KINEMATIC WAVE MODELS FOR OVERLAND FLOW*

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1. Introduction. We consider the following problem: there is rainfall over a plane rectangular ground area (catchment basin) inclined at a slight angle to the horizontal, and there is infiltration into the ground. The ground is initially dry, the higher end of the ground remains dry at all times, and the rainfall ceases after time T. At the bottom end of the rectangle there is a stream or channel into which the ground flow discharges. We will assume that the rainfall and infiltration rate are time-dependent but their space dependency is only on the distance x from the higher edge of the rectangle. It is clear then that the streamlines of the overland flow are lines parallel to the inclined sides of the rectangle and the flow is the same on each streamline. We may ask the following questions: What is the depth $h(x, t)$ and velocity $u(x, t)$ of the flow? Secondly, it is clear that, starting at time T, a free boundary $x(t)$ will develop; $x(t)$ is the time history of the water edge as it recedes, because of downflow and infiltration, from the upper edge of the rectangle. What is $x(t)$?

An analogous problem can be formulated for a converging catchment basin; this is part of the inner surface of a cone with a very large angle at the vertex. More precisely, let segment PQ of length L intersect a horizontal plant at point P. The angle between PQ and the plane is small. Let P_1 be between PQ, $P_1Q = a$. Rotate PQ around the perpendicular to the plane at P through an angle less than 2π. Then the surface generated by P_1Q is the basin. Flow is along the generating segments.

The continuity and momentum equations for these problems are easily formulated [1, chapter 15]; we will obtain the continuity equation for each problem in Sec. 2. It is not possible, in general, to obtain explicit solutions, so it is a question of what reasonable simplifications can be made in order to obtain such solutions. There are two such simplifications. The first is to replace the momentum equation by $u = \alpha h^m$, where $m > 0$ and α is a friction coefficient which may depend on x. The justification for this equation is given in [1, chapter 15]; essentially, these terms are dominant, from the point of view of order of magnitude, in the general momentum equation. The second simplification is to restrict the rainfall and infiltration to be time-independent but still dependent on x. Under these conditions explicit solutions can be obtained for both problems, and this is the purpose of the paper. In Sec. 2 we formulate the problem, in Sec. 3 we consider the case infiltration term 0, and in Sec. 4 we consider the case infiltration term not 0.

The kinematic wave models studied in this paper have been discussed in a number of papers (see references). We have extended the results of those papers by allowing spatial dependency in the rainfall and infiltration rates, in the derivation of the free boundary, and in the unified approach to the two problems described above. There is a concomitant increase in the mathematical complication, but that does not prevent the obtaining of explicit solutions.

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2. Formulation of the problem. We consider first the plane rectangular catchment basin. Let x be the distance from the higher edge, $0 \le x \le a$, with $x = a$ the lower edge. Let $q(x, t)$ be the volume of rainfall per unit area per unit time, $f(x, t)$ the volume of infiltration per unit area per unit time, $h(x, t)$ the depth of water, and $u(x, t)$ the velocity, Flow is along lines perpendicular to $x = a$. We assume that, for a specified T,

$$q(x, t) = 0, \quad t > T, \quad q(x, t) > f(x, t), \quad 0 \le t \le T.$$

We assume that f depends on h in the following sense:

$$f(x, t) > 0 \quad \text{if} \quad h(x, t) > 0; \qquad f(x, t) = 0 \quad \text{if} \quad h(x, t) = 0.$$

If dy is perpendicular to dx than, referring to the volume of water on the rectangle $dxdy$, we have

inflow: $uh\, dy\, dt + q\, dx\, dy\, dt$,

outflow: $(uh + (uh)_x\, dx)\, dy\, dt + f\, dx\, dy\, dt$,

change in storage: $h_t\, dx\, dy\, dt$,

from which we get the continuity equation

$$h_t = (uh)_x = q - f. \tag{2.1}$$

The momentum equation [1, chapter 15] is

$$Q = uh = \alpha(x)h^n, \tag{2.2}$$

where $n > 1$ and $\alpha(x) > 0$ measures the surface roughness. From (2.1) and (2.2) we get

$$h_t + (\alpha h^n)_x = q - f. \tag{2.3}$$

We assume that h is 0 initially, and also at $x = 0$ for all t:

$$h(x, 0) = 0, \quad 0 \le x \le a, \quad h(0, t) = 0, \quad 0 \le t \le T. \tag{2.4}$$

