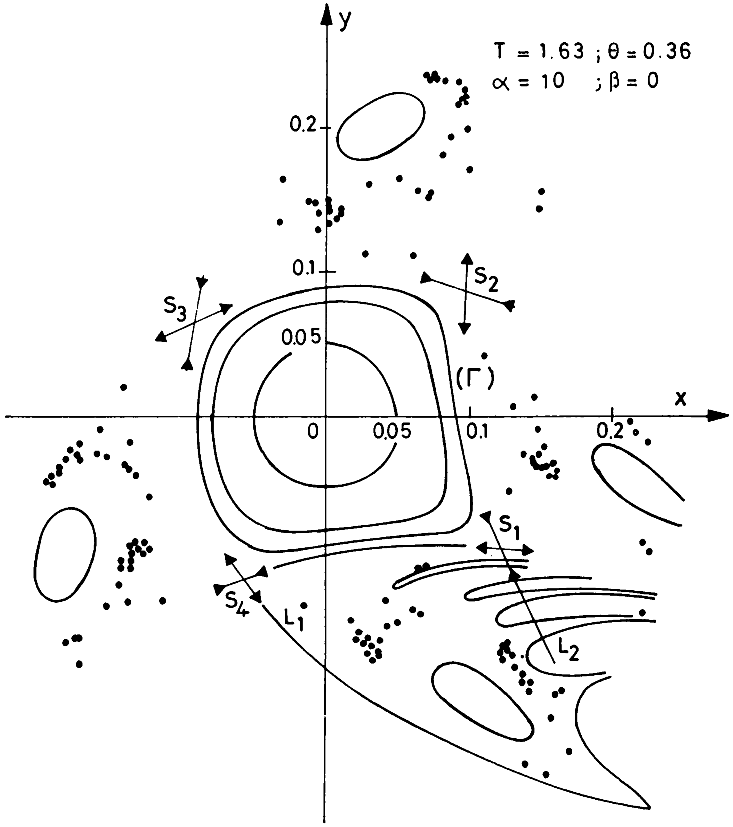

FIG. 2.

of a satellite subject to periodic variations in its moments of inertia; these variations can be caused by the relative motion of the moving parts (antennae, etc.) in relation to the main body [2]. For the sake of brevity, the calculations necessary to obtain the equations of motion (cf. [2]) will not be developed here.

The equations (Euler equations) can be written:

$$\dot{\alpha} + H\left(\frac{C - B}{CB}\right)\beta\gamma = 0$$

$$\dot{\beta} + H\left(\frac{A - C}{AC}\right)\alpha\gamma = 0$$

$$\dot{\gamma} + H\left(\frac{B - A}{AB}\right)\alpha\beta = 0 \tag{29}$$

where α, β, γ are the direction cosines of the kinetic moment **H**; H is the modulus of the kinetic moment **H** (constant of motion). A, B, C are the main moments of inertia of the satellite.

Eqs. (29) are written in an orthonormal axis set, placed according to the axes of the main moments of inertia of the satellite. If we take as the law of variation of the inertial

