TEMPERATURES OF LINEARLY AND NONLINEARLY RADIATING SEMI-INFINITE RODS*

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1. Introduction. Let $D = \{(x, t): x > 0, t > 0\}$, $S = \{(0, t): t > 0\}$, and $C = \{(x, 0) : x \geq 0\}$ with D^- and S^- denoting the closures of D and S respectively. Also let $u(x, t)$ be the temperature distribution in a homogeneous, isotropic and semi-infinite rod, which is radiating heat along its entire length at the rates proportional to u and $g(x, t; u)$, and also at the end $x = 0$ at the rates proportional to u and $B(t; u)$. Without loss of generality in the main results, we assume the diffusivity of the rod to be one. If energy is supplied to the end $x = 0$ at a rate proportional to some function $f(t)$, and the initial distribution of temperature is given by $\phi(x)$, then $u(x, t)$ is determined by the following initial boundary value problem:

$$Lu \equiv u_{xx}(x, t) - hu(x, t) - u_t(x, t) = kg(x, t; u) \quad \text{in } D, \tag{1.1}$$

$$u(x, 0) = \phi(x) \quad \text{on } C, \tag{1.2}$$

$$Au \equiv u_x(0, t) - bu(0, t) = aB(t; u) - f(t) \quad \text{on } S, \tag{1.3}$$

where u is assumed to tend to zero as x tends to infinity for $t \geq 0$. Here h, k, b and a are given constants with h, k and a being nonnegative while the given functions g, ϕ, B and f are piecewise continuous with ϕ and f being bounded and nonnegative, and $f \equiv 0$ for $T < t < \infty$, where T is a nonnegative constant.

When $k = \phi = 0$, and $B(t; u) = u^n(0, t)$, where n is a positive constant, our problem (1.1)–(1.3) reduces to the one studied recently by Hartka [2]. If in addition $h = b = 0$, then we have the problem considered by Keller and Olmstead [3]. For further references, we refer to these papers, where existence of the nonnegative solution for each problem was established by constructing the surface temperature $u(0, t)$. On the other hand, if $B(t; u) = u^n(0, t)$, then $n = 1$ corresponds to the Newton law of cooling, and $n = 4$ corresponds to the Stefan radiation law for black bodies.

The main purpose of this paper is to establish existence of the maximal and the minimal nonnegative solutions of the problem (1.1)–(1.3), to give conditions under which these coincide, and to construct upper and lower bounds for the nonnegative solutions. Our quest for nonnegative solutions is motivated by the physical concept of the absolute temperature. The methods used here are different from those in the above-mentioned papers.

Instead of treating the special case $B(t; u) = u^n(0, t)$, we shall require B to satisfy some or all of the following conditions:

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(B_1) there exists a bounded, nonnegative and piecewise continuous function $q(x, t)$ such that for bounded functions $u(x, t)$ and $v(x, t)$ in D^-, $B(t; u(0, t)) - B(t; v(0, t)) \leq q[u(0, t) - v(0, t)]$ if $u > v$ at $(0, t)$;

(B_2) $B(t; 0) = 0$;

(B_3) $u > 0$ at the point $(0, t)$ implies $B(t; u(0, t)) \geq 0$;

(B_4) $u > v$ at the point $(0, t)$ implies $B(t; u(0, t)) \geq B(t; v(0, t))$.

Also we shall need the function g to satisfy some or all of the following assumptions:

(g_1) there exists a bounded, nonnegative and piecewise continuous function $p(x, t)$ such that for bounded functions $u(x, t)$ and $v(x, t)$ in D^-, $g(x, t; u(x, t)) - g(x, t; v(x, t)) \leq p[u(x, t) - v(x, t)]$ if $u > v$ at (x, t);

(g_2) $g(x, t; 0) = 0$;

(g_3) $u > 0$ at the point (x, t) implies $g(x, t; u(x, t)) \geq 0$;

(g_4) $u > v$ at the point (x, t) implies $g(x, t; u(x, t)) \geq g(x, t; v(x, t))$.

