THE INSTABILITY OF SUPERPOSED FLOW*

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Abstract. The interfacial instability of a stratified two-phase flow is studied. Two independent cases are investigated: first, the instability of an idealized atmosphere (density decreasing exponentially with height) over a liquid; second, the instability of an ideal gas over a liquid. The unbounded and incompressible cases may be obtained by specializing the above results.

The purpose of this paper is to investigate the influence of the combined effect of compressibility or stratification, and rigid boundaries, on the instability of a superposed flow.

1. **Introduction.** A superposed two-phase flow, for instance a gas blowing over a liquid surface, is a very common phenomenon in nature as well as in engineering apparatus. The wind blowing over the ocean surface; the entry of a meteor into the atmosphere; the flow system inside a rocket combustion chamber; and the fluid inside containment and piping systems, for example in the petroleum industry and in nuclear power plants, are a few cases of interest.

In all these interfacial stability problems, the lower fluid of the system can often be treated as a liquid, where by "liquid" we mean that the fluid is incompressible. The upper fluid, on the other hand, is usually in a gaseous state, and can be considered as a perfect gas or an idealized atmosphere.

In studying the instability of a bounded incompressible two-phase flow, Liang and Seidel [6] found that the characteristics of the upper fluid play a major role in determining the stability of a system. Thus, the compressibility or stratification of the upper fluid should be taken into consideration in a stability analysis.

The interfacial instability problem has long attracted the attention of numerous investigators [2–5]. The stability of a system without an upper boundary was studied by Sontowski and Seidel [7] for the idealized atmosphere, and by Chang and Russell [8] for the compressible case.

Sontowski and Seidel found two types of instability. As the velocity of the gas relative to the liquid increases from zero, there first appears an instability of a selective and relatively weak nature referred to as the initial instability. This is followed, at higher velocities, by a stronger type of instability called the gross instability. However, the band width of the initial instability has never been found.

The upper boundary is also an important factor in instability analysis. It has been

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shown [6] that a system with an upper solid boundary may decrease the critical velocity considerably.

In the present work, the idealized atmosphere and the compressible gas are investigated separately. The combined solid boundary and compressibility, or stratification, effects are studied. The unbounded flow, namely the flow system without an upper boundary, is included as a special case. The band width of the initial instability is also studied.

2. Idealized atmosphere. A rectangular coordinate system is used. The system is bounded by an upper and a lower solid boundary at $z = h_a$ and $z = h_b$, as shown in Fig. 1.

An idealized atmosphere in the stationary (undisturbed) state is a fluid whose density decreases exponentially with height [1], i.e.

$$\rho_0 = \rho_a \exp(-\beta z). \tag{2.1}$$

In the disturbed state, the density does not change following the motion of the fluid; the motion of the fluid is isochoric and satisfies the equation

$$\nabla \cdot \mathbf{V} = 0. \tag{2.2}$$

Accordingly, the equation of continuity is

$$D\rho/Dt = 0 \tag{2.3}$$

for stationary and disturbed states.

The lower fluid is an incompressible liquid, i.e.

$$\rho_0 = \rho_b \quad \text{(constant)} \tag{2.4}$$

which also satisfies Eqs. (2.2) and (2.3). Both fluids are assumed to be inviscid, and the equation of motion can be written as

$$D\mathbf{V}/Dt = \mathbf{B} - (\nabla p/\rho), \tag{2.5}$$

where **B** is the body force per unit mass.

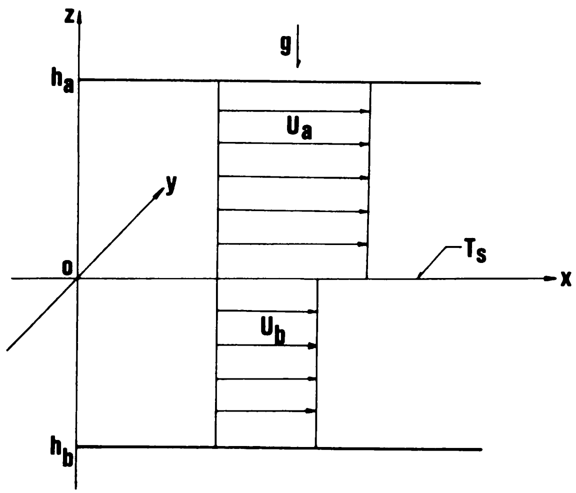

FIG 1. Flow geometry.