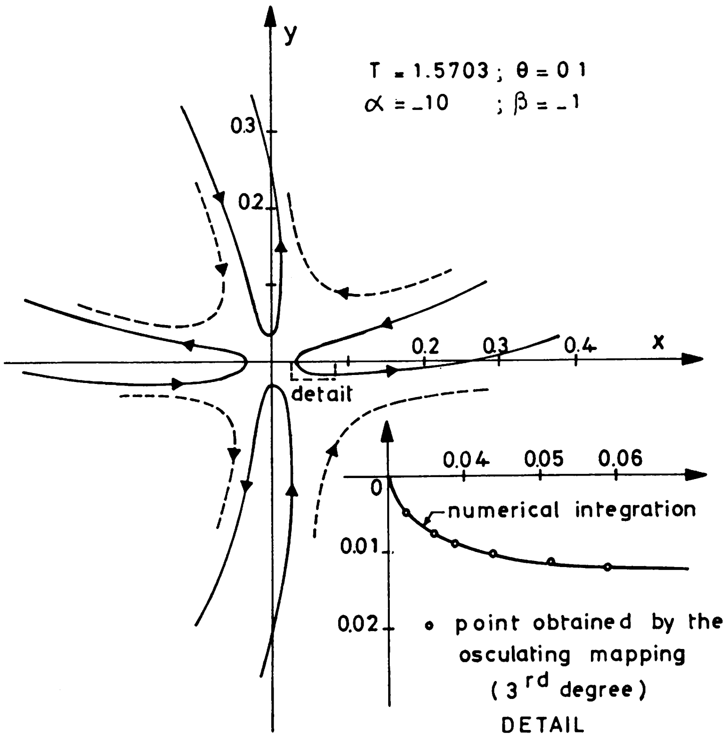

Fig. 3.

TABLE 3.

Parameter values for the examples treated in Sec. 4.

Example			
Parameter	*A*	*B*	*C*
μ	1.150	1.10	1.00
φ_{20}	3.37347	3.37698	0.89872
r	0.99912	1.00746	1.05171
γ	0.03744	0.03705	0.00079
δ	0.03910	0.03954	0.00092
τ_1	0.01549	0.01551	0.00029
τ_2	0.01657	0.01697	0.00049
ν	0.00176	0.00173	0.
ϵ	0.00146	0.00150	0.
a_3	1.38967	1.39126	0.05685

forces:

$$A = A_0, \qquad B = B_0 + \epsilon \cos \omega_0 t, \qquad C = C_0 + \epsilon \cos \omega_0 t$$

and combine (29) with $(\alpha^2 + \beta^2 + \gamma^2 = 1)$ so as to eliminate γ, we get:

$$\begin{aligned} \dot{\alpha} &= f(t)\beta/Q(\alpha, \beta) \\ \dot{\beta} &= g(t)\alpha/Q(\alpha, \beta) \end{aligned} \tag{30}$$

where

$$f(t) = \frac{B - C}{BC} H, \qquad g(t) = \frac{C - A}{AC} H, \qquad Q(\alpha, \beta) = (1 - \alpha^2 - \beta^2)^{-1/2} .$$

The pointwise transformation associated with the differential system (30) is conservative with a quasi-invariant function $Q(\alpha, \beta)$ (in the sense of Birkhoff [6]); i.e., it preserves the integral invariant:

$$\iint Q(\alpha, \beta)\, d\alpha\, d\beta.$$

The mapping is not area-preserving with respect to the variables α, β, and the method given in Sec. 2 cannot be applied directly. However, it is possible by a change of variables

$$\begin{aligned} u &= u(\alpha, \beta) \\ v &= v(\alpha, \beta) \end{aligned} \tag{31}$$

to make the transformation expressed in these variables area-preserving i.e. to make

$$\iint_D du\, dv$$

invariant:

$$Q(u, v) \begin{bmatrix} \dfrac{\partial u}{\partial \alpha} & \dfrac{\partial u}{\partial \beta} \\ \dfrac{\partial v}{\partial \alpha} & \dfrac{\partial v}{\partial \beta} \end{bmatrix} = Q(\alpha, \beta). \tag{32}$$

If $Q(u, v) = 1$ in (32) this enables one to find a suitable change (31), for example:

$$\begin{aligned} v &= -\alpha \\ u &= \int_\alpha^\beta Q(x, s)\, ds = \arcsin(\beta/(1 - \alpha^2)^{1/2}). \end{aligned} \tag{33}$$

The differential system (30) is written:

$$\begin{aligned} du/dt &= [g(t) - f(t) \sin^2 u]v \\ dv/dt &= f(t) \sin u \cos u(1 - v^2) \end{aligned} \tag{34}$$

with respect to the variables u, v.

It is easy to see that (34) is derived from the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}[f(t) \sin^2 u(1 - v^2) + g(t)v^2].$$

TABLE 4.

Satellite: coefficients of the mapping.