Here we note in particular that q and p in conditions (B_1) and (g_1) respectively can be replaced by appropriate nonnegative constants. Also, conditions (B_2) and (B_4) taken together imply condition (B_3) while assumption (g_3) follows from assumptions (g_2) and (g_4).

Let

$$K(x, t; \xi, \tau) = \{2^{-1}[\pi(t - \tau)]^{-1/2}\} \exp \{-(x - \xi)^2/[4(t - \tau)]\}.$$

Also let $Z(x, t; \xi, \tau)$ be the solution of

$$lZ \equiv Z_{xx}(x, t; \xi, \tau) - \lambda Z(x, t; \xi, \tau) - Z_t(x, t; \xi, \tau) = 0 \quad \text{for} \quad x, \xi > 0, t > \tau,$$

$$Z(x, \tau; \xi, \tau) = \delta(x - \xi),$$

$$\gamma Z \equiv Z_x(0, t; \xi, \tau) - \theta Z(0, t; \xi, \tau) = 0,$$

where δ is the Dirac distribution, λ and θ are given constants. The function Z is a fundamental solution called the Neumann function (cf. Chan [1]) of $lw(x, t) = 0$ in D and $\gamma w(0, t) = 0$ on S. Let

$$Z(x, t; \xi, \tau) = z(x, t; \xi, \tau) \exp [-\lambda(t - \tau)]. \tag{1.4}$$

It follows from Stakgold [5, pp. 209–210] that

$$z(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau) - \theta[\pi(t - \tau)]^{-1/2} \int_0^\infty \exp \{-\theta y - (x + \xi + y)^2/[4(t - \tau)]\}\, dy. \tag{1.5}$$

We shall need the following two positivity lemmas.

LEMMA 1. $Z(x, t; \xi, \tau) > 0$ for $x, \xi \geq 0, t > \tau \geq 0$.

Proof. If $\theta = 0$, then it follows from (1.4) and (1.5) that the lemma is proved. If $\theta \neq 0$, then by integrating the third term on the right-hand side of (1.5) by parts, we have

$$z(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(-x, t; \xi, \tau) + (t - \tau)^{-1} \int_0^\infty [\exp(-\theta y)](x + \xi + y)K(-x, t; \xi + y, \tau)\, dy,$$

from which the Lemma follows with the use of (1.4).

The solution $w(x, t)$ of the problem

$$lw = G(x, t) \quad \text{in } D, \tag{1.6}$$

$$w(x, 0) = \Phi(x) \quad \text{on } C, \tag{1.7}$$

$$\gamma w = F(t) \quad \text{on } S, \tag{1.8}$$

where w tends to zero as x tends to infinity for $t \geq 0$, is given by

$$w(x, t) = \int_0^\infty Z(x, t; \xi, 0)\Phi(\xi)\, d\xi - \int_0^t \int_0^\infty Z(x, t; \xi, \tau)G(\xi, \tau)\, d\xi\, d\tau - \int_0^t Z(x, t; 0, \tau)F(\tau)\, d\tau.$$

This together with Lemma 1 gives the following result.

Lemma 2. If in (1.6)–(1.8), $G \leq 0$, $\Phi \geq 0$, and $F \leq 0$, then $w \geq 0$ in D^-.

In Sec. 2, we establish existence of the maximal and the minimal nonnegative solutions, which are constructed respectively as the limits of a monotone nonincreasing sequence of upper bounds and a monotone nondecreasing sequence of lower bounds. Hence the error involved in using a certain approximate solution can be estimated. Also we give conditions which ensure uniqueness of the solution. With uniqueness, the rate of convergence for each of the above sequences is shown to be geometrical. In Sec. 3, we show that an iteration scheme of the Picard type gives an alternating sequence consisting of two monotone subsequences bounding any nonnegative solution from above and below. Thus each successive iteration gives a more accurate pointwise upper or lower bound. Under additional assumptions, this sequence is shown to converge uniformly and geometrically to a solution.