In the stationary state, $z = 0$ is the interface; the velocity vector and the body force vector are

$$\mathbf{V} = [U_0, 0, 0] \tag{2.6}$$

$$\mathbf{B} = [0, 0, -g] \tag{2.7}$$

where U_0 is a constant and $U_0 = U_a$ $(z > 0)$, $U_0 = U_b$ $(z < 0)$.

Substituting (2.6) and (2.7) into Eqs. (2.2), (2.3) and (2.5), we find that in the stationary state

$$\rho_0 = \rho_0(z), \tag{2.8}$$

$$p_0 = p_0(z), \tag{2.9}$$

$$dp_0/dz = -\rho_0 g, \tag{2.10}$$

From Eqs. (2.1) and (2.10) we obtain

$$p_0 = \frac{\rho_a g}{\beta} \exp(-\beta z) \tag{2.11}$$

provided $p_0 = 0$ as $z \to \infty$. Usually, β is very small. With the earth's atmosphere, for example, substituting $p_0 = 1$ atm, $\rho_a = 0.0012$ gm/cm^3 at $z = 0$ into (2.11), we have $\beta = 1.16 \times 10^{-6}$ cm^{-1}.

At the beginning of a perturbed state the disturbances are small and can be expressed as

$$\rho' = \rho_0 + \delta\rho, \tag{2.12}$$

$$p' = p_0 + \delta p, \tag{2.13}$$

$$\mathbf{V}' = [U_0 + \delta U, 0, \delta W], \tag{2.14}$$

where the small perturbations $\delta\rho$, δp, δU, and δW are functions of x, z and t. A system is said to be stable if all possible perturbations are found to diminish as time increases; otherwise, it is unstable.

Substituting (2.12) to (2.14) into Eqs. (2.2), (2.3), (2.5), and neglecting the higher-order terms, we obtain the equations governing the perturbations:

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) \delta\rho = -\delta W(d\rho_0/dz), \tag{2.15}$$

$$\frac{\partial \delta U}{\partial x} + \frac{\partial \delta W}{\partial z} = 0, \tag{2.16}$$

$$\rho_0 \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) \delta U = - \partial \delta p/\partial x,$$

$$\rho_0 \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) \delta W = - \frac{\partial \delta p}{\partial z} - g\delta\rho + T_s \left(\frac{\partial^2 \delta z_s}{\partial x^2}\right) \delta_D z. \tag{2.18}$$

The last term of Eqs. (2.18) is the pressure jump across the disturbed interface $z = \delta z$; δ_D is the Dirac delta function.

At the interface the velocity δW and the displacement δz have the following relationship:

$$\delta W = D\delta z/Dt. \tag{2.19}$$

Applying the normal mode method, all the perturbations are expressed in the form

$$\delta W = \overline{\delta W}(z) \exp [i(kx + nt)] \tag{2.20}$$

where $\overline{\delta W}(z)$ is a complex function of z, k is real, and n is complex. These perturbations propagate in the flow direction (x-direction), since this direction gives the most stringent requirement for stability [6].

Eq. (2.19) and the normal mode method give

$$\overline{\delta z} = -i\left(\frac{\overline{\delta W}}{n + kU_0}\right). \tag{2.21}$$

Notice that at the interface

$$\overline{\delta z_a} = \overline{\delta z_b} \quad . \tag{2.22}$$

Combining Eqs. (2.15) to (2.18) and (2.21), we have

$$\frac{d}{dz}\left[\rho_0(n + kU_0)\frac{d\overline{\delta W}}{dz}\right] - \rho_0 k^2(n + kU_0)\,\overline{\delta W} = gk^2\frac{d\rho_0}{dz}\frac{\overline{\delta W}}{n + kU_0} - k^4 T_s\left(\frac{\overline{\delta W}}{n + kU_0}\right)\delta_D z. \tag{2.23}$$