T u_1 / v_1	a / c	b / d	φ_{30} / ψ_{30}	φ_{21} / ψ_{21}	φ_{12} / ψ_{12}	φ_{03} / ψ_{03}
5	−1,00071	0,02451	0,00016	−1,64503	0,08196	−0,84313
	0,04885	−1,00049	3,26647	−0,16019	1,64459	0,00042
5,1	−0,99877	0,06964	−0,14889	−1,71776	0,01120	−0,84254
	−0,03853	−0,99855	3,31583	−0,31331	1,73966	−0,07664

It can be written as:

$$\partial u/\partial t = \partial H/\partial v, \qquad \partial v/\partial t = -\partial H/\partial u.$$

The coefficients of the pointwise transformation associated with (34) were calculated using a program described in [8], which approximates the periodic signals by piecewise constant functions by dividing the period into 500 equal intervals. The parameters chosen are those given by [2], that is $A_0 = 7$, $B_0 = 6$, $C_0 = 8, 9$, $H = 18, 66$, $\epsilon = 0, 05$.

Two values for the time period, $T = 5$ and $T = 5.1$, were selected from the many tried. The values of the coefficients of the truncated recurrence are given in Table 4. Figs. 4 and 5 show the results obtained with respect to the variables α and β. Of course, the nature of the origin fixed point is the same as in [2] since it is fixed by the lowest-order terms of the recurrence. In addition, it can be verified that there is good agreement between the topological portraits obtained by total pointwise transformation and those resulting from approximating pointwise transformation. The only difference to note, which is particularly marked in Fig. 5, comes from the introduction of second-degree terms (coefficient C, Table 3) in order to obtain the approximating pointwise transformation. This spoils the symmetry of the figure. Finally, in contrast to the previous case, there are no zones with stochastic behavior in evidence. This is due to the low amplitude (0.05) of the periodic disturbances.

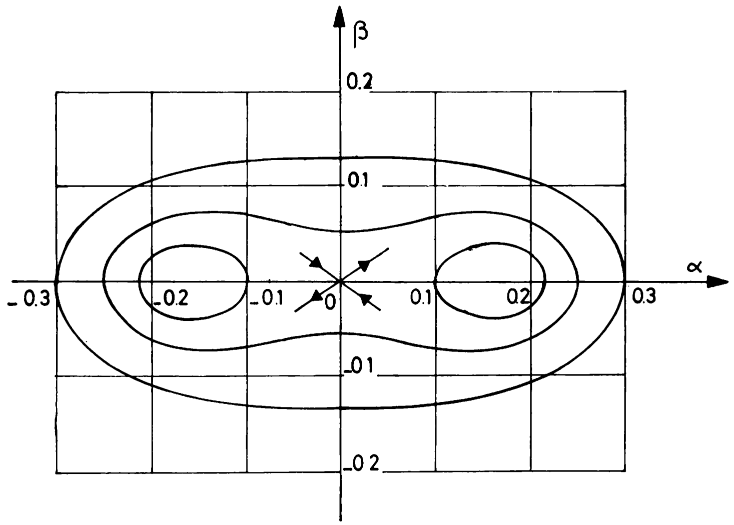

FIG. 4.

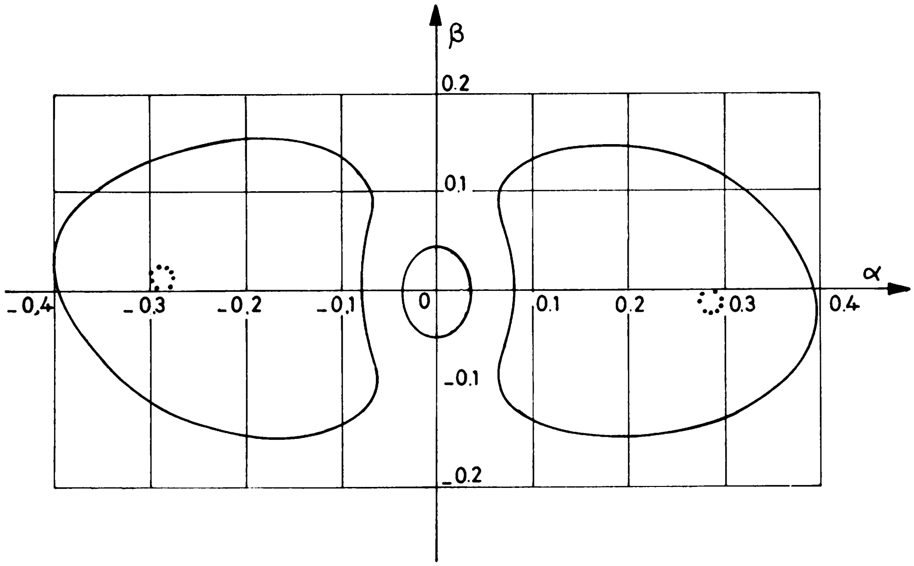

FIG. 5.