2. Maximal and minimal solutions. Since we are interested in nonnegative solutions here, we shall use the following definition.

Definition. A solution $M(x, t)$ ($m(x, t)$) of the problem (1.1)–(1.3) is said to be maximal (minimal) if $u(x, t) \leq M(x, t)$ ($m(x, t) \leq u(x, t)$) for any nonnegative solution $u(x, t)$.

Let us construct a sequence $\{M_i(x, t)\}$ by $LM_0 = 0$ in D, $M_0 = \phi$ on C, $AM_0 = -f$ on S, and for $i = 0, 1, 2, \cdots$,

$$LM_{i+1} = kg(x, t; M_i) + r(M_{i+1} - M_i) \quad \text{in} \quad D, \tag{2.1}$$

$$M_{i+1} = \phi \quad \text{on} \quad C, \tag{2.2}$$

$$AM_{i+1} = aB(t; M_i) + s(M_{i+1} - M_i) - f \quad \text{on} \quad S, \tag{2.3}$$

where M_i tends to zero as x tends to infinity for $t \geq 0$, r and s are constants chosen to be $r \geq kp$ and $s \geq aq$ with p and q being given in conditions (g_1) and (B_1) respectively. The following theorem shows that this constructed sequence forms a uniformly bounded and monotone nonincreasing sequence of upper bounds for solutions of the problem (1.1)–(1.3), and also establishes existence of the maximal solution.

Theorem 1. Under assumptions (g_1)–(g_3) and (B_1)–(B_3), and $b^2 - h \leq 0$ if $b < 0$, the sequence $\{M_i\}$ satisfies the inequalities

$$0 \leq M_{i+1} \leq M_i \leq c_1 \quad \text{in} \quad D^-, \qquad i = 0, 1, 2, \cdots \tag{2.4}$$

where c_1 is a nonnegative constant; furthermore, it converges to the maximal solution of the problem (1.1)–(1.3).

Proof. By Lemma 2, $M_0 \geq 0$ in D^-. On the other hand,

$$M_0(x, t) = \int_0^\infty R(x, t; \xi, 0)\phi(\xi)\, d\xi + \int_0^t R(x, t; 0, \tau)f(\tau)\, d\tau,$$

where R is the Neumann function of $Lw = 0$ in D and $Aw = 0$ on S. From (1.4) and Stakgold [5, pp. 209–210], R can be written in the form

$$\begin{aligned} R(x, t; \xi, \tau) = \{\exp[-h(t - \tau)]\}[K(x, t; \xi, \tau) + K(-x, t; \xi, \tau)] \\ - b\{\exp[b(x + \xi) + (b^2 - h)(t - \tau)]\} \\ \cdot\{1 - \operatorname{erf}[(x + \xi)/(2(t - \tau)^{1/2}) + b(t - \tau)^{1/2}]\}, \end{aligned} \tag{2.5}$$

where $\operatorname{erf} y = 2\pi^{-1/2} \int_0^y \exp(-\xi^2)\, d\xi$ is the error function. Since ϕ and f are bounded, let c_2 and c_3 be nonnegative constants such that $\phi \leq c_2$ and $f \leq c_3$. Also, let

$$\begin{aligned} E &= t^{1/2} \quad \text{if} \quad t \leq T, \\ &= t^{1/2} - (t - T)^{1/2} \quad \text{if} \quad t > T, \\ F &= (h - b^2)^{-1}\{\exp[T(h - b^2)] - 1\} \quad \text{if} \quad b < 0 \quad \text{and} \quad b^2 - h < 0, \\ &= T \quad \text{if} \quad b < 0 \quad \text{and} \quad b^2 - h = 0. \end{aligned}$$

