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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

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The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{}$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx)\cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n]\cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n)\cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

—BOOK REVIEW SECTION—

Generalized inverses: theory and application. By Adi Ben-Israel and Thomas N. E. Greville. New York, John Wiley and Sons, 1974. xi + 395 pp. $22.50.

The generalized inverse, *par excellence,* of a matrix is probably the one which has come to be known as the Moore-Penrose inverse. Its definition and basic elementary properties are as follows. Let all the matrices in question have complex elements and let * designate the conjugate transpose. Given *any* $m \times n$ matrix A, there is a *unique* $n \times m$ matrix X which satisfies the conditions (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, (4) $(XA)^* = XA$. X is called the Moore-Penrose inverse of A and will be designated by A^+. If A is square and nonsingular then $A^+ = A^{-1}$. The system of linear equations $AX = B$ (A is $m \times n$) has a solution if and only if $AA^+B = B$. In this case, the general solution is $X = A^+B + (I - A^+A)Y$ for an arbitrary vector Y. The matrix A^+A is a projection. The system $AX = B$ always has a least-squares solution. This least-squares solution is unique if and only if the columns of A are linearly independent in which case it is given by A^+B. In any case, A^+B is the least-squares solution of minimum (Euclidean) norm. Through this least-squares property a variety of applications arise. If the column of A are independent (covering, perhaps, 80% of the applications) then A^+ is given explicitly by $A^+ = (A^*A)^{-1}A^*$, which comes from the solution of the normal equations $A^*AX = A^*B$. In this case, computer languages (such as *APL*) have found it convenient to deliver A^+ when A^{-1} is asked for.

Finally, we mention the relationship of the Moore-Penrose inverse to eigenvalues. If A is $m \times n$ and has rank r, then by the "diagonal decomposition theorem", one can find unitary matrices U and V of orders m and n such that $UAV = D = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ where D_r is a diagonal matrix of order r, diagonal elements are positive and are the singular values of A, i.e., the eigenvalues of A^*A. The matrix D is $m \times n$. Then, $A^+ = VD^+U$ where D^+ is the $n \times m$ matrix

$$\begin{pmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

After numerous rumbles extending over a period of a half century or so embracing both finite matrices as well as operators, the volcano of generalized matrix inverse erupted in full intensity in the mid-fifties. The topic has been pursued vigorously since then by numerous skillful and devoted mathematicians, applied mathematicians, statisticians, etc. This book presents a systematic development of this work, concentrating mainly on the matrix-theoretic aspects. It is beautifully written, and, assuming that your matrix muscles are in good shape, is a pleasure to read. Most of the topics presented should be of wide interest, though here and there a few hobbyhorses are ridden.

The table of contents is 0. Introduction, 1. Existence and construction of generalized inverses, 2. Linear systems and characterization of generalized inverses, 3. Minimal properties of generalized inverses, 4. Spectral generalized inverses, 5. Generalized inverses of partitioned matrices, 6. A spectral theory for rectangular matrices, 7. Computational aspects of generalized processes, 8. Generalized inverses of linear operators between Hilbert spaces.

This book is ideal for a course on generalized inverses and toward that end numerous exercises have been included. The one reservation I have is that the sub-title is somewhat misleading. I should like to have seen more "hard core" applications included. But it is true that one man's theory is very often another man's application.

PHILIP J. DAVIS (*Providence*)

The experimental foundations of solid mechanics. By James F. Bell. Springer-Verlag, Berlin, 1973. 813 pp.

This volume, which covers three centuries of experimental work in solid mechanics, contains a vast body of information which has been assembled with scholarly skill by Professor Bell. The labor

involved in the compilation of this work must have been immense, and the author must be congratulated, not only on his zeal in pursuing the multitude of references, many of which must have been very elusive, but also on the interesting way in which the volume is presented. In recent years Professor Bell has contributed much to the study of several branches of solid mechanics, and it is clear that this has to some extent colored his choice of contemporary experimental work on which to comment and, to a lesser extent, his choice of experimental work of the past which he considers significant. This results in some loss of objectivity but, in the opinion of the reviewer, this loss is more than compensated for by the lively character that it has given to the whole work.