If we integrate this equation from $-\epsilon$ to ϵ and let $\epsilon \to 0$, we achieve the dynamic boundary condition at the interface

$$\Delta\left[\rho_0(n + kU_0)\frac{d\overline{\delta W}}{dz}\right] = gk^2\left[\Delta\rho_0 - k^2\frac{T_s}{g}\right]\left(\frac{\overline{\delta W}}{n + kU_0}\right), \tag{2.24}$$

where $\Delta f = \lim_{\epsilon\to 0}\{(f)_\epsilon - (f)_{-\epsilon}\}$

Recalling Eqs. (2.1) and (2.4), we can obtain the differential equations governing the upper and lower fluid from Eq. (2.23):

$$\frac{d^2\overline{\delta W_a}}{dz^2} - k^2\left[1 - \frac{\beta g}{(n + kU_a)^2}\right]\overline{\delta W_a} = 0 \qquad (z > 0), \tag{2.25}$$

$$\frac{d^2\overline{\delta W_b}}{dz^2} - k^2\overline{\delta W_b} = 0 \qquad (z < 0). \tag{2.26}$$

The boundary conditions at the upper and lower solid boundary are

$$\overline{\delta W_a}(h_a) = 0, \tag{2.27}$$

$$\overline{\delta W_b}(h_b) = 0. \tag{2.28}$$

Eqs. (2.25), (2.26) and the boundary conditions (2.22), (2.24), (2.27) and (2.28) give the characteristic equation of the system

$$\rho_a(n + kU_a)^2\left[1 - \frac{\beta g}{(n + kU_a)^2}\right]^{1/2}\coth\left\{\left[1 - \frac{\beta g}{(n + kU_a)^2}\right]^{1/2} kh_a\right\} - \rho_b(n + kU_b)^2\coth(kh_b) - k(\rho_b g - \rho_a g + T_s k^2) = 0. \tag{2.29}$$

Eq. (2.29) is a transcendental equation. The Nyquist criterion is used in this analysis. Essentially, this method is based on Cauchy's principle of the argument [9–11], which states: if (1) s is a complex variable, and $F(s)$ is regular within and on a closed contour C on the s-plane, save for some poles within C, and (2) $F(s)$ does not vanish on C but has zeros within C, then the excess of the number of zeros over the number of poles of $F(s)$ within C is $1/2\pi$ times the increase in arg $F(s)$ as s goes once around C.

In order to use this principle let us transform Eq. (2.29) into

$$F(s) = \sigma\kappa(s+1)^2\left[1-\frac{b^2}{(s+1)^2}\right]^{1/2}\coth\left\{\left[1-\frac{b^2}{(s+1)^2}\right]^{1/2}H_a\right\} + \kappa(s-1)^2\coth H_b + \sigma - 1 - \tau = 0 \qquad (2.30)$$

where

$$Y = \tfrac{1}{2}(U_b - U_a),\ b^2 = \beta g/k^2Y^2, \qquad \tau = k^2T_s/\rho_b g,\ H_a = kh_a,\ H_b = kh_b,$$

$$\sigma = \rho_a/\rho_b, \qquad \kappa = kY^2/g,$$

$$s = \left[-\frac{n}{k} - \tfrac{1}{2}(U_a + U_b)\right]\Big/ Y.$$

For convenience, we assume $Y > 0$.

2.1 *Unbounded upper flow.* If the upper boundary is moved to infinity ($h_a \to \infty$), Eqs. (2.30) becomes

$$F(s) = \sigma\kappa(s+1)^2\left[1-\frac{b^2}{(s+1)^2}\right]^{1/2} + \kappa(s-1)^2\coth H_b + \sigma - 1 - \tau = 0. \qquad (2.31)$$

Since the disturbances have to be zero as $z \to \infty$, the boundary condition is now replaced by the requirement that the real part of the square root appearing in Eq. (2.31) must be positive (see Eqs. (2.25) and (2.27)).

The singular points of (2.31) are located at $s = -1 \pm b$.

If $I(n) < 0$ ($I(s) > 0$), for some k, the system is unstable. Thus for a stable system, the roots of $F(s)$ must not have a positive imaginary part; A closed curve as shown in Fig. 2 (with $L \to \infty$, $l \to 0$) can be taken as the closed contour C for Cauchy's principle in this test. D, E and F represent $-1 - b$, -1, and $-1 + b$ respectively.