Using $f \equiv 0$ for $t > T$, and the facts that $0 \leq \operatorname{erf} y \leq 1$ and $e^{-y} \leq 1$ for $y \geq 0$, we obtain

$$\begin{aligned} M_0(x, t) &\leq [\exp(-ht)][3c_2/2 + 2c_3E\pi^{-1/2}\exp(hT)] \quad \text{if} \quad b \geq 0, \\ M_0(x, t) &\leq [\exp(-ht)]\{c_2[3/2 + 2\exp(b^2t)] \\ &\quad + 2c_3[\pi^{-1/2}E\exp(hT) + |b|\, F\exp(b^2t)]\} \quad \text{if} \quad b < 0. \end{aligned} \tag{2.6}$$

Since $b^2 - h \leq 0$ if $b < 0$, we have

$$M_0(x, t) \leq 7c_2/2 + 2c_3[\pi^{-1/2}T^{1/2}\exp(hT) + |b|\, F],$$

irrespective of the sign of b. Denoting the right-hand side by c_1, we have $M_0(x, t) \leq c_1$.

From assumptions (g_1) and (g_2), we have

$$g(x, t; M_0) - pM_0(x, t) \leq g(x, t; 0) = 0.$$

Similarly, from assumptions (B_1) and (B_2), we have

$$B(t; M_0) - qM_0(0, t) \leq B(t; 0) = 0.$$

It follows from (2.1) and (2.3) respectively that

$$\begin{aligned} (L - r)M_1 \leq k[g(x, t; M_0) - pM_0] \leq 0 \quad &\text{in} \quad D, \\ (A - s)M_1 \leq a[B(t; M_0) - qM_0] - f \leq 0 \quad &\text{on} \quad S. \end{aligned}$$

Since $\phi \geq 0$, it follows from Lemma 2 that $M_1 \geq 0$ in D^-.

Next, we show that $M_0 \geq M_1$ in D^-. Since $M_0 \geq 0$ in D^-, it follows from assumptions (g_2), (g_3), (B_2) and (B_3) that $g(x, t; M_0) \geq 0$ and $B(t; M_0) \geq 0$. Thus,

$$\begin{aligned} (L - r)(M_0 - M_1) = -kg(x, t; M_0) \leq 0 \quad &\text{in} \quad D, \\ (A - s)(M_0 - M_1) = -aB(t; M_0) \leq 0 \quad &\text{on} \quad S. \end{aligned}$$

Since $M_0 - M_1 = 0$ on C, we have $M_0 \geq M_1$ in D^-.

To establish (2.4), let us assume that for a particular value of i (say $j \geq 1$),

$$0 \leq M_i \leq M_{i-1} \leq \cdots \leq M_0 \leq c_1 \quad \text{in} \quad D^-.$$

Then it follows from assumptions (g_1) and (g_2) that

$$g(x, t; M_i) \leq pM_i , \tag{2.7}$$

and from assumptions (B_1) and (B_2) that

$$B(t; M_i) \leq qM_i . \tag{2.8}$$

Thus,

$$(L - r)M_{i+1} \leq (kp - r)M_i \leq 0 \quad \text{in} \quad D,$$

$$(A - s)M_{i+1} \leq (aq - s)M_i - f \leq 0 \quad \text{on} \quad S.$$

Hence, $M_{i+1} \geq 0$ in D^-. Using assumptions (g_1) and (B_1), we have

$$(L - r)(M_i - M_{i+1}) \leq (kp - r)(M_{i-1} - M_i) \leq 0 \quad \text{in} \quad D,$$

$$(A - s)(M_i - M_{i+1}) \leq (aq - s)(M_{i-1} - M_i) \leq 0 \quad \text{on} \quad S.$$

Since $M_i - M_{i+1} = 0$ on C, it follows that $M_i \geq M_{i+1}$ in D^-. From the principle of mathematical induction, we have (2.4).