3.2.2. *Pendulum under parametric excitation.* This example, taken from [1], is concerned with the motion of a simple pendulum which is excited parametrically by vertical oscillations of its support. Its motion is governed by:

$$\ddot{\theta} + \left(\omega_0^2 + \frac{\xi}{L}\right) \sin \theta = 0 \tag{35}$$

with $\xi = \xi_0 \cos \omega t$. After the following changes of variable and parameter:

$$\tau = \omega_0 t, \qquad \epsilon = \xi_0/L, \quad \theta = \epsilon^{1/2} X, \quad \eta = \omega/\omega_0 ,$$

(35) can be written:

$$\ddot{X} + (1 - \eta^2 \epsilon \cos \eta\tau)\left(X - \frac{\epsilon X^3}{6} + \cdots\right) = 0. \tag{36}$$

As before, the coefficients of the pointwise transformation were determined numerically. The results are given in Table 5. Finally, for the value $\epsilon = 0.05$ a simulation was carried

TABLE 5.

Pendulum: coefficients of the mapping.

η u_1 / v_1	a / c	b / d	φ_{30} / ψ_{30}	φ_{21} / ψ_{21}	φ_{12} / ψ_{12}	φ_{03} / ψ_{03}
2,105	−1,00326	−0,01047	0,00125	0,00877	0,00485	0,01023
	−0,31346	−1,00002	−0,00701	−0,00123	−0,00723	0,00156
2	−1,01389	−0,16337	−0,00005	0,00823	0,00119	0,01033
	−0,15258	−1,01088	−0,00833	−0,00263	−0,00905	0,00011
1,91	−1,00136	−0,30676	−0,00145	0,00775	0,00119	0,01091
	0,00022	−0,99857	−0,00940	−0,00431	−0,00905	−0,00132
1,905	−0,99996	−0,31503	−0,00153	0,00771	0,00108	0,01017
	0,00909	−0,99718	−0,00945	−0,00442	−0,00909	−0,00147

