—NOTES—

ON GREENSPAN'S TRANSFORMATION*

BY CHIH-BING LING (*Virginia Polytechnic Institute and State University*)

1. In an early issue of the Quarterly, Greenspan [1] used the transformation

$$x = p^* \cos\theta + r^* \cos 3\theta, \qquad y = q^* \sin\theta - r^* \sin 3\theta \tag{1}$$

to approximate an ovaloid hole by taking

$$p^* = 2.063, \qquad q^* = 1.108, \qquad r^* = -0.079 \tag{2}$$

and also to approximate a square hole by taking

$$p^* = q^* = 1, \qquad r^* = -0.14. \tag{3}$$

The ovaloid hole is of the shape of a square with a semicircle erected on each of the two opposite sides. The resulting square hole has rounded corners with the sides parallel to the coordinate axes. By taking $r^* = 0.14$ instead, the same square but with the diagonals parallel to the coordinate axes is obtained.

The purpose of the note is to generalize the transformation to approximate first a general ovaloid hole and second a hole in the shape of a regular polygon. The general ovaloid hole has a rectangle in the middle of the hole instead of a square. The resulting regular polygon is a regular curvilinear polygon. Besides, a new feature is introduced into each such transformation in that the sides of the closed curve have zero curvature at the midpoints.

2. For the general ovaloid hole, consider the mapping function

$$z = \lambda\left(\zeta + \frac{2pq}{\zeta} + \frac{q^2}{\zeta^3}\right), \qquad (|q^2| < 1) \tag{4}$$

where p and q are two parameters either both real or both purely imaginary. The function transforms the exterior region of a closed curve in the z-plane into the exterior of a unit circle in the ζ-plane with the center at the origin. The closed curve has two curved sides and two rounded ends and is symmetrical with respect to both the x-axis and y-axis. Define a pair of polar coordinates (ρ, θ) in the ζ-plane by $\zeta = \rho \exp(i\theta)$. The unit circle is thus represented by $\rho = 1$. The center of the closed curve is at the origin. Let the major axis be on the x-axis and the minor axis be on the y-axis. Further, let the semi-minor axis be of length unity. Then $z = \pm i$ correspond to $\theta = \pm\pi/2$, respectively, on the unit circle $\rho = 1$. Consequently,

$$\lambda = 1/(1 - 2pq + q^2), \tag{5}$$

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which is positive when $0 < pq < (1 + q^2)/2$. The point $\theta = 0$ on the unit circle corresponds to $z = \gamma$, where γ is the semimajor axis of the closed curve given by

$$\gamma = (1 + 2pq + q^2)/(1 - 2pq + q^2), \tag{6}$$

which is greater than unity when $\lambda > 0$. Note that γ here is the ratio of the semimajor axis to the semiminor axis of the hole. The radius of curvature of the closed curve is found to be equal to

$$R = \frac{\lambda\{1 + 4p^2q^2 + 9q^4 - 4pq(1 - 3q^2)\cos 2\theta - 6q^2 \cos 4\theta\}^{3/2}}{1 - 4p^2q^2 - 27q^4 - 24pq^3 \cos 2\theta + 6q^2 \cos 4\theta}. \tag{7}$$

Suppose that the curvature is zero at $z = \pm i$ or $\theta = \pm\pi/2$ so that the curve is straight at the midpoints of the sides. At these points, the denominator of the preceding fraction vanishes. This leads to

$$(1 + 2pq - 3q^2)(1 - 2pq + 9q^2) = 0. \tag{8}$$

Thus two quadratic equations are obtained. One of the roots of the second equation suitable for our purpose is

$$q = (p - i(9 - p^2)^{1/2})/9. \tag{9}$$

With this relation of p and q, we have for a prescribed value of the ratio γ,

$$\lambda = \tfrac{1}{8}(4\gamma + 5), \qquad q^2 = -\frac{1}{4\gamma + 5}, \qquad pq = \frac{2(\gamma - 1)}{4\gamma + 5}. \tag{10}$$