It is physically plausible that there will be a free boundary $t = t^0(x)$ such that, in the semi-infinite strip $D = \{0 < x \le a, t \ge 0\}$, $h(x, t) = 0$ above and on $t = t^0(x)$ and $h(x, t) > 0$ below $t = t^0(x)$. $t = t^0(x)$ is the time history of the interface between the covered and uncovered areas of the catchment basin. Thus h is subject to (2.3) in D below $t = t^0(x)$, and satisfies (2.4) and the condition $h(x, t^0(x)) = 0$ for $0 \le x \le a$.

Using the same notation for the converging basin, we consider the volume of water on the area $(L - x)\, d\theta\, dx$:

inflow: $uh(L - x)\, d\theta\, dt + q(L - x)\, dx\, d\theta\, dt$,

outflow: $\{uh(L - x) + [uh(L - x)]_x\, dx\}\, d\theta\, dt + f(L - x)\, dx\, d\theta\, dt$,

change in storage: $h_t(L - x)\, dx\, d\theta\, dt$.

The continuity equation is, therefore,

$$h_t + (uh)_x = q - f + \frac{uh}{L - x}. \tag{2.5}$$

From (2.5) and (2.2) we get

$$h_t + (\alpha h_n)_x = q - f + \frac{\alpha h^n}{L - x}. \tag{2.6}$$

Again we have the boundary and initial conditions (2.4), and the free boundary $t = t^0(x)$.

We will obtain the explicit solutions of problems (2.3)–(2.4) and (2.6)–(2.4) when q and f depend only on x. We consider the case $f(x) = 0$ in Sec. 3 and the case $f(x) \neq 0$ in Sec. 4.

3. Infiltration term 0. In this case (2.3) is

$$h_t + (\alpha h^n)_x = q(x). \tag{3.1}$$

Taking x as parameter, the characteristic curves are given by

$$\frac{dt}{dx} = \frac{1}{n\alpha(x)h^{n-1}}, \quad \frac{dh}{dx} = \frac{q(x) - \alpha'(x)h^n}{n\alpha(x)h^{n-1}}. \tag{3.2}$$

The solution of (3.1)–(2.4) is the surface formed by the characteristics passing through $0 \leq x \leq a$ on the x-axis and $0 \leq t \leq T$ on the t-axis. Thus we add to (3.2) the initial conditions

$$t(x_0) = 0, \quad h(x_0) = 0, \tag{3.3}$$

or

$$t(0) = t_0, \quad h(0) = 0. \tag{3.4}$$

The solution of (3.2)–(3.3) is, indicating dependence on x_0,

$$t = t(x, x_0), \quad h = h(x, x_0),$$

and the solution of (3.2)–(3.4) is

$$t = t(x, t_0), \quad h = h(x, t_0).$$

The curve $t = t(x, 0)$ may (case B) or may not (case A) intersect $t = T$ in D. It will be seen, from (3.6) below, that these two cases are distinguished according as

$$T = \int_0^x \frac{d\xi}{n\alpha(\xi)^{1/n}p(\xi)^{(n-1)/n}}, \tag{3.5}$$

where

$$p(x) = \int_0^x q(\xi)\, d\xi$$

does (case B) or does not (case A) have a root between 0 and a. Since the right side of (3.5) is an increasing function of x, (3.5) has no root or exactly one root in $0 < x < a$. If $F(x)$ is the right side of (3.5) then $F(a) \leq T$ implies case A and $F(a) > T$ implies case B.