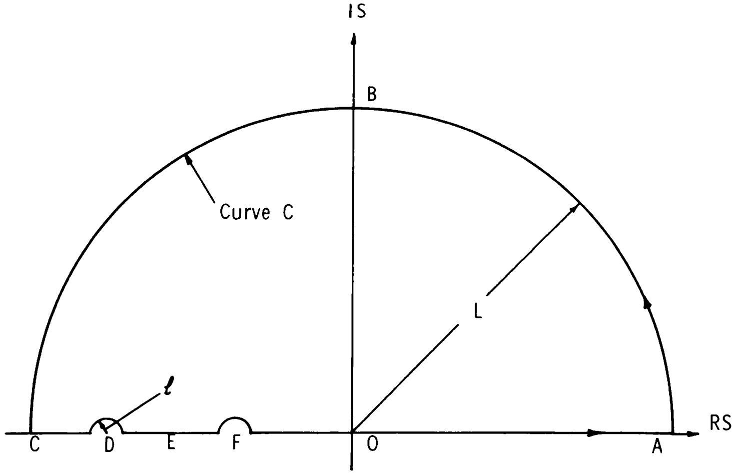

FIG. 2. s-plane.

For large s,

$$F(s) = \sigma\kappa(s + 1)^2 + \kappa(s - 1)^2 \coth H_b + \sigma - 1 - \tau. \tag{2.32}$$

s real is especially important. If s is real and $(s + 1)^2 < b^2$,

$$F(s) = i\sigma\kappa(s + 1)\,[b^2 - (s + 1)^2]^{1/2} + \kappa(s - 1)^2 \coth H_b + \sigma - 1 - \tau. \tag{2.33}$$

If s is real and $(s + 1)^2 > b^2$,

$$F(s) = \sigma\kappa\,|s + 1|\,[(s + 1)^2 - b^2]^{1/2} + \kappa(s - 1)^2 \coth H_b + \sigma - 1 - \tau.$$

Eq. (2.33) shows that, if s is real and $-1 - b < s < -1 + b$, $F(s)$ always has an imaginary part (except at $s = -1$), and the imaginary part is negative if $s < -1$, positive if $s > -1$.

Since β is very small, if s is not in the vicinity of $s = -1$, the minimum of $F(s)$ can be found at $s = G$, where

$$G = (\coth H_b - \sigma)/(\coth H_b + \sigma). \tag{2.34}$$

$F(G)$ is a minimum since $F''(G) > 0$.

Now the Nyquist diagram can be constructed as shown in Fig. 3.

Points A, B, C, D, E and F in the s-plane are transformed into points A', B', C', D', E', and F' in the $F(s)$-plane. G' represents $F(G)$. The location of the origin depends on the values of b, σ, κ, H_b, and τ.

According to the Nyquist criterion, the system is definitely stable if the origin is located to the right of E' and definitely unstable if it is located to the left of G'. On the other hand, the system is also stable if it is located between F' and G', and unstable if it is in the area between E' and F'.

In other words, the sufficient condition for stability is

$$F(E) \leq 0; \tag{2.35}$$

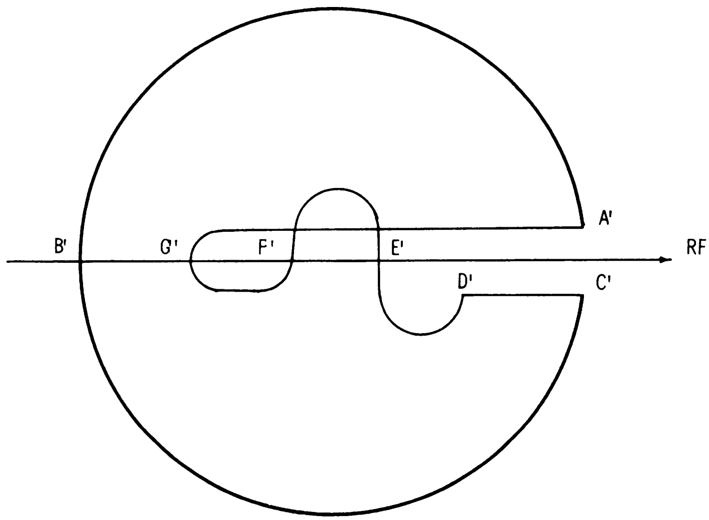

FIG 3. Nyquist diagram for a flow without upper boundary ($F(s)$-plane).