Since the sequence $\{M_i\}$ is monotone nonincreasing and uniformly bounded, there exists a function M to which the sequence converges pointwise. To show that M is a solution of the problem (1.1)–(1.3), let us rewrite the iteration scheme (2.1)–(2.3) equivalently as

$$\begin{aligned} M_{i+1}(x, t) = &\int_0^\infty R(x, t; \xi, 0)\phi(\xi)\, d\xi \\ &- \int_0^t \int_0^\infty R(x, t; \xi, \tau)[kg(\xi, \tau; M_i) + r(M_{i+1} - M_i)]\, d\xi\, d\tau \\ &- \int_0^t R(x, t; 0, \tau)[aB(\tau; M_i) + s(M_{i+1} - M_i) - f(\tau)]\, d\tau. \end{aligned} \tag{2.9}$$

By Lemma 1, (2.4), (2.7) and (2.8), the integrands in the second and the third integrals on the right-hand side of (2.9) are bounded respectively by

$$(kp + r)M_0(\xi, \tau)R(x, t; \xi, \tau),$$

$$[aqM_0(0, \tau) + sM_0(0, \tau) + c_3]R(x, t; 0, \tau),$$

both of which are integrable over their respective regions of integration. As i tends to infinity in (2.9), it follows from the Lebesgue convergence theorem (cf. Royden [4, p. 200]) that we can interchange the limit and the integration processes. Hence,

$$\begin{aligned} M(x, t) = \int_0^\infty R(x, t; \xi, 0)\phi(\xi)\, d\xi - k \int_0^t \int_0^\infty R(x, t; \xi, \tau)g(\xi, \tau; M)\, d\xi\, d\tau \\ - \int_0^t R(x, t; 0, \tau)[aB(\tau; M) - f(\tau)]\, d\tau. \end{aligned}$$

This implies that M is a solution of the problem (1.1)–(1.3).

To show that M is maximal, let u be any nonnegative solution of the problem (1.1)–(1.3). Then,

$$L(M_0 - u) = -kg(x, t; u) \leq 0 \quad \text{in} \quad D,$$
$$A(M_0 - u) = -aB(t; u) \leq 0 \quad \text{on} \quad S.$$

As $M_0 - u = 0$ on C, we have $M_0 \geq u$ in D^-. Let us assume that $u \leq M_j$ in D^- for some j. Using assumptions (g_1) and (B_1), we have

$$(L - r)(M_{j+1} - u) \leq (kp - r)(M_j - u) \leq 0 \quad \text{in} \quad D,$$
$$(A - s)(M_{j+1} - u) \leq (aq - s)(M_j - u) \leq 0 \quad \text{on} \quad S.$$

Since $M_{j+1} - u = 0$ on C, it follows that $M_{j+1} \geq u$ in D^-. From the principle of mathematical induction, we have $u \leq M_i$ in D^- for $i = 0, 1, 2, \cdots$. Hence $u \leq M$ in D^-. This shows that M is the maximal solution.

If $b^2 - h < 0$, then it follows from (2.6) that $M_0(x, t)$ tends to zero as t tends to infinity; this is physically obvious since the temperature must approach zero if heat is lost along the length of the rod faster than it can be absorbed at the end (cf. Hartka [2]).

Our next theorem gives a monotone nondecreasing sequence of lower bounds for nonnegative solutions of the problem (1.1)–(1.3), and also existence of the minimal solution. We omit its proof here since it is similar to that of Theorem 1 with some obvious modifications.

THEOREM 2. Under the hypotheses of Theorem 1, the sequence $\{m_i(x, t)\}$ constructed by $m_0 \equiv 0$ in D^-, and for $i = 0, 1, 2, \cdots$,

$$Lm_{i+1} = kg(x, t; m_i) + r(m_{i+1} - m_i) \quad \text{in} \quad D,$$
$$m_{i+1} = \phi \quad \text{on} \quad C,$$
$$Am_{i+1} = aB(t; m_i) + s(m_{i+1} - m_i) - f \quad \text{on} \quad S,$$

where m_i tends to zero as x tends to infinity for $t \geq 0$, satisfies

$$0 \leq m_i \leq m_{i+1} \leq M_0 \leq c_1 \quad \text{in} \quad D^-, \qquad i = 0, 1, 2, \cdots, \tag{2.10}$$

and converges to the minimal solution m of the problem (1.1)–(1.3).