In case A, D is divided into three regions D_1, D_2, and D_3; these are the intersections with D of, respectively, $\{t \geq T\}$, $\{t(x, 0) \leq t \leq T\}$, and $\{0 \leq t \leq t(x, 0)\}$. In D_2

$$h(x, t_0) = \left[\frac{p(x)}{\alpha(x)}\right]^{1/n}, \quad t(x, t_0) = t_0 + \int_0^x \frac{d\xi}{n\alpha(\xi)^{1/n}p(\xi)^{(n-1)/n}}. \tag{3.6}$$

Here $0 \leq t_0 \leq T$. The curves $t = t(x, t_0)$ do not intersect in D_2, $t_x(x, t_0) > 0$, and $t_{t_0}(x, t_0) = 1$. In D_2

$$h(x, t) = \left[\frac{p(x)}{\alpha(x)}\right]^{1/n}.$$

In D_3

$$h(x, x_0) = \left[\frac{p(x) - p(x_0)}{\alpha(x)}\right]^{1/n}, \tag{3.7}$$

$$t(x, x_0) = \int_{x_0}^{x} \frac{d\xi}{n\alpha(\xi)^{1/n}[p(\xi) - p(x_0)]^{(n-1)/n}}.$$

In (3.7) $0 \le x_0 \le a$. By partial integration we get, from (3.7),

$$t(x, x_0) = \left[\frac{p(x) - p(x_0)}{\alpha(x)}\right]^{1/n} \frac{1}{q(x)} - \int_{x_0}^{x} [p(\xi) - p(x_0)]^{1/n} \frac{d}{d\xi} \frac{1}{\alpha(\xi)^{1/n} q(\xi)}\, d\xi.$$

We will have $t_{x_0}(x, x_0) < 0$ provided

$$\frac{d}{dx} \frac{1}{\alpha^{1/n} q} \le 0,$$

or equivalently

$$\frac{d}{dx}(\alpha q^n) \ge 0. \tag{3.8}$$

We note that (3.8) includes the case $\alpha(x)$ and $q(x)$ both constant. Thus, under condition (3.8), $t_{x_0}(x, x_0) < 0$. Also $t_x(x, x_0) > 0$. The curves $t = t(x, x_0)$ therefore do not intersect in D_3, and (3.7) defines $h(x, t)$ in D_3.

To obtain the solution in D_1 we set $q(x) = 0$ in (3.2) and impose initial conditions

$$t(x_0^*) = T, \quad h(x_0^*) = \left[\frac{p(x_0^*)}{\alpha(x_0^*)}\right]^{1/n}, \qquad 0 \le x_0^* \le a.$$

Then in D_1

$$h(x, x_0^*) = \left[\frac{p(x_0^*)}{\alpha(x)}\right]^{1/n},$$

$$t(x, x_0^*) = T + \frac{1}{n}[p(x_0^*)]^{-(n-1)/n} \int_{x_0^*}^{x} \alpha(\xi)^{-1/n}\, d\xi. \tag{3.9}$$

It is clear from (3.9) that $t_{x_0^*}(x, x_0^*) < 0$, so the curves $t = t(x, x_0^*)$ do not intersect in D_1. For fixed x $t(x, x_0^*)$ is defined on the interval $0 < x_0^* \le x$ and maps that interval in a one to one manner on $t \ge T$. Thus (3.9) defines $h(x, t)$ on $D(T) = \{0 < x \le a, t \ge T\}$. Since, for fixed x, $t \to \infty$ is equivalent to $x_0^* \to 0$, we get $h(x, t) \to 0$ as $t \to \infty$. From (3.9) we get

$$h(x, t) = \frac{\psi(x)}{(t - T)^{1/(n-1)}} + O(t^{-n/(n-1)}),$$

where

$$\psi(x) = \frac{\left[\int_0^x \alpha(\xi)^{-1/n}\, d\xi\right]^{1/(n-1)}}{n^{1/(n-1)}\alpha(x)^{1/n}}.$$

We note also from (3.9) that $h(x, t) > 0$ in $D(T)$ and that, for fixed t, $h(x, t) \to 0$ as $x \to 0$. Thus the free boundary $t^0(x)$ coincides with the t-axis above $t = T$.