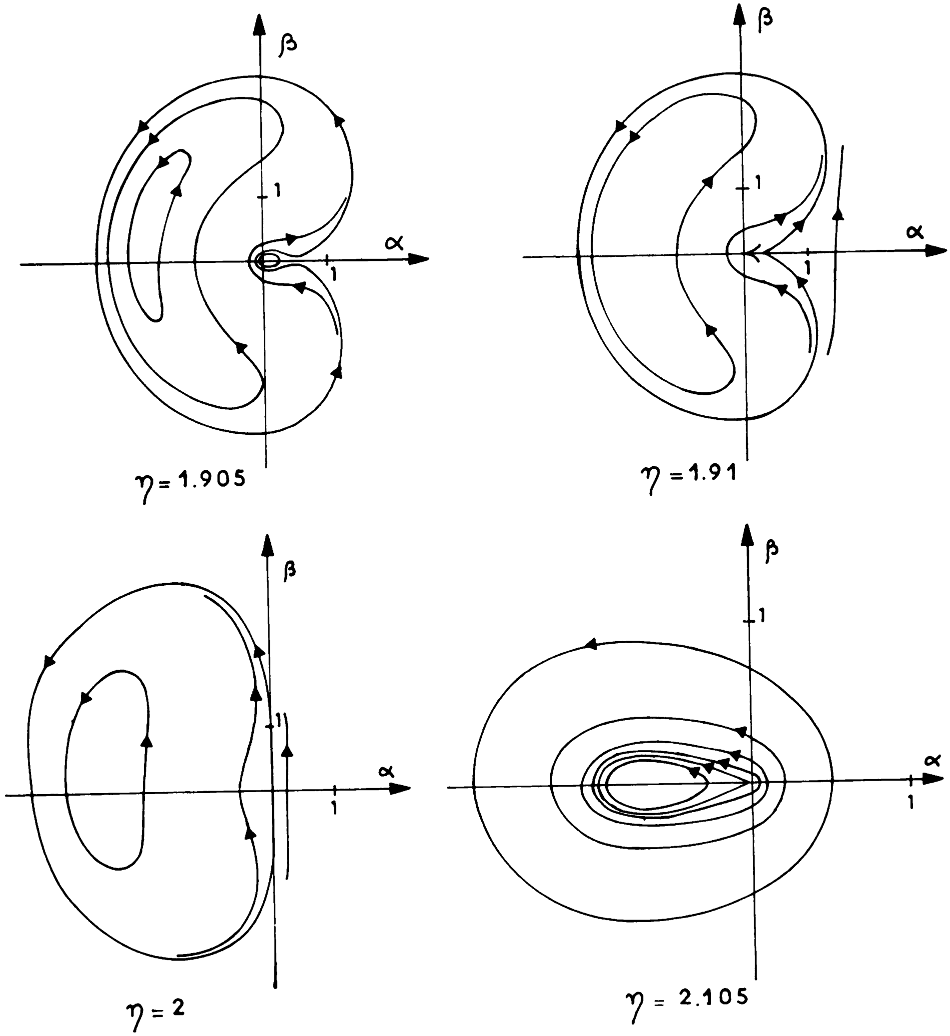

FIG. 6.

out for values around $\eta = 2$. The results are shown in Fig. 6 which was plotted, as in [1], with respect to the variables:

$$\alpha = \left[x^2 - \left(\frac{2y}{\eta}\right)^2\right]/\Delta; \qquad \beta = 2y/\eta\Delta; \qquad \Delta = \left(x^2 + \left(\frac{2y}{\eta}\right)^2\right)^{1/2}.$$

Some results from [1] are shown in Fig. 7 for comparison. A qualitative comparison was not possible since the corresponding values for η were not given there. The absence of zones with stochastic behavior can be explained, as in example 1, by the low value chosen for the parametric excitation.

Conclusion. If the numerical study of the solutions of a Hamiltonian system with periodic coefficients is tackled by means of numerical integration, then one comes up against the problem of the sensitivity of the system to the discretization of the differential

