ON DUAL INTEGRAL EQUATIONS WITH TRIGONOMETRIC KERNELS*

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1. Introduction. Dual integral equations arise when integral transforms are used to solve mixed boundary value problems of elasticity. Formal techniques for solving such integral equations have been developed vigorously during the last four decades, but it appears that sufficient effort has not been made to determine the conditions for the validity of various procedures. Usually the heuristic treatment proceeds on the assumption that the conditions for interchanging the order of various limiting processes are satisfied. These consist of term-by-term differentiation or integration of an infinite series and differentiation or integration of an integral with respect to a parameter. Analytical techniques required to deal with these problems often tend to be abstruse. Hence an existence and uniqueness theorem whose proof is simple to present appears to be of some interest.

Of concern here is the pair of dual integral equations

$$\int_0^\infty \Psi(\xi) \sin \xi x \, d\xi = f(x), \qquad 0 < x < 1, \tag{1}$$

$$\int_0^\infty \Psi(\xi) \cos \xi x \, d\xi = 0, \qquad x > 1. \tag{2}$$

$f(x)$ is real-valued and belongs to $L_2(0, 1)$. A square-integrable Ψ is to be determined. The integrals are in the usual L_2-sense.

2. Existence and uniqueness theorems.

THEOREM 1 (Existence): The pair of dual integral equations (1) and (2) has a solution if and only if there exist $p \in L_2(0, 1)$ and $q \in L_2(1, \infty)$ so that

$$\frac{2}{\pi} \int_0^1 f(x) \sin xy \, dx = \frac{2}{\pi} \int_0^1 p(t) \cos ty \, dt - \frac{2}{\pi} \int_1^\infty q(t) \sin ty \, dt. \tag{3}$$

Proof: If (3) is satisfied, then

$$\Psi(\xi) = \frac{2}{\pi} \int_0^1 p(t) \cos t\xi \, dt \tag{4}$$

is a solution of the dual integral equations, as can be directly verified. On the other hand, if a Ψ satisfies Eq. (1) and (2), let

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$$\int_0^\infty \Psi(\xi) \cos \xi x \, d\xi = p_1(x), 0 < x < 1, \tag{5}$$

$$\int_0^\infty \Psi(\xi) \sin \xi x \, d\xi = q_1(x), x > 1. \tag{6}$$

From the inversion theorems for Fourier cosine and sine transforms, it follows that

$$\Psi(\xi) = \frac{2}{\pi} \int_0^1 p_1(t) \cos t\xi \, dt \tag{7}$$

as well as

$$\Psi(\xi) = \frac{2}{\pi} \int_0^1 f(t) \sin t\xi \, dt + \frac{2}{\pi} \int_1^\infty q_1(t) \sin t\xi \, dt. \tag{8}$$

Combining (7) and (8) we get, to complete the proof,

$$\frac{2}{\pi} \int_0^1 f(t) \sin t\xi \, dt = \frac{2}{\pi} \int_0^1 p_1(t) \cos t\xi \, dt - \frac{2}{\pi} \int_1^\infty q_1(t) \sin t\xi \, dt.$$

Existence theorems in this form for dual integral equations with other Fourier kernels can be obtained in the same manner.

THEOREM 2 (Uniqueness): The solution of Eqs. (1) and (2), if it exists, is unique.

Proof: Suppose, if possible, that $\Psi_1(\xi) \neq \Psi_2(\xi)$ where $\Psi_1(\xi)$ and $\Psi_2(\xi)$ are in $L_2(0, \infty)$, and both $\Psi_1(\xi)$ and $\Psi_2(\xi)$ satisfy Eqs. (1) and (2). Let $\Psi(\xi) = \Psi_1(\xi) - \Psi_2(\xi)$. Then

$$\int_0^\infty \Psi(\xi) \sin \xi x \, d\xi = 0, 0 < | x | < 1, \tag{9}$$

$$\int_0^\infty \Psi(\xi) \cos \xi x \, d\xi = 0, | x | > 1. \tag{10}$$

Define

$$F(z) = \int_0^\infty \Psi(\xi) \exp (i\xi z) \, d\xi$$

where $z = x + iy$ is a complex variable. Since the integral converges uniformly, $F(z)$ is analytic in the upper half plane $\mathrm{Im}(z) > 0$ and on the real line $y = 0$, $F(x)$ is purely imaginary for $| x | > 1$ and $F(x)$ is real for $| x | < 1$. Hence $\{F(z)\}^2$ is an analytic function whose imaginary part vanishes for $y = 0$. Thus $\{F(z)\}^2$ is a real constant and $F(z)$ is either a purely real constant or a purely imaginary constant. If it is purely real, since $\mathrm{Re}\{F(z)\}$ vanishes for $| x | > 1$, it must be identically zero. Likewise, if it is purely imaginary since $\mathrm{Im}\{F(z)\}$ vanishes for $| x | < 1$, $F(z)$ must be zero identically once again. Hence, $\Psi(\xi) \equiv 0$.

3. Concluding remarks. How are we to determine the functions p and q in relation (3) for a given $f(x)$? Taking the Fourier sine transforms of both sides of (3), we obtain, using the results of Steiner [1, 2] on repeated Fourier transforms, the following equations:

$$\frac{1}{\pi} \int_0^1 p(t) \left(\frac{1}{w + t} + \frac{1}{w - t} \right) dt = q(w), w > 1. \tag{11}$$

and

$$\frac{1}{\pi}\int_0^1 p(t)\left(\frac{1}{w+t}+\frac{1}{w-t}\right)dt = f(w),\ 0 < w < 1. \qquad (12)$$

Eq. (12) is a singular integral equation of the Cauchy type. If it has a L_2-solution, then together with (11) it gives us the appropriate decomposition required for Eq. (3), and Eq (7) or Eq. (8) can be used to determine the solution $\psi(\xi)$ of the pair of dual integral equations (1) and (2).

Tricomi [3] has examined and obtained precise conditions for the solution of the singular integral equation of the Cauchy type, viz.

$$\frac{1}{\pi}\int_{-1}^1 \frac{\Phi(x)\,dx}{x-y} = f(y),\ -1 < y < 1. \qquad (13)$$

If $f \in L_p(-1, 1)$ with $p > 4/3$, then (13) has the solution

$$\Phi(x) = -\frac{1}{\pi}\int_{-1}^1 \frac{(1-y^2)^{1/2}}{(1-x^2)^{1/2}}\frac{f(y)}{y-x}\,dy + \frac{C}{(1-x^2)^{1/2}}\,.$$

C is an arbitrary constant. The first term belongs to $L_p(-1, 1)$ for every $p < 4/3$. A simple calculation shows that for $f(y) \equiv 1$, there is no L_2-solution. Thus it is not surprising that no simple conditions for the existence of a L_2-solution are well known. Indeed, the only result known to me is due to Söhnigen [4] and depends on the properties of the functions of a complex variable s defined by

$$\int_0^1 \left\{\left(\frac{1-x}{1+x}\right)^s + \left(\frac{1+x}{1-x}\right)^s\right\} f(x)\,dx.$$

More investigation of this problem is certainly of interest.

References

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