the sufficient condition for instability is

$$F(G) \geq 0. \tag{2.36}$$

The system is also stable if

$$F(G) \leq 0 \quad \text{and} \quad F(F) \geq 0 \tag{2.37}$$

and unstable if

$$F(E) \geq 0 \quad \text{and} \quad F(F) \leq 0. \tag{2.38}$$

Point F' is of special interest. From Eq. (2.31), we find

$$F(F) = 4\kappa \coth H_b \left(1 - b + \frac{b^2}{4}\right) + \sigma - 1 - \tau. \tag{2.39}$$

$F(F) = 0$ can be rewritten as

$$Y^2 - \frac{(\beta g)^{1/2}}{k} Y + \frac{\beta g}{4k^2} - \frac{\rho_b g - \rho_a g + T_s k^2}{4k\rho_b \coth H_b} = 0, \tag{2.40}$$

which gives

$$2Y = \frac{(\beta g)^{1/2}}{k} \pm \left(\frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b \coth H_b}\right)^{1/2}. \tag{2.41}$$

The stability condition (2.37) becomes

$$\frac{\beta g}{k^2} + \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b \coth H_b} + 2\left(\frac{\beta g(\rho_b g - \rho_a g + T_s k^2)}{k^3 \rho_b \coth H_b}\right)^{1/2}$$
$$\leq (U_a - U_b)^2 \leq \frac{(\rho_a + \rho_b \coth H_b)(\rho_b g - \rho_a g + T_s k^2)}{k\rho_a\rho_b \coth H_b}. \tag{2.42}$$

The instability condition (2.38) becomes

$$\frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b \coth H_b} \leq (U_a - U_b)^2$$
$$\leq \frac{\beta g}{k^2} + \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b \coth H_b} + 2\left(\frac{\beta g(\rho_b g - \rho_a g + T_s k^2)}{k^2 \rho_b \coth H_b}\right)^{1/2}. \tag{2.43}$$

The sufficient condition for stability (2.35) is

$$(U_a - U_b)^2 \leq \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b \coth H_b} \tag{2.44}$$

and the sufficient condition for instability (2.36)

$$(U_a - U_b)^2 \geq \frac{(\rho_a + \rho_b \coth H_b)(\rho_b g - \rho_a g + T_s k^2)}{k\rho_a\rho_b \coth H_b}. \tag{2.45}$$

The stability of a system can thus be illustrated as shown in Fig. 4.

2.2 *Unbounded upper and lower flow.* If h_b as well as h_a tends to infinity, conditions (2.42)–(2.45) become:

stable:

$$(U_a - U_b)^2 \leq \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b}; \tag{2.46}$$

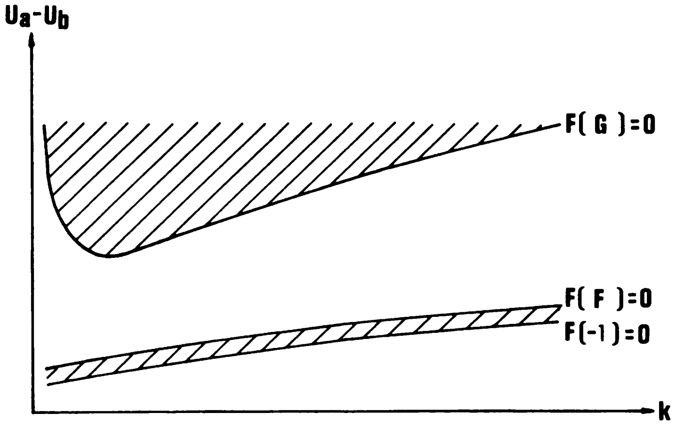

FIG 4. Stability diagram for a flow without upper boundary. ☐ stable, ▨ unstable.