The first chapter is a short introduction which discusses the two main purposes of experimental work in solid mechanics, namely the experimental determination of constitutive relations for real solids and the verification of theoretical predictions based on the application of such relations to various physical situations. The second chapter describes the long history of the measurement of nonlinear elastic response in materials as varied as wood, metals, glass, paints, concrete and animal tissue, and discusses the extent to which linearity can be assumed when the deformations are sufficiently small. The third chapter deals with experiments in the region of such very small deformations and describes acoustic measurements as well as very careful quasi-static stress-strain measurements. Professor Bell very obviously greatly admires the work of Guillaume Wortheim, whom he describes as a "Faraday without a Maxwell," Although most of Wertheim's discoveries have since been confirmed, during his lifetime he had to contend with violent attacks on his work because his experimental results did not conform with Cauchy's uniconstant theory, which predicts a universal 'value' of Poisson's ratio of 1/4. Wertheim's experimental results disproved this assertion, and indicated that if Poisson's ratio had a single value, it was 1/3. As a result of this generalization, Wertheim has since often been described as a proponent of a uniconstant theory, although as Bell shows he was not in fact at all wedded to such a view.

The last chapter of this volume deals with work on finite deformation, and in addition to a full description of Bell's own extensive experimental work in this field, gives a lucid account of work on rubber-like elasticity with an account of the experimental work on this material from "Joule to Rivlin." The results are described and assessed with clarity and enthusiasm.

Some parts of the book, particularly the comments on contemporary work, are likely to provoke controversy and most readers will find subjects which they feel should have been dealt with more fully; for example, the present reviewer would have liked to find a fuller treatment of contemporary experimental work on visco-elastic response. All will agree, however, that this volume is a valuable one and fills a gap which has long been apparent in the contemporary literature.

H. KOLSKY (*Providence*)

Inners and stability of dynamic systems. By E. I. Jury. John Wiley & Sons, New York, 1974. xii + 308 pp. $17.95.

This book is an exhaustive treatment of the problem of obtaining information on the distribution of the roots of polynomials and, in particular, the eigenvalues of matrices. Among the many applications discussed and illustrated are the stability of linear, time-invariant, continuous and discrete dynamical systems (ordinary linear autonomous differential and difference equations), the evaluation of complex integrals that arise in problems of communication, control and filtering, absolute stability (the Lur'e problem), optimality, sensitivity, data and image processing, and numerical analysis. The unifying theme is the expression of criteria in terms of the determinants of the inners of a matrix. These determinants have been called "disencumbered remainders" (1862), "subresultants" (1907), and "bigradients" (1968). The inners of a matrix are obtained by successively dropping the outer rows and columns. The last chapter (Chapter 7) is devoted to computational algorithms for evaluating these determinants and for using the various criteria given. Much of the book is based on the author's own research. It is painstakingly thorough and contains a wealth of information, and results are carefully stated and illustrated. Many references are given to more complete discussions of the theory and applications. Because of the large number of significant applications and the practicality of the methods this is an important and useful book.

J. P. LASALLE (*Providence*)

—BOOKS RECEIVED—

Notice in this section does not preclude later full review in the Book Reviews Section.

Dimensionsanalyse. By H. Görtler. Springer-Verlag, 1975. ix + 247 pp. $23.80 (in German).

This work explores the theory of physical dimensions with applications to dimensional analysis and model theory, with numerous examples and an account of the history of the subject.

Group theory and quantum mechanics. By B. L. van der Waerden. Springer-Verlag, 1974. viii + 211 pp. $23.00.

This is the English-language version—and not just a translation—of the German 1932 edition. The entire book has been rewritten and brought up to date by the author.

Diffusion processes and their sample paths. By K. Ito and H. P. McKean, Jr. Second printing (corrected). Springer-Verlag, 1974. xiv + 321 pp. $27.00.

This book investigates the stochastic processes associated with diffusion when the assumption of spatial homogeneity is dropped. The theory is illustrated with numerous examples.