From the above discussion we see that the behavior of $h(x, t)$ for fixed x is as follows: if $(x, t) \in D_3$, then, since

$$h_t(x, t) = \frac{h_{x_0}(x, x_0)}{t_{x_0}(x, x_0)}, \tag{3.10}$$

$h_t(x, t) > 0$. If $(x, t) \in D_2$ then $h_t(x, t) = 0$. If $(x, t) \in D_1$ then, since (3.10) applies with x_0 replaced by x_0^*, $h_t(x, t) < 0$.

We consider now case B. Let x^* be the root of (3.5). Let D_{11} be the part of D above $t = T$ and $t = t(x, 0)$, D_{12} the region bounded by $t = T$, $t = t(x, 0)$, and $x = a$, D_2 the region bounded by $t = T$, $t = t(x, 0)$, and $x = 0$, and D_3 the part of D below $t = t(x, 0)$ and $t = T$. The solution in D_{11} is given by (3.9), where $0 \leq x_0^* \leq x^*$, the solution in D_2 is given by (3.6), and the solution in D_3 is given by (3.7). To obtain the solution in D_{12}, let x_0^* be defined by $T = t(x_0^*, x_0)$, where $t(x, x_0)$ is given by (3.7):

$$T = \int_{x_0}^{x_0^*} \frac{d\xi}{n\alpha(\xi)^{1/r}[p(\xi) - p(x_0)]^{(n-1)/n}}, \qquad x^* \leq x_0^* \leq a. \tag{3.11}$$

We get, in D_{12},

$$h(x, x_0) = \left[\frac{p(x_0^*) - p(x_0)}{\alpha(x)}\right]^{1/n},$$

$$t(x, x_0) = T + \frac{1}{n}[p(x_0^*) - p(x_0)]^{-(n-1)/n} \int_{x_0^*}^{x} \alpha(\xi)^{-1/n}\, d\xi. \tag{3.12}$$

We prove now that $t_{x_0}(x, x_0) < 0$; this implies that the curves $t = t(x, x_0)$ do not intersect in D_{12}, and therefore (3.12) defines $h(x, t)$. A partial integration of (3.11) yields

$$T = \frac{[p(x_0^*) - p(x_0)]^{1/n}}{\alpha(x_0^*)^{1/n}q(x_0^*)} - \int_{x_0}^{x_0^*} [p(\xi) - p(x_0)]^{1/n} \frac{d}{d\xi}\frac{1}{\alpha(\xi)^{1/n}q(\xi)}\, d\xi. \tag{3.13}$$

Differentiating (3.13) with respect to x_0, we get

$$0 = \frac{[p(x_0^*) - p(x_0)]^{(1/n)-1}}{\alpha(x_0^*)^{1/n}q(x_0^*)}\left[q(x_0^*)\frac{dx_0^*}{dx_0} - q(x_0)\right]$$

$$+ \int_{x_0}^{x_0^*} [p(\xi) - p(x_0)]^{(1/n)-1} q(x_0) \frac{d}{d\xi}\frac{1}{\alpha(\xi)^{1/n}q(\xi)}\, d\xi. \tag{3.14}$$

From (3.14) and (3.8) we get

$$\frac{dx_0^*}{dx_0} > 0, \quad q(x_0^*)\frac{dx_0^*}{dx_0} - q(x_0) \geq 0,$$

and therefore

$$\frac{d}{dx_0}[p(x_0^*) - p(x_0)] = q(x_0^*)\frac{dx_0^*}{dx_0} - q(x_0) \geq 0,$$

$$\frac{d}{dx_0}\int_{x_0^*}^{x} \alpha(\xi)^{-1/n}\, d\xi < 0.$$

Thus $t_{x_0}(x, x_0) < 0$.

The behavior of $h(x, t)$ for fixed x in case B is the same as in case A if $0 < x \le x^*$; thus $h_t(x, t) > 0$ if $(x, t) \in D_3$, $h_t(x, t) = 0$ if $(x, t) \in D_2$, and $h_t(x, t) < 0$ if $(x, t) \in D_{11}$. If $x^* < x \le a$ then the same inequalities apply in D_3 and D_{11}, but in D_{12} we have $h_t(x, t) \le 0$ (since $h_{x_0}(x, x_0) \ge 0$ and $t_{x_0}(x, x_0) < 0$).