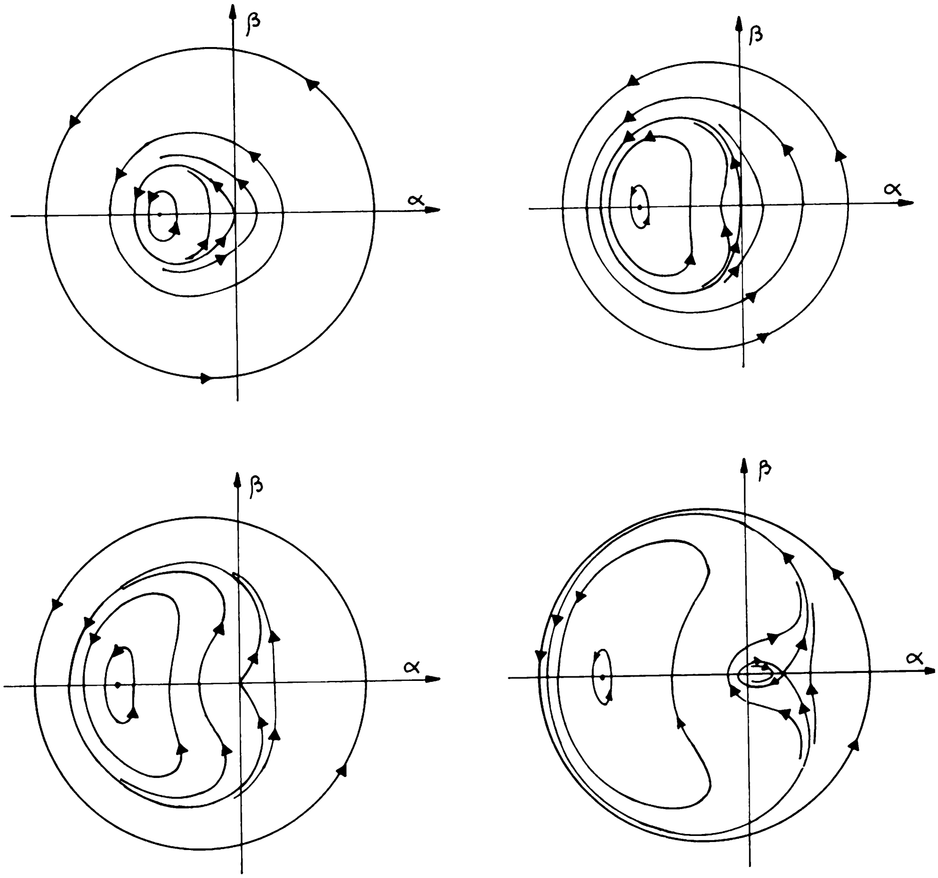

FIGURE 7

equations. This can be explained by the conservative nature of the system under consideration. This problem can, broadly speaking, be solved when it is possible to associate with the system a pointwise transformation. Unfortunately, however, it is not usually possible to formulate such a transformation. It is possible, on the other hand, to determine its first terms, but the resulting truncated transformation loses its conservative nature. From the truncated pointwise transformation, however, one can construct an area-preserving polynomial pointwise transformation which has in a sufficiently wide neighborhood of the origin the same topological portrait as the exact transformation. In addition, the numerical study is much quicker, being carried out using a mathematical model made up of recurrence equations. This last point is an important one since these numerical experiments are at present an important tool in the study of conservative dynamical systems of this kind [16–20].

Appendix 1. Determination of the coefficients of $\varphi_{mn}^{(k)}$.

1) Determine the difference equations:

$$\varphi_{20}^{(k+1)} - \rho\varphi_{20}^{(k)} = \rho^{2k}\varphi_{20} ,$$

$$\varphi_{11}^{(k+1)} - \rho\varphi_{11}^{(k)} = \varphi_{11} ,$$

$$\varphi_{02}^{(k+1)} - \rho\varphi_{02}^{(k)} = \rho^{-2k}\varphi_{02} ,$$

$$\varphi_{30}^{(k+1)} - \rho\varphi_{30}^{(k)} = 2\varphi_{20}\rho^{k}\varphi_{20}^{(k)} + \varphi_{11}\rho^{k}\varphi_{02}^{(k)} + \rho^{3k}\varphi_{30} ,$$

$$\varphi_{21}^{(k+1)} - \rho\varphi_{21}^{(k)} = 2\varphi_{20}\rho^{k}\varphi_{11}^{(k)} + \varphi_{11}\rho^{k}\varphi_{11}^{*(k)} + 2\varphi_{02}\rho^{-k}\varphi_{02}^{*(k)} + \varphi_{11}\rho^{-k}\varphi_{20}^{(k)} + \rho^{k}\varphi_{21} ,$$

$$\varphi_{12}^{(k+1)} - \rho\varphi_{12}^{(k)} = 2\varphi_{30}\rho^{k}\varphi_{02}^{(k)} + \varphi_{11}\rho^{k}\varphi_{20}^{*(k)} + \varphi_{11}\rho^{-k}\varphi_{11}^{(k)} + 2\varphi_{02}\rho^{-k}\varphi_{11}^{*k} + \rho^{-k}\varphi_{12} ,$$

$$\varphi_{03}^{(k+1)} - \rho\varphi_{03}^{(k)} = \varphi_{11}\rho^{-k}\varphi_{02}^{(k)} + 2\varphi_{02}\rho^{-k}\psi_{02}^{(k)} + \rho^{-3k}\varphi_{03} .$$

2) Solve the difference equations:

$$\varphi_{20}^{(k)} = \varphi_{20}\frac{\rho^{2k} - \rho^{k}}{\rho^{2} - \rho} ,$$

$$\varphi_{11}^{(k)} = \varphi_{11}\frac{\rho^{k} - 1}{\rho - 1} ,$$

$$\varphi_{02}^{(k)} = \varphi_{02}\frac{\rho^{k} - \rho^{-2k}}{\rho - \rho^{-2}} ,$$

$$\varphi_{30}^{(k)} = (\varphi_{30} - A_2 - A_0)\frac{\rho^{3k} - \rho^{k}}{\rho^{3} - \rho} + A_2\frac{\rho^{2k} - \rho^{k}}{\rho^{2} - \rho} + A_0\frac{1 - \rho^{k}}{1 - \rho} ,$$