Random processes. Part II: Poisson and jump point processes. Edited by Anthony Ephremides. Benchmark Papers in Electrical Engineering and Computer Science, vol. 11. Dowden, Hutchison & Ross, Inc., distributed by Halsted Press, a division of John Wiley & Sons, 1975. x + 351 pp. $24.00.

This volume contains classical papers, with comments by the editor, in the following fields: Poisson models in communications (Bar-David, Rubin, Snyder), statistical analysis and application of shot noise (Beutler and Leneman, Bar-David and Nemirovsky), general point processes (Leadbetter, Cox and Lewis, Solomon and Wang), martingale representation (Boel and Varaiya and Wong).

Floating-point computation. By P. H. Sterbenz. Prentice-Hall, Inc., 1974. xiv + 316 pp. $15.00.

This book treats floating-point number systems, overflow and underflow, error analysis, double-precision arithmetic rounding, radix conversion and, in general, the arithmetic performed in digital computers as the result of programs written in high-level languages.

Nonlinear theory of elastic stability. By K. Huseyin. Noordhoff International Publishing, Leyden, Netherlands, 1975. xxii + 220 pp.

The theory treated in this book falls into three main categories: a phenomenological treatment of instabilities in conservative systems; fundamental theorems of theoretical and practical value; systematic methods of analysis such as the multiple-parameter perturbation technique.

Theories of elastic plates. By V. Panc. Noordhoff International Publishing, Leyden, 1975. xx + 716 pp.

This book presents theories of elastic plates in which both bending and transverse shear effects are taken into account. In contrast to the Kirchoff–Love theory, which leads to fourth-order differential

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equations, these theories result in sixth-order equations, so that three boundary conditions can be accommodated at each edge. Practically important numerical examples are included.

Small deformations of thin shells. By P. Seide. Noordhoff International Publishing, Leyden, 1975. xvi + 654 pp.

This book provides a rigorous derivation of elastic thin shell theory and a unified approach to the various approximations which have been proposed. Extensions of the theory to include thermal stresses, nonuniformly thick shells, anisotropic and layered materials, closely spaced stiffeners and ribs, and dynamic effects are treated and illustrated. Numerical methods, including the finite-element method, are discussed.

Mechanics of visco-elastic media and bodies. Edited by Jan Hult. Springer-Verlag, 1975. viii + 391 pp. $29.30.

These are the proceedings of a symposium held in Gothenburg, Sweden, September 2–6, 1974. The topics discussed include the following: 1. Limits of linearized theories and effective versions of the nonlinear theory of viscoelasticity; 2. Theory of experimental studies in linear and nonlinear viscoelasticity aiming at the determination of constitutive equations and conditions of strength and failure under an arbitrary deformation process; 3. Experimental and theoretical studies of thermal effects in viscoelasticity taking into account thermomechanical coupling; 4. Novel analytical and numerical techniques for the solution of problems in viscoelasticity and thermo-viscoelasticity.

Three-dimensional crack problems. By M. K. Kassir and G. C. Sih. Noordhoff International Publishing, 1975. liii + 452 pp.

This is Volume II of *Mechanics of Fracture*, of which Volume I dealt mostly with two-dimensional cracks. An introductory chapter by G. C. Sih expounds a three-dimensional strain energy density factor theory of crack propagation and the ten chapters provide a systematic approach for obtaining stress intensity factors of three-dimensional cracks. Chapters 1 and 2 deal with embedded and external circular cracks that are loaded non-axisymmetrically. Cracks occupying the interior and exterior regions of a plane ellipse are treated in chapters 3 and 4. In chapter 5, a closed-form solution for the problem of an arbitrarily located concentrated force on a semi-infinite plane crack is found. Chapter 6 discusses a multilayered body of dissimilar materials. Chapter 7 is concerned with the influence of the specimen boundary on the intensity of crack-border stress fields. Anisotropic materials whose elastic properties are direction-sensitive are considered in chapter 8 and nonhomogeneous materials whose elastic properties vary from one location to another in chapter 9. The last chapter briefly introduces some aspects of dynamic crack propagation.