We consider now (2.6)–(2.4) when $f = 0$. Then

$$((L - x)h)_t + ((L - x)\alpha h^n)_x = (L - x)q(x). \tag{3.15}$$

If

$$(L - x)h(x) = \bar{h}(x), (L - x)\, q(x) = \bar{q}(x), \frac{\alpha(x)}{(L - x)^{n-1}} = \bar{\alpha}(x), \tag{3.16}$$

then (3.15) is the same as (3.1). Thus the entire discussion above applies to (3.15)–(2.4), with the appropriate modification of the various expressions for the solutions. Here we have

$$\bar{p}(x) = \int_0^x q(\xi)(L - \xi)\, d\xi.$$

We note, however, that condition (3.8) becomes

$$\frac{d}{dx}\, \alpha(x)(L - x(q)x)^n \ge 0. \tag{3.17}$$

This is unsatisfactory since it excludes the case $\alpha(x)$ and $q(x)$ both constant. The situation in $D_2 \cup D_1$ (case A) and in $D_2 \cup D_{11}$ (case B) is unaffected since (3.17) is not required in order that the curves $t(x, t_0)$ and $t(x, x_0^*)$ not intersect. In D_3 we have, in both cases,

$$t(x, x_0) = \int_{x_0}^x \frac{1}{n\alpha(\xi)^{1/n}} \left[\frac{L - \xi}{p(\xi) - p(x_0)}\right]^{(n-1)/n} d\xi.$$

If $\eta = (L - \xi)/(L - x_0)$,

$$t(x, x_0) = \int_{(L-x)/(L-x_0)}^1 \frac{L - x_0}{n\alpha(L - (L - x_0)\eta)^{1/n}} \left[\frac{(L - x_0)\eta}{\int_{x_0}^{L-(L-x_0)\eta} q(\xi)(L - \xi)\, d\xi}\right]^{(n-1)/n} d\eta,$$

and if $\sigma = (L - \xi)/(L - x_0)$,

$$t(x, x_0) = \int_{(L-x)/(L-x_0)}^1 \frac{(L - x_0)^{1/n}}{n\alpha(L - (L - x_0)\eta)^{1/n}} \left[\frac{\eta}{\int_\eta^1 q(L - \sigma(L - x_0))\sigma\, d\sigma}\right]^{(n-1)/n} d\eta.$$

In order that $t_{x_0}(x, x_0) < 0$ in D_3 it is sufficient that $f_{x_0}(x_0, x, \eta) < 0$, where

$$f(x_0, x, \eta) = \left[\frac{(L - x_0)}{\alpha(L - (L - x_0)\eta)}\right]^{1/n} \left[\int_\eta^1 q(L - \sigma(L - x_0))\sigma\, d\sigma\right]^{-(n-1)/n}.$$

If $z = L - (L - x_0)\eta$,

$$\frac{L - x_0}{\alpha(L - (L - x_0)\eta)} = \frac{L - z}{\eta\alpha(z)}.$$

Therefore $((L - x)/\alpha(x))' < 0$ implies

$$\frac{\partial}{\partial x_0} \frac{L - x_0}{\alpha(L - (L - x_0)\eta)} < 0.$$

Also $q'(x) \geq 0$ implies

$$\frac{\partial}{\partial x_0} \int_\eta^1 q(L - \sigma(L - x_0))\sigma \, d\sigma \geq 0.$$

Thus $((L - x)/\alpha(x))' < 0$ and $q'(x) \geq 0$ imply $f_{x_0}(x_0, x, \eta) < 0$, and therefore either (3.17) or

$$\frac{d}{dx} \frac{L - x}{\alpha(x)} < 0 \quad \text{and} \quad q'(x) \geq 0 \tag{3.18}$$

implies $t_{x_0}(x, x_0) < 0$. Condition (3.18) is satisfied when $\alpha(x)$ and $q(x)$ are both constant. Under conditions (3.17) or (3.18) the curves $t = t(x, x_0)$ do not intersect in D_3.