That the maximal solution is nonnegative follows from its definition. Because of (2.10), the minimal solution is also nonnegative. The following result gives the conditions under which we have a unique solution.

THEOREM 3. Under the hypotheses of Theorem 1 and the assumptions (g_4) and (B_4), there exists a unique nonnegative solution of the problem (1.1)–(1.3).

Proof. By Theorems 1 and 2, we have existence of the maximal solution M and the minimal solution m. Thus,

$$0 \leq M(x, t) - m(x, t) = -k \int_0^t \int_0^\infty R(x, t; \xi, \tau)[g(\xi, \tau; M) - g(\xi, \tau; m)]\, d\xi\, d\tau$$
$$- a \int_0^t R(x, t; 0, \tau)[B(\tau; M) - B(\tau; m)]\, d\tau \quad \text{in} \quad D^-. \tag{2.11}$$

It follows from Lemma 1 and assumptions (g_4) and (B_4) that the integrands in the last two integrals are nonnegative, and hence from (2.11) we have

$$0 \le M(x, t) - m(x, t) \le 0 \text{ in } D^-.$$

Thus, $M \equiv m$ in D^-, and hence the solution is unique.

Our next result gives the rate of convergence of each of the sequences $\{M_i\}$ and $\{m_i\}$ to the solution.

Theorem 4. Under the hypotheses of Theorem 3, each of the sequences $\{M_i\}$ and $\{m_i\}$ converges uniformly and geometrically to the unique nonnegative solution

$$u = \lim_{i\to\infty} M_i = \lim_{i\to\infty} m_i \tag{2.12}$$

of the problem (1.1)–(1.3).

Proof. Since $M_{i+1} \le M_i$ in D^-, $k \ge 0$, and $a \ge 0$, it follows from (2.9), Lemma 1, and assumptions (B_4) and (g_4) that

$$M_i(x, t) - M_{i+1}(x, t) \le r \int_0^t \int_0^\infty R(x, t; \xi, \tau)(M_{i-1} - M_i)\, d\xi\, d\tau$$
$$+ s \int_0^t R(x, t; 0, \tau)(M_{i-1} - M_i)\, d\tau$$

after having dropped out the nonpositive terms. Let

$$\rho_i = \sup_{(x,t)\in D^-} |M_{i+1} - M_i|,$$

and $c_4 = \max\{r, s\}$. Then

$$\rho_i \le c_4\rho_{i-1}\left[\int_0^t \int_0^\infty R(x, t; \xi, \tau)\, d\xi\, d\tau + \int_0^t R(x, t; 0, \tau)\, d\tau\right]. \tag{2.13}$$

Using (2.5), and the facts that $0 \le \operatorname{erf} y \le 1$ and $e^{-y} \le 1$ for $y \ge 0$, we have

$$\rho_i \le c_4[(7/2 + 2\,|b|)t + 2\pi^{-1/2}t^{1/2}]\rho_{i-1}\,. \tag{2.14}$$

Let the quantity inside the square brackets be denoted by $\mu(t)$, which is nonnegative. From (2.4), $\rho_0 \le c_1$. Thus it follows from induction that

$$\rho_n \le [c_4\mu(t)]^n\rho_0 \le c_1[c_4\mu(t)]^n.$$

Let us choose the time interval $[0, \sigma]$ such that $c_4\mu(t) < 1$ so that the sequence $\{M_i\}$ converges uniformly and geometrically to a solution on $[0, \sigma]$ with the use of the Lebesgue convergence theorem. Since $M_i \ge 0$, this solution is nonnegative.