$$\varphi_{21}^{(k)} = (\varphi_{21} - B_2 - B_0 - B_{-2})k\rho^{k-1} + B_2\frac{\rho^{2k} - \rho^{k}}{\rho^{2} - \rho} + B_0\frac{1 - \rho^{k}}{1 - \rho} + B_{-2}\frac{\rho^{-2k} - \rho^{k}}{\rho^{-2} - \rho} ,$$

$$\varphi_{12}^{(k)} = (\varphi_{12} - C_2 - C_0 - C_{-2})\frac{\rho^{-k} - \rho^{k}}{\rho^{-1} - \rho} + C_2\frac{\rho^{2k} - \rho^{k}}{\rho^{2} - \rho} + C_0\frac{1 - \rho^{k}}{1 - \rho} + C_{-2}\frac{\rho^{-2k} - \rho^{k}}{\rho^{-2} - \rho} ,$$

$$\varphi_{03}^{(k)} = (\varphi_{03} - D_0 - D_{-2})\frac{\rho^{-3k} - \rho^{k}}{\rho^{-3} - \rho} + D_0\frac{1 - \rho^{k}}{1 - \rho} + D_{-2}\frac{\rho^{-2k} - \rho^{k}}{\rho^{-2} - \rho}.$$

Writing the coefficient of φ_{mn} with respect to those of the differential equation:

$$\varphi_{20} = \alpha_{20}(\rho - \rho^{2}),$$

$$\varphi_{11} = \alpha_{11}(1 - \rho),$$

$$\varphi_{02} = \frac{\alpha_{02}}{3}(\rho^{-2} - \rho),$$

$$\varphi_{30} = \frac{\rho - \rho^{3}}{2}\alpha_{30} + \frac{\rho^{3}}{2}\left(2\alpha_{20}^{2} - \frac{\alpha_{11}\alpha_{02}^{*}}{3}\right) - 2\alpha_{20}^{2}\rho^{2} + \rho\left(\alpha_{20}^{2} + \frac{\alpha_{11}\alpha_{02}^{*}}{2}\right) - \frac{\alpha_{11}\alpha_{02}^{*}}{3}$$

$$\varphi_{21} = jT\rho\left(\alpha_{21} + \alpha_{11}\alpha_{11}^{*} + \frac{2}{3}\alpha_{02}^{*} + \alpha_{20}\alpha_{11}\right) + 2\rho^{2}\alpha_{20}\alpha_{11} + \rho\left(\alpha_{11}\alpha_{11}^{*}\frac{2}{9}\alpha_{02}\alpha_{02}^{*} - 3\alpha_{20}\alpha_{11}\right) + \alpha_{11}\alpha_{20} - \alpha_{11}\alpha_{11}^{*} - \frac{2\rho^{-2}}{9}\alpha_{02}\alpha_{02}^{*},$$

$$\varphi_{12} = \frac{\rho^{-1} - \rho}{2}\alpha_{12} + \frac{2\rho^{2}}{3}\alpha_{02}\alpha_{20} - \frac{\rho}{2}\left(2\alpha_{20}\alpha_{02} - \alpha_{11}^{2} + \frac{2\alpha_{02}\alpha_{11}^{*}}{3} + \alpha_{11}\alpha_{20}\right) + \alpha_{11}\alpha_{20}^{*} - \alpha_{11}^{2} + \frac{\rho^{-1}}{2}\left(\frac{2}{3}\alpha_{20}\alpha_{02} + 2\alpha_{02}\alpha_{11}^{*} + \alpha_{11}^{2} - \alpha_{11}\alpha_{20}^{*}\right) - \frac{2\rho^{-2}}{3}\alpha_{20}\alpha_{11}^{*},$$

$$\varphi_{03} = \frac{\rho^{-3} - \rho}{4}\alpha_{03} + \left(\frac{\alpha_{11}\alpha_{02}}{12} - \frac{\alpha_{20}^*\alpha_{02}}{2}\right)\rho^{-3} + 2\frac{\alpha_{02}\alpha_{20}^*}{3}\rho^{-2} + \left(\frac{\alpha_{11}\alpha_{02}}{4} - \frac{\alpha_{20}^*\alpha_{02}}{6}\right)\rho - \frac{\alpha_{11}\alpha_{02}}{3}.$$