In regard to D_{12} the curves $t = t(x, x_0)$ do not intersect under condition (3.17), according to our earlier discussion. We show that the same is true under condition (3.18). We have, in D_{12},

$$t(x, x_0) = T + \int_{x_0^*}^{x} \frac{(L - \xi)^{(n-1)/n}}{n\alpha(\xi)^{1/n}} \, d\xi \left[\int_{x_0}^{x_0^*} q(\xi)(L - \xi)\, d\xi\right]^{-(n-1)/n},$$

$$T = \int_{x_0}^{x_0^*} \frac{(L - \xi)^{(n-1)/n}}{n\alpha(\xi)^{1/n}} \left[\int_{x_0}^{\xi} q(\xi)(L - \xi)\, d\xi\right]^{-(n-1)/n} d\xi.$$

These equations become, on introducing η and σ as above,

$$t(x, x_0) = T + \frac{1}{n} \int_{(L-x)/(L-x_0)}^{z} \left[\frac{L - x_0}{\alpha(L - (L - x_0)\eta)}\right]^{1/n} \cdot d\eta \left[\int_z^1 q(L - (L - x_0)\eta)\eta \, d\eta\right]^{-(n-1)/n},$$

$$nT = \int_z^1 \left[\frac{L - x_0}{\alpha(L - (L - x_0)\eta)}\right]^{1/n} \left[\int_\eta^1 q(L - (L - x_0)\sigma)\sigma \, d\sigma\right]^{-(n-1)/n} \eta^{(n-1)/n} \, d\eta,$$

where $z = (L - x_0^*)/(L - x_0)$. From the second equation it is clear that $dz/dx_0 < 0$. Then, from the first equation, $t_{x_0}(x, x_0) < 0$.

The expressions for the solution of (3.15)–(2.4) may be obtained from the solution of (3.1)–(2.4) by using (3.16).

4. Infiltration term not 0. We consider first

$$h_t + (\alpha h^n)_x = q(x) - f(x), \tag{4.1}$$

together with (2.4). The characteristic curves are

$$\frac{dt}{dx} = \frac{1}{n\alpha(x)h^{n-1}}, \quad \frac{dh}{dx} = \frac{q(x) - f(x) - \alpha'(x)h^n}{n\alpha(x)h^{n-1}}, \tag{4.2}$$

and the initial conditions are (3.3) or (3.4). We now have three cases depending on the relative disposition of $t = t^0(x)$, $t = T$, and $t = t(x, 0)$:

A. $t^0(x) > T > t(x, 0), \quad 0 < x \leq a$.

B_1. $t^0(x) > T$ and $t^0(x) > t(x, 0)$, but $t = T$ and $t = t(x, 0)$ intersect at x^*, i.e., $T = t(x^*, 0)$ and $0 < x^* < a$.

B_2. $t^0(x) > T$, but $t = T$ and $t = t(x, 0)$ intersect at $x = x^*$ and $t = t^0(x)$ and $t = t(x, x^*)$ ($t = t(x, x^*)$ is the prolongation $t = t(x, 0)$ to the right of $x = x^*$) intersect at $x = \bar{x}$, i.e., $t^0(\bar{x}) = t(\bar{x}, x^*)$ and $0 < \bar{x} < a$.

These cases can be distinguished prior to the solution of (4.1)–(2.4). We have case A when

$$T = \int_0^x \frac{d\xi}{n\alpha(\xi)^{1/n} r(\xi)^{(n-1)/n}}, \tag{4.3}$$

where

$$r(x) = \int_0^x (q(\xi) - f(\xi))\, d\xi,$$

does not have a root between 0 and a; otherwise we are in case B_1 or B_2. This is clear from (4.5) below. The prior distinction between B_1 and B_2 will appear in the discussion below. As in the discussion in Sec. 3, if $G(x)$ is the right side of (4.3) then $G(a) \leq T$ implies case A and $G(a) > T$ implies B_1 or B_2.