In particular, when $\theta = 0$ or π, the radius of curvature at the crown of the rounded ends is

$$R_0 = -9/4(\gamma - 1). \tag{11}$$

3. For a curvilinear regular polygon, consider the mapping function

$$z = \lambda^*\left(a\zeta - \frac{p}{a^s\zeta^s}\right), \tag{12}$$

where λ^*, a and p are real parameters. When the parameters are properly chosen, the mapping function transforms conformally the exterior region of a hypotrochoid in the z-plane with the center at the origin into the exterior of a unit circle $\rho = 1$ in the ζ-plane. The hypotrochoid is a regular curvilinear polygon of $(s + 1)$ sides provided that s is an integer not less than 2. When θ varies from $-\pi$ to π on the unit circle $\rho = 1$, the corresponding curve in the z-plane is a complete hypotrochoid. In particular, when $|p| = 1/s$ and $a = 1$, the hypotrochoid becomes a hypocycloid with $(s + 1)$ cusps. We shall henceforth let $|p| = 1/s$ and $a > 1$ so that the curve is a hypotrochoid with rounded vertices. The sides of the hypotrochoid are somewhat curved. The curvature at each midpoint changes from concavity to convexity when a increases from unity up to a certain critical value. We let a assume this critical value so that the sides are straight or flattened at each midpoint with zero curvature.

Let the midpoint of one of the sides of the hypotrochoid be on the x-axis at $z = 1$ and the corresponding point on the unit circle $\rho = 1$ be at $\theta = 0$. Then the mapping function becomes

$$z = \lambda^*\left(a\zeta - \frac{1}{sa^s\zeta^s}\right), \tag{13}$$

where

$$\lambda^* = sa^s/(sa^{s+1} - 1). \tag{14}$$

The radius of curvature of the hypotrochoid is found to be equal to

$$R = -\frac{\lambda^*\{a^{2s+2} + 1 + 2a^{s+1}\cos(s+1)\theta\}^{3/2}}{a^s\{a^{2s+2} - s - (s-1)a^{s+1}\cos(s+1)\theta\}}. \tag{15}$$

Hence, the midpoints of the sides have zero curvature if the denominator of the preceding fraction vanishes when $\theta = 0$. This leads to

$$a^{s+1} = s. \tag{16}$$

With this value of a, the mapping function finally becomes

$$z = \lambda\left(\zeta - \frac{1}{s^2\zeta^s}\right), \tag{17}$$

where

$$\lambda = s^2/(s^2 - 1). \tag{18}$$

In particular, when $\theta = \beta$, where

$$\beta = \pi/(s+1), \tag{19}$$

the radius of curvature at the vertex is

$$R_0 = -(s-1)/2(s+1). \tag{20}$$

Suppose that the hypotrochoid is rotated about its center through an angle β so that one of the adjacent vertices is now on the x-axis. Let the point on the unit circle corresponding to this vertex be at $\theta = 0$. Then the point corresponding to the midpoint of the side originally on the x-axis is at $\theta = \beta$. The mapping function becomes

$$z = \lambda\left(\zeta + \frac{1}{s^2\zeta^s}\right). \tag{21}$$

In either orientation, the hypotrochoid is symmetrical with respect to the x-axis. When s is an odd integer, it possesses a further symmetry with respect to the y-axis.

4. It is seen that the sides of the closed curve in the foregoing transformations have zero curvature at the midpoints. Greenspan's first transformation was intended for $\gamma = 2$ and his second corresponded to $s = 3$. The reason for his choice of the parameters was not given. The sides of the closed curve in his first transformation were slightly convex, but those in the second transformation were slightly concave, at the midpoints.

Reference

[1] M. Greenspan, *Effect of a small hole on the stresses in a uniformly loaded plate*, Quart. Appl. Math. **2**, 60–71 (1944)