OPTIMIZATION OF ELASTOHYDRODYNAMIC CONTACTS*

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Introduction. Hydrodynamic lubrication is concerned with a particular form of creeping flow between surfaces in relative motion. The momentum and continuity equations for this situation can be combined into a single equation—the Reynolds equation—derived by O. Reynolds near the end of the last century [1]. In this note a class of optimization problems associated with these flows will be discussed.

Analysis. The one-dimensional Reynolds equation governing the lubrication problem is given by the two-point elliptic boundary value problem:

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = \frac{dh}{dx}, \tag{1}$$

$$p(0) = p(1) = 0.$$

In Eq. (1) p is the dimensionless film pressure and $h(x) \geq 1$ is the film profile. We observe that for each $h \geq 1$ a film pressure p_h can be obtained and the load capacity functional $W = W[h]$ can be written as

$$W[h] = \int_0^1 p_h dx. \tag{2}$$

Lord Rayleigh [2] investigated the effect of different forms of h on W and discovered the optimum profile, i.e., the profile which maximized $W[h]$ over all profiles satisfying $h(x) \geq 1$. That profile is called the Rayleigh step and is shown in Fig. 1.

Due to the generation of high pressures, it is recognized that the bearing components in a practical situation may deflect and hence the film thickness becomes functionally related to the film pressure; i.e., we write

$$h(x) = h_g(x) + \mathcal{L}p \tag{3}$$

where $\mathcal{L}$ is a linear operator relating the bearing component deformations to the applied pressure distribution. This situation is called elastohydrodynamic lubrication (EHD) [1]. Note that by substituting Eq. (3) into Eq. (1) a nonlinear integrodifferential equation typically results. In this note we consider the optimization problem considered by Rayleigh for the EHD case. We again require that h as given by (3) satisfies $h \geq 1$. For notational convenience we will denote p_h, where h is given by Eq. (3), by $\tilde{p}_{h_g}$ and $W[h]$ by $\tilde{W}[h_g]$.

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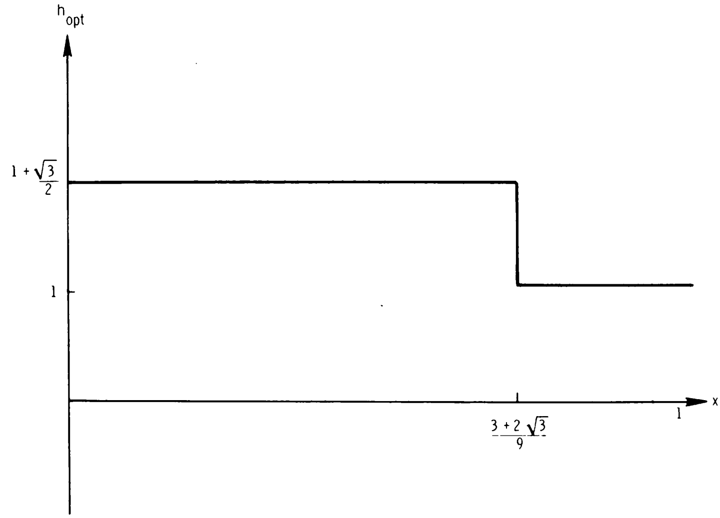

FIG. 1. Optimum slider profile (Rayleigh [2]).

Denote by h_{opt} the optimum film thickness when $\mathcal{L} \equiv 0$ (Rayleigh's result) and by p_{opt} the corresponding film pressure:

$$W_{opt} = W[h_{opt}] = \int_0^1 p_{opt}\, dt. \tag{4}$$

Let $h_1(x)$ be such that

$$h_{opt} = h_1 + \mathcal{L} p_{opt}. \tag{5}$$

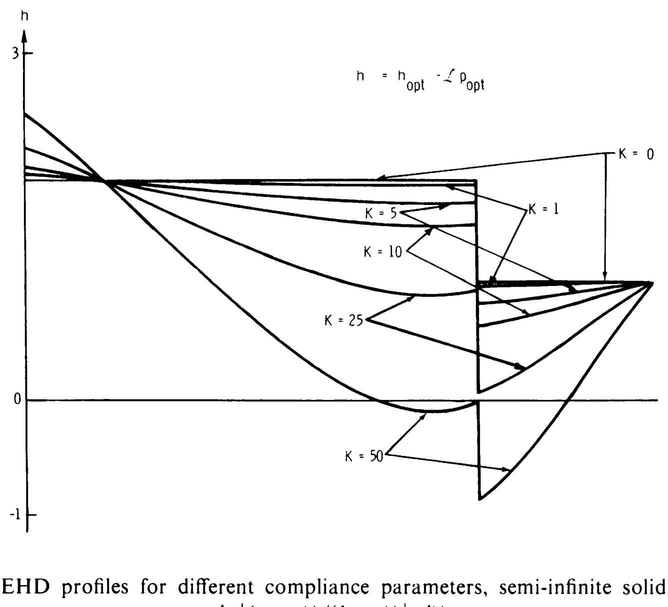

FIG. 2. Optimum EHD profiles for different compliance parameters, semi-infinite solid ($\mathcal{L}p = -K \int_0^1 p(\xi) \ln|(x - \xi)/(1 - \xi)|\, d\xi$).

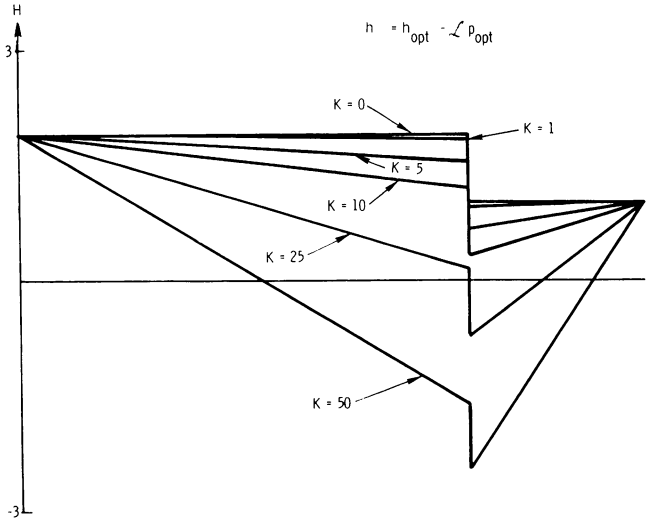

FIG. 3. Optimum EHD profile for different compliance parameters, Winkler solid ($\mathcal{L}p$ = kp).

Then we have $\tilde{p}_{h_1} = p_{\text{opt}}$ and hence the optimum solution h^* for the EHD case satisfies

$$\tilde{W}[h^*] \geq \tilde{W}[h_1] = W_{\text{opt}}. \tag{6}$$

On the other hand, consider the problem of finding an $a^* \geq 1$ such that $W[a]$ is maximized, where we do not require that $a(x) = h_g(x) + \mathcal{L}p_a$. We thus have

$$W[a^*] \geq \tilde{W}[h^*]. \tag{7}$$

But clearly $a^* = h_{\text{opt}}$ and $W[a^*] = W_{\text{opt}}$. Hence, combining Eqs. (6) and (7), we have

$$W_{\text{opt}} = \tilde{W}[h^*], \tag{8}$$

and we may set $h^* = h_1$. In view of Eq. (5) we have the simple result that

$$h^* = h_{\text{opt}} - \mathcal{L}p_{\text{opt}} . \tag{9}$$

Eq. (9) allows us to trivally calculate h^* for different operators $\mathcal{L}$. Figs. 2 and 3 show some typical results. Note that nowhere in our derivation have we made use of the fact that Eq. (1) is one-dimensional. Hence the result (9) applies equally well to two-dimensional problems and we may use the recent results [3,4] to calculate two-dimensional EHD optimum profiles. The extension to the case in which the viscosity is pressure-dependent is more complicated and will be the topic of a future note.

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