In case A there are three domains D_1, D_2, and D_3: D_2 and D_3 are as in case A, Sec. 2, and D_1 is the part of D between $t = T$ and $t = t^0(x)$. The solution of (4.2)–(3.4) in D_2 is

$$h(x, t_0) = \left[\frac{r(x)}{\alpha(x)}\right]^{1/n}, \quad t(x, t_0) = t_0 + \int_0^x \frac{d\xi}{n\alpha(\xi)^{1/n} r(\xi)^{(n-1)/n}}, \quad 0 \leq t_0 \leq T. \tag{4.4}$$

The curves $t(x, t_0)$ do not intersect in D_2, $t_x(x, t_0) > 0$, $t_{x_0}(x, t_0) = 1$, and

$$h(x, t) = \left[\frac{r(x)}{\alpha(x)}\right]^{1/n}.$$

In D_3

$$h(x, x_0) = \left[\frac{r(x) - r(x_0)}{\alpha(x)}\right]^{1/n},$$

$$t(x, x_0) = \int_{x_0}^x \frac{d\xi}{n\alpha(\xi)^{1/n}[r(\xi) - r(x_0)]^{(n-1)/n}}, \qquad 0 \leq x_0 \leq a. \tag{4.5}$$

As in Sec. 3 we impose the condition

$$\frac{d}{dx}[\alpha(x)(q(x) - f(x))^n] \geq 0 \tag{4.6}$$

in order that $t_{x_0}(x, x_0) < 0$. (4.6) holds if α, q, and f are constant. Then the curves $t = t(x, x_0)$ do not intersect in D_3, and (4.5) defines $h(x, t)$ in D_3. The solution in D_1 is obtained by solving (4.2) with $q(x) = 0$, subject to the conditions

$$t(x_0^*) = T, \quad h(x_0^*) = \left[\frac{r(x_0^*)}{\alpha(x_0^*)}\right]^{1/n}.$$

The solution in D_1 is

$$h(x, x_0^*) = \left[\frac{p(x_0^*) - s(x)}{\alpha(x)}\right]^{1/n},$$

$$t(x, x_0^*) = T + \int_{x_0^*}^x \frac{d\xi}{n\alpha(\xi)^{1/n}[p(x_0^*) - s(\xi)]^{(n-1)/n}}, \tag{4.7}$$

where

$$s(x) = \int_0^x f(\xi)\, d\xi$$

and $0 \leq x_0{}^* \leq a$. (4.7) is valid so long as $p(x_0{}^*) - s(x) > 0$; thus the free boundary $t^0(x)$ is obtained by solving for $x_0{}^*$ in $p(x_0{}^*) - s(x) = 0$ and inserting in $t = t(x, x_0{}^*)$. We have then

$$t^0(x) = T + \frac{1}{n} \int_{\psi(x)}^x \frac{d\xi}{\alpha(\xi)^{1/n}[s(x) - s(\xi)]^{(n-1)/n}}, \tag{4.8}$$

where $\psi(x)$ is the root of

$$\int_0^{\psi(x)} q(\xi)\, d\xi = \int_0^x f(\xi)\, d\xi. \tag{4.9}$$

Since $q(x) > f(x)$ there is always a unique root $\psi(x) < x$ of (4.9) when $0 \leq x \leq a$; $\psi(x)$ is an increasing function of x and $\psi(0) = 0$. It is clear from (4.7) that $t_{x_0{}^*}(x, x_0{}^*) < 0$. Then the curves $t = t(x, x_0{}^*)$ do not intersect in D_1, and therefore (4.7) defines $h(x, t)$ in D_1. The behavior of $h(x, t)$ for fixed x is as in Sec. 3: $h_t(x, t) < 0$ if $(x, t) \epsilon D_3$, $h_t(x, t) = 0$ if $(x,t) \epsilon D_2$, $h_t(x, t) > 0$ if $(x, t) \epsilon D_1$.

Case B_1 or B_2 occur when (4.3) has a root x^*, $0 < x^* < a$. To distinguish between B_1 and B_2 we obtain $t = t(x, x^*)$, the prolongation of $t = t(x, 0)$ above $t = T$ and to the right of $x = x^*$, by setting $x_0{}^* = x^*$ in (4.7). We are in case B_1 if $h(x, x^*) > 0$ for $x^* < x < a$, and we are in case B_2 if $h(\bar{x}, x^*) = 0$, $x^* < \bar{x} < a$, and $h(x, x^*) > 0$ when $x^* < x < \bar{x}$. Thus

$$\int_0^{x^*} q(\xi)\, d\xi > \int_0^x f(\xi)\, d\xi$$

for $x^* < x < a$ implies case B_1, and the existence of $\bar{x}$, $x^* < \bar{x} < a$, such that

$$\int_0^{x^*} q(\xi)\, d\xi = \int_0^x f(\xi)\, d\xi$$

implies case B_2.