Next, we start from $t = \sigma - \eta$, where η is an arbitrarily chosen positive constant such that $\sigma - \eta > 0$. Using an argument similar to the above, we obtain the inequality

$$c_4[(7/2 + 2\,|b|)(t - \sigma + \eta) + 2\pi^{-1/2}(t - \sigma + \eta)^{1/2}] < 1$$

restricting the time interval for convergence. To satisfy this inequality, we can choose the time interval to be $[\sigma - \eta, 2\sigma - \eta]$. Thus, $\{M_i\}$ converges uniformly and geometrically to a nonnegative solution for $0 \le t \le 2\sigma - \eta$. By repeating the above procedures, we have uniform and geometrical convergence of $\{M_i\}$ to a nonnegative solution of the problem (1.1)–(1.3).

A similar argument applied to the sequence $\{m_i\}$ shows that $\{m_i\}$ converges uniformly and geometrically to a nonnegative solution. From Theorem 3, the nonnegative solution of the problem (1.1)–(1.3) is unique. Thus, (2.12) holds.

3. Alternating bounds. Let us construct the sequence $\{u_i(x, t)\}$ by $u_0 \equiv M_0$ in D^-, and for $i = 0, 1, 2, \cdots$,

$$Lu_{i+1} = kg(x, t; u_i) \quad \text{in} \quad D,$$

$$u_{i+1} = \phi \quad \text{on} \quad C,$$

$$Au_{i+1} = aB(t; u_i) - f \quad \text{on} \quad S,$$

where u_i tends to zero as x tends to infinity for $t \geq 0$. This gives an alternating sequence, which is different from that by Keller and Olmstead [3].

THEOREM 5. If assumptions (g_2), (g_4), (B_2) and (B_4) hold, then any nonnegative solution u of the problem (1.1)–(1.3) satisfies

$$u_1 \leq \cdots \leq u_{2i+1} \leq \cdots \leq u \leq \cdots \leq u_{2i} \leq \cdots \leq u_0 \text{ in } D^-. \tag{3.1}$$

Proof. First, we show that $u \leq u_0$ in D^-. Since assumptions (g_2) and (g_4) imply (g_3), and $u \geq 0$ in D^-, we have $g(x, t; u) \geq 0$ in D. Similarly, assumptions (B_2) and (B_4) imply (B_3), and we have $B(t; u) \geq 0$ on S. Thus,

$$L(u_0 - u) = -kg(x, t; u) \leq 0 \quad \text{in} \quad D,$$

$$A(u_0 - u) = -aB(t; u) \leq 0 \quad \text{on} \quad S.$$

Since $u_0 - u = 0$ on C, we have $u_0 \geq u$ in D^- by using Lemma 2.

Because $u \leq u_0$ in D^-, it follows from assumptions (g_4) and (B_4) respectively that $L(u - u_1) \leq 0$ in D, and $A(u - u_1) \leq 0$ on S. As $u - u_1 = 0$ on C, we have $u \geq u_1$ in D^-.

Let us assume that for a particular value of i, say j,

$$u_1 \leq \cdots \leq u_{2j+1} \leq u \leq u_{2j} \leq \cdots \leq u_0 \quad \text{in} \quad D^-.$$

Then for $i = j + 1$, it follows from assumptions (g_4) and (B_4) respectively that $L(u_{2j+2} - u) \leq 0$ in D, and $A(u_{2j+2} - u) \leq 0$ on S. Since $u_{2j+2} - u = 0$ on C, we have $u_{2j+2} \geq u$ in D^-. By repeating the argument for $u_{2j} - u_{2j+2}$, $u - u_{2j+3}$, and $u_{2j+3} - u_{2j+1}$ respectively, we have $u_{2j} \geq u_{2j+2}$, $u \geq u_{2j+3}$, and $u_{2j+3} \geq u_{2j+1}$, From the principle of mathematical induction, we have (3.1).