unstable:

$$\frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b} \leq (U_a - U_b)^2 \leq \frac{\beta g}{k^2} + \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b} + 2\left(\frac{\beta g(\rho_b g - \rho_a g + T_s k^2)}{k^3 \rho_b}\right)^{1/2}; \quad (2.47)$$

stable:

$$\frac{\beta g}{k^2} + \frac{\rho_b g - \rho_a g + T_s k^2}{k\rho_b} + 2\left(\frac{\beta g(\rho_b g - \rho_a g + T_s k^2)}{k^3 \rho_b}\right)^{1/2} \leq (U_a - U_b)^2 \leq \frac{(\rho_a + \rho_b)(\rho_b g - \rho_a g + T_s k^2)}{k\rho_a \rho_b}; \quad (2.48)$$

unstable:

$$(U_a - U_b)^2 \geq \frac{(\rho_a + \rho_b)(\rho_b g - \rho_a g + T_s k^2)}{k\rho_a \rho_b}. \quad (2.49)$$

The criteria (2.46) and (2.49) are referred to as the initial instability and gross instability [7] respectively. Now we can see that the band width of the initial instability is defined by (2.47). The band width is zero if $\beta = 0$.

As a particular example, consider air blowing over the water. Take

$$\rho_a = 0.0012 \text{ gm/cm}^3, \qquad \rho_b = 0.9982 \text{ gm/cm}^3,$$
$$T_s = 72.8 \text{ dynes/cm}, \qquad g = 980 \text{ cm/sec}^2,$$
$$\beta = 1.1 \times 10^{-6} \text{ 1/cm}.$$

Conditions (2.46) to (2.49) give the instability diagram shown in Fig. 5.

2.3 *Bounded flow.* For the flow with an upper and lower boundary, we recall Eq. (2.30):

$$F(s) = \sigma\kappa(s+1)^2\left[1 - \frac{b^2}{(s+1)^2}\right]^{1/2} \coth\left\{\left[1 - \frac{b^2}{(s+1)^2}\right]^{1/2} H_a\right\} + \kappa(s-1)^2 \coth H_b + \sigma - 1 - \tau. \quad (2.50)$$

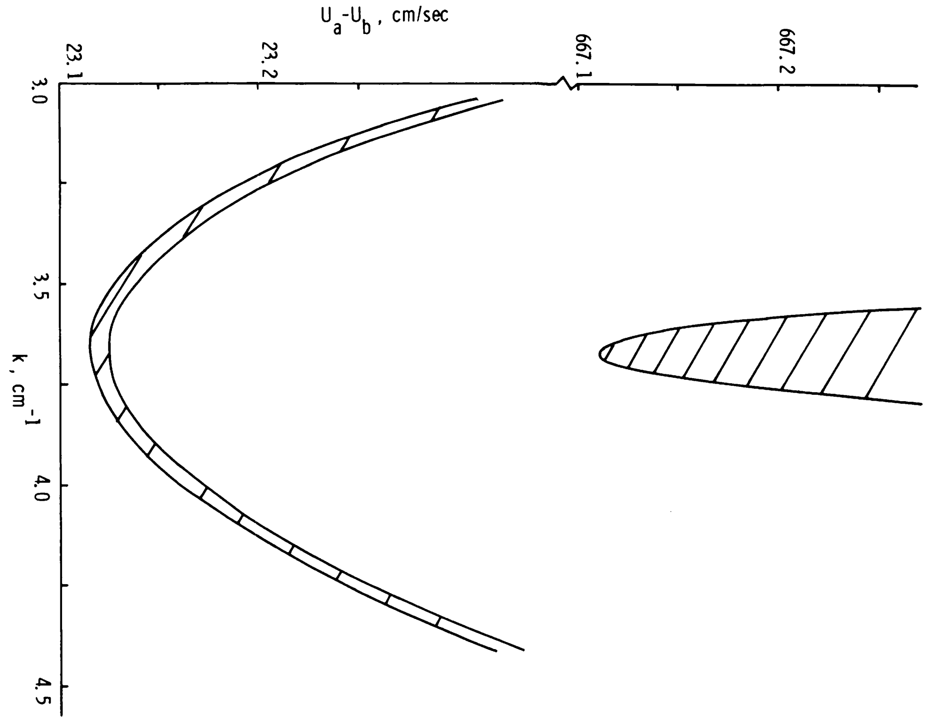

FIG 5. Stability diagram for air blowing over water.