Introduction to mathematical statistics. By L. Schmetterer. Springer-Verlag, 1975. vi + 502 pp. $50.00.

This book—volume 202 of the "Grundlehren" series— is a translation, by Kenneth Wickwire, of the 1966 original. References and bibliography and some sections have been upgraded and enlarged. It is a rigorous and mathematically demanding account of probability theory, sampling, hypothesis testing, confidence sets, estimation theory, regression theory, sampling theory of multidimensional normal distributions, and nonparametric statistics.

Integral equations: a reference text. By P. P. Zabreyko, A. I. Koshelev, M. A. Krasnoselskii, S. G. Mikhlin, L. S. Rakovshchik and V. Ya. Stetsenko. Noordhoff International Publishing, 1975. xix + 443 pp.

This is a translation, by T. O. Shaposhnikova, R. S. Anderssen and S. G. Mikhlin, of the original published in Moscow in 1968. The following areas are treated: Fredholm theory, Hilbert–Schmidt theory, equations which can be solved in closed form, and equations with non-negative kernels. Also,

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the generalization of the Fredholm theory for equations with compact operators and the theory of integral equations of convolution type (including Wiener–Hopf and dual equations) are treated. Nonlinear integral equations are discussed in the last chapter.

Finite elements in fluids. Volume 2: Mathematical foundations, Aerodynamics and lubrication. Edited by R. H. Gallagher, J. T. Oden, C. Taylor and O. G. Zienkiewicz. John Wiley and Sons, 1975. xi + 287 pp. $26.50.

The thirteen papers in this book were presented at the International Symposium in the Finite Element Method in Flow Problems, held at University College of Wales, Swansea, July 7–11, 1974.

Applied multiple regression and correlation analysis for the behavioral sciences. By Jacob Cohen and Patricia Cohen. Halsted Press, a division of John Wiley and Sons, 1975. xxi + 490 pp. $19.95.

A non-mathematical exposition, with emphasis on concrete problems, of the subject.

Basic concepts in information theory and statistics—axiomatic foundations and applications. By A. M. Mathai and P. N. Rathie. Halsted Press, a division of John Wiley and Sons, 1975. x + 137 pp. $9.95.

This is a research monograph, the contents drawn mainly from research papers by the authors on axiomatic foundations of information theory and statistics.

Positional astronomy. By D. McNally. Halsted Press, a division of John Wiley and Sons, 1975. xiii + 375 pp. $15.95.

This book—conceived as an undergraduate text and thus eschewing advanced mathematics—sets down in a logical manner the principles upon which astronomical coordinate systems are based. The phenomena affecting star positions are discussed and formulae to allow for their effects are derived. Orbit theory is considered to the extent that elementary procedures are given for determining the elements of visual spectroscopic and eclipsing binary stars. The method of determining the place of a planet in the sky is also discussed, as is a direct method for determining the elements of orbits within the solar system. The prediction of eclipses of both the sun and the moon are treated, as are the prediction and reduction of occultations.

The analysis of time series: theory and practice. By C. Chatfield. Halsted Press, a division of John Wiley and Sons, 1975. xiv + 263 pp. $19.75.

This is a volume in the series *Monographs on applied probability and statistics* (formerly Methuen's Monographs). It assumes a knowledge of basic probability and elementary statistical inference, and is written with the purpose of bridging the gap between theory and practice. The chapter headings are: introduction, simple descriptive techniques, probability models for time series, estimation in the time domain, forecasting, stationary processes in the frequency domain, spectral analysis, bivariate processes, linear systems.

Sequential methods in statistics. By G. Barrie Wetherill. Halsted Press, a division of John Wiley and Sons, 1975. x + 232 pp. $10.95.

This is a second edition of a book first published in 1966 as a Methuen's Monograph on Applied Probability and Statistics. It is a survey, with emphasis on methods of practical importance. It discusses the sequential t-test, estimation of points on quantal response curves, sequential estimation (with Neymann–Pearson and Bayesian approaches). The last chapter briefly discusses plant selection trials and screening procedures for drugs. There are problems on each topic, some of research character.