In case B_1 there are four domains D_{11}, D_{12}, D_2, and D_3: D_2, D_3, and D_{12} are as in case B of Sec. 3, and D_{11} is the region bounded by $t = t^0(x)$, $t = T$, $t = t(x, x^*)$, and $x = a$. The free boundary $t^0(x)$ is given by (4.8). The solution in D_{11} is given by (4.7), with $0 \leq x_0{}^* \leq x^*$. The solution in D_2 is given by (4.4), and the solution in D_3 is given by (4.5). In D_{12}

$$h(x, x_0) = \left[\frac{p(x_0{}^*) - r(x_0) - s(x)}{\alpha(x)}\right]^{1/n},$$

$$t(x, x_0) = T + \int_{x_0{}^*}^x \frac{d\xi}{n\alpha(\xi)^{1/n}\, [p(x_0{}^*) - r(x_0) - s(\xi)]^{(n-1)/n}}, \tag{4.10}$$

where

$$T = \int_{x_0}^{x_0{}^*} \frac{d\xi}{n\alpha(\xi)^{1/n}\, [r(\xi) - r(x_0)]^{(n-1)/n}}. \tag{4.11}$$

Analogous to the discussion in Sec. 3, we have $t_{x_0}(x, x_0) < 0$ in D_{12} if we impose condition (4.6). Thus (4.10) defines $h(x, t)$ in D_{12}. The behavior of $h(x, t)$ for $0 < x \leq x^*$, and also when $x^* < x < a$ and $(x, t)\epsilon D_{11}$ or $(x, t)\epsilon D_3$, is as in case A. When $(x, t)\epsilon D_{12}$ it can be seen from (4.10) that $h_{x_0}(x, x_0) \geq 0$, so $h_t(x, t) < 0$.

The discussion for the case B_2 is the same as for the case B_1. In case B_2 the upper boundary of D_{12} consists partly of $t(x, x^*)$ and partly of $t^0(x)$; we need an explicit determination of the latter when $x \geq \bar{x}$. This is obtained from (4.10) and (4.11); from (4.10) we get

$$p(x_0^*) - r(x_0) - s(x) = 0. \tag{4.12}$$

Then

$$t^0(x) = T + \int_{\bar{\psi}(x)}^{x} \frac{d\xi}{n\alpha(\xi)^{1/n}\,[s(x) - s(\xi)]^{(n-1)/n}}, \tag{4.13}$$

where $\bar{\psi}(x) = x_0^*(x)$ is obtained from (4.11) and (4.12).

We consider next

$$((L - x)h)_t + ((L - x)\alpha h^n)_x = (L - x)(q(x) - f(x)). \tag{4.14}$$

If to (3.16) we add $f(x)(L - x) = \bar{f}(x)$ then (4.14) is the same as (4.1). If we impose the condition

$$\frac{d}{dx}\alpha(x)\,(L - x)\,(q(x) - f(x))^n \geq 0, \tag{4.15}$$

then the entire discussion above applies to (4.14) and (4.4). But, as in Sec. 3, condition (4.15) is unsatisfactory since it excludes the case $\alpha(x)$, $q(x)$, and $f(x)$ constant. Again the situation in $D_2 \cup D_1$ or $D_2 \cup D_{11}$ is not affected. As in Sec. 3 we can replace (4.15) by

$$\frac{d}{dx}\frac{L - x}{\alpha(x)} < 0 \quad \text{and} \quad q'(x) - f'(x) \geq 0. \tag{4.16}$$

Then (4.16) includes the case $\alpha(x)$, $q(x)$, $f(x)$ constant, and implies that the solution in D_3 is given by appropriately modified expressions in the discussion above. This is also true in D_{12}, but we omit the detailed argument.

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