Appendix 2. Calculation for the approximation to the third-degree terms. Consider the transformation:

$$\begin{aligned} U_1 &= U + \sum \varphi_{mn} U^m V^n \\ V_1 &= V + \sum \psi_{mn} U^m V^n \end{aligned} \qquad m + n = 2, 3 \tag{1}$$

in which the changes of variable (t_1) and (t_2) (expression (23) and (24)) are carried out with respect to the variables ξ, η. It can be written:

$$\begin{aligned} \xi_1 &= \xi + \varphi_{20}\xi^2 + \varphi_{11}\xi\eta + \varphi_{02}\eta^2 + \sum_{mn} \varphi_{mn}'\xi^m\eta^n \\ \eta_1 &= \eta + \psi_{20}\xi^2 + \psi_{11}\xi\eta + \psi_{02}\eta^2 + \sum_{mn} \psi_{mn}'\xi^m\eta^n \end{aligned} \qquad m + n = 3 \tag{2}$$

where

$$\begin{aligned} \varphi_{30}' &= \varphi_{30} - t_1\varphi_{11} , \\ \varphi_{21}' &= \varphi_{21} - 2t_1\varphi_{02} + 2t_2\psi_{20} , \\ \varphi_{12}' &= \varphi_{12} - 2t_2\varphi_{02} + 2t_2\psi_{11} , \\ \varphi_{03}' &= \varphi_{03} - t_2\varphi_{11} + 2t_2\psi_{02} , \\ \psi_{30}' &= \psi_{30} - t_1\psi_{11} + 2t_1\varphi_{20} , \\ \psi_{21}' &= \psi_{21} + 2t_1\varphi_{11} - 2t_1\psi_{02} , \\ \psi_{12}' &= \psi_{12} + 2t_1\varphi_{02} - 2t_2\psi_{20} , \\ \psi_{03}' &= \psi_{03} - t_2\psi_{11} . \end{aligned}$$

By identifying the coefficients of the second- and third-order terms between (22) and expression (2) above, the second-order terms being provided by (19), we get the following system of equations which is linear in relation to the unknown quantities a, t_1, t_2, ϵ, ν:

$$\begin{aligned} a\alpha^3\beta &= \varphi_{30} - t_1\varphi_{11} \\ 3a\alpha^2\beta^2 - 2\beta^2\alpha\delta &= \varphi_{21} - 2t_1\varphi_{02} + 2t_2\psi_{20} \\ 3a\alpha\beta^3 - 4\beta\alpha^2\delta &= \varphi_{12} + 2t_2\psi_{11} - 2t_2\varphi_{20} \\ a\beta^4 - 2\alpha^3\delta + \epsilon &= \varphi_{03} + 2t_2\psi_{02} - t_2\varphi_{11} \\ -a\alpha^4 + 2\beta\alpha^2\gamma + \nu &= \psi_{30} - t_1\psi_{11} + 2t_1\varphi_{20} . \end{aligned} \tag{3}$$

a, t_1, t_2 can be deduced from the first three equations of (3): setting $r = \varphi_{11}/2\varphi_{20}$, we get

$$\begin{aligned} t_1 &= [2(\psi_{11} - \varphi_{20})a_1 - 2\psi_{20}b_1]/\Delta \\ t_2 &= [(3\varphi_{11}r - 2\varphi_{02})b_1 - 3\varphi_{11}r^2a_1]/\Delta, \end{aligned}$$

where

$$\Delta = 2(\psi_{11} - \varphi_{20})(2\varphi_{11}r - 2\varphi_{02}) - 6\psi_{20}\varphi_{11}r^2,$$
$$a_1 = 3\varphi_{30}r - 2\ \delta\varphi_{20}/r - \varphi_{21}\ ,$$
$$b_1 = 3\varphi_{30}r^2 - 4\ \delta\varphi_{20} - \varphi_{12}\ ,$$
$$a = (\varphi_{30} - t_1\varphi_{11})/\alpha^3\beta.$$

ϵ and ν can easily be deduced from the last two equations of the system (3).

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