The subsequence $\{u_{2i+1}\}$ is monotone nondecreasing and bounded above by u_0 while the subsequence $\{u_{2i}\}$ is monotone nonincreasing and bounded below by u_1. It has not been proved above that the alternating sequence converges to a solution of the problem (1.1)–(1.3). Even if the odd and the even subsequences converge respectively to a lower bound and an upper bound, neither of them may be a solution. Our next result shows that under additional conditions, the sequence $\{u_i\}$ does converge to a solution, which need not be nonnegative.

THEOREM 6. Under the hypotheses of Theorem 5, assumptions (g_1) and (B_1), $b^2 - h \leq 0$ if $b < 0$, and $g \equiv B \equiv 0$ for $t > c$ where c is a nonnegative constant, the sequence $\{u_i\}$ converges uniformly and geometrically to a solution of the problem (1.1)–(1.3).

Proof. First, let us show that u_1 is bounded below by a constant. From (2.4),

$0 \leq u_0 \leq c_1$ in D^-. Thus, $g(x, t; u_0) \geq 0$ and $B(t; u_0) \geq 0$. It follows that

$$u_1(x, t) \geq -k \int_0^t \int_0^\infty R(x, t; \xi, \tau) g(\xi, \tau; u_0)\, d\xi\, d\tau - a \int_0^t R(x, t; 0, \tau) B(\tau; u_0)\, d\tau$$

after having dropped out the nonnegative terms on the right hand side. Let

$$\begin{aligned} d &= t \quad \text{if} \quad t \leq c, \\ &= c \quad \text{if} \quad t > c. \end{aligned}$$

Using assumptions (g_1), (g_2), (B_1) and (B_2), $r \geq kp$, $s \geq aq$, and $c_4 = \max\{r, s\}$, we have

$$u_1(x, t) \geq -c_1 c_4 \left[\int_0^d \int_0^\infty R(x, t; \xi, \tau)\, d\xi\, d\tau + \int_0^d R(x, t; 0, \tau)\, d\tau \right].$$

A reasoning analogous to that in arriving at (2.14) from (2.13) gives

$$u_1(x, t) \geq -c_1 c_4 \{(7/2 + 2\,|b|)d + 2\pi^{-1/2}[t^{1/2} - (t - d)^{1/2}]\}.$$

Since

$$t^{1/2} - (t - d)^{1/2} \leq c^{1/2},$$

we have

$$u_1(x, t) \geq -c_1 c_4 [(7/2 + 2\,|b|)c + 2\pi^{-1/2} c^{1/2}].$$

Thus, u_1 is bounded below. An argument similar to that in proving the convergence of $\{M_i\}$ in Theorem 4 shows that the sequence $\{u_i\}$ converges uniformly and geometrically to a solution of the problem (1.1)–(1.3).

We note that in each step of the constructions of the maximal and the minimal solutions, the same Neumann function of $(L - r)w = 0$ in D and $(A - s)w = 0$ on S is used, except in the initial step of constructing M_0 when we use the Neumann function of $Lw = 0$ in D and $Aw = 0$ on S. This latter function is also used in the constructions of the alternating bounds.

References

[1] C. Y. Chan, *Positive solutions for nonlinear parabolic second initial boundary value problems*, Quart. Appl. Math. **31**, 443–454 (1974)

[2] J. E. Hartka, *Temperature of a semi-infinite rod which radiates both linearly and nonlinearly*, Quart. Appl. Math. **32**, 101–111 (1974)

[3] J. B. Keller and W. E. Olmstead, *Temperature of a nonlinearly radiating semi-infinite solid*, Quart. Appl. Math. **29**, 559–566 (1972)

[4] H. L. Royden, *Real analysis*, Macmillan Co., New York, 1963, p. 200

[5] I. Stakgold, *Boundary value problems of mathematical physics*, vol. 2, Macmillan Co., New York, 1968, pp. 209–210