If s is in the range $(-1-b,\ -1+b)$ then $(s+1)^2 \le b^2$, and (2.50) can be written as

$$F(s) = \sigma\kappa(s+1)^2\left[\frac{b^2}{(s+1)^2}-1\right]^{1/2}\cot\left\{\left[\frac{b^2}{(s+1)^2}-1\right]^{1/2}H_a\right\} + \kappa(s-1)^2\coth H_b + \sigma - 1 - \tau. \qquad (2.51)$$

In this expression, $F(s)$ is real as long as s is real. Compare this with the behavior of $F(s)$ in Eq. (2.33), where $F(s)$ has both positive and negative imaginary parts in this range of s.

A system with large H_a has been studied in the preceding section. For small H_a

$$F(s) = \sigma\kappa(s+1)^2 H_a^{-1} + \kappa(s-1)^2\coth H_b + \sigma - 1 - \tau \qquad (2.52)$$

and the minimum is at $s = H$, where

$$H = \frac{\coth H_b - \sigma H_a^{-1}}{\coth H_b + \sigma H_a^{-1}} \qquad (2.53)$$

The Nyquist diagram of small H_a is shown in Fig. 6. Hence the system is stable if $F(H) \le 0$, or

$$(U_a - U_b)^2 \le \frac{[\rho_a + kh_a\rho_b\coth(-kh_b)](\rho_b g - \rho_a g + T_s k^2)}{k\rho_a\rho_b\coth(-kh_b)}\ . \qquad (2.54)$$

For small H_a the distinction between initial instability and gross instability does not exist. Physically speaking, for small h_a and small β, the change of the density from $z = 0$ to

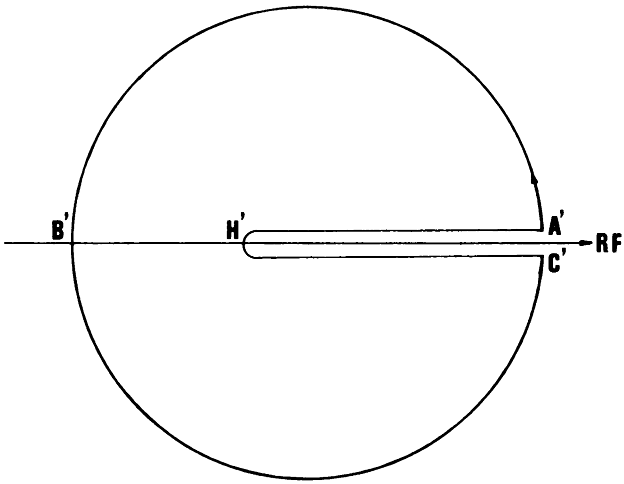

FIG 6. Nyquist diagram for bounded flow.

$z = h_a$ is essentially zero; hence the effect of density heterogeneity on the instability is negligible and the band width of initial instability disappears.

For medium H_a, the behavior of $F(s)$ is undefined as $s \to -1$. The stability of such a system is undetermined in this paper.

3. **Perfect gas.** The flow geometry is shown in Fig. 1. The upper fluid is assumed to be a compressible, inviscid, perfect gas; while the lower fluid is incompressible and inviscid liquid. The gas flow is considered to be isentropic.

In the undisturbed state, the velocity profiles are assumed to be uniform. The body force is $\mathbf{B} = [0, 0, -g]$.

The governing equations, namely the equation of state, the continuity equation, the equation of motion, and the velocity vector for the upper fluid, are:

$$p = C\rho\gamma \tag{3.1}$$

where C is a constant and γ is the ratio of specific heats;

$$D\rho/Dt + \rho(\nabla\cdot\mathbf{V} = 0, \tag{3.2}$$

$$D\mathbf{V}/Dt = \mathbf{B} - \nabla p/\rho \tag{3.3}$$

$$\mathbf{V} = [U_a , 0, 0]. \tag{3.4}$$

For the lower fluid,

$$\rho = \rho_b , \tag{3.5}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{3.6}$$

$$D\mathbf{V}/Dt = \mathbf{B} - \nabla p/\rho, \tag{3.7}$$

$$\mathbf{V} = [U_b , 0, 0]. \tag{3.8}$$

Substituting the velocity and body force vector into the above governing equations, we obtain these relationships among the stationary variables:

$$\rho_0 = \left(\frac{a_a^2}{C\gamma} - \frac{\gamma - 1}{C\gamma} gz\right)^{1/(\gamma-1)}, \qquad (z > 0)$$

$$\rho_0 = \rho_b, \qquad (z < 0)$$

$$p_0 = C\left(\frac{a_a^2}{C\gamma} - \frac{\gamma - 1}{C\gamma} gz\right)^{\gamma/(\gamma-1)}, \qquad (z > 0)$$

$$p_0 = C\left(\frac{a_a^2}{C\gamma}\right)^{\gamma/(\gamma-1)} - \rho_b gz, \qquad (z < 0)$$

where a is the speed of sound.

The perturbed state can be expressed as

$$\rho' = \rho_0 + \delta\rho, \; p' = p_0 + \delta p,$$

$$\mathbf{V}' = [U_0 + \delta U, \delta V, \delta W].$$

The small perturbations δU, δV, δW, $\delta\rho$ and δp are functions of x, y, z and t.

Linearizing the governing equations, we find that the small perturbations, for the gas phase, are related as follows:

$$\delta\rho = a^2 \delta p, \tag{3.9}$$

$$\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x}\right)\delta\rho = -\rho_0\left(\frac{\partial \delta U}{\partial x} + \frac{\partial \delta V}{\partial y} + \frac{\partial \delta W}{\partial z}\right), \tag{3.10}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x}\right)\delta U = -\frac{\partial \delta p}{\partial x}, \tag{3.11}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x}\right)\delta V = -\frac{\partial \delta p}{\partial y}, \tag{3.12}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x}\right)\delta W = -\frac{\partial \delta p}{\partial z}. \tag{3.13}$$

For the liquid phase,

$$\delta\rho = 0, \tag{3.14}$$

$$\frac{\partial \delta U}{\partial x} + \frac{\partial \delta V}{\partial y} + \frac{\partial \delta W}{\partial z} = 0, \tag{3.15}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x}\right)\delta U = -\frac{\partial \delta p}{\partial x}, \tag{3.16}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x}\right)\delta V = -\frac{\partial \delta p}{\partial y}, \tag{3.17}$$

$$\rho_0\left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x}\right)\delta W = -\frac{\partial \delta p}{\partial z}. \tag{3.18}$$

By the two-dimensional normal mode method we see that

$$\delta x = \overline{\delta x} \exp\left[i(k_x x + k_y y + nt)\right]$$

and, eliminating four of the unknowns, we achieve

$$d^2\,\overline{\delta W}/dz^2 - m^2\,\overline{\delta W} = 0 \quad \text{(gas)}, \tag{3.19}$$

$$d^2\,\overline{\delta W}/dz^2 - k^2\,\overline{\delta W} = 0 \quad \text{(liquid)} \tag{3.20}$$

where $k^2 = k_x^2 + k_y^2$ and

$$m = k\left[1 - \left(\frac{n}{k} + \frac{k_x}{k}U_a\right)^2 \frac{1}{a^2}\right]^{1/2}.$$

The upper and the lower boundary conditions are

$$\overline{\delta W}(h_a) = 0,$$
$$\overline{\delta W}(h_b) = 0.$$

Consider the uniqueness of the interface displacement; the kinematic interface boundary condition is obtained as

$$\overline{\delta W}_a(0)/(n + k_xU_a), = \overline{\delta W}_b(0)/(n + k_xU_b).$$

Since the pressure jump across the interface is

$$\Delta p = T_s\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta z,$$

we have the dynamic interface boundary condition

$$-i\,\frac{\rho_a(n + k_xU_a)}{m_a^2}\,\frac{d\overline{\delta W}_a(0)}{dz} + i\,\frac{\rho_b(n + k_xU_b)}{k^2}\,\frac{d\overline{\delta W}_b(0)}{dz} + (\rho_bg - \rho_ag + T_sk^2)\,\overline{\delta z}_s = 0.$$

In deriving the above condition the linearized Taylor's series is used, i.e.

$$p'(\delta z) = p'(0) + \left(\frac{dp_0}{dz}\right)\delta z,\ \overline{\delta W}(\delta z) = \overline{\delta W}(0).$$

The characteristic equation of the above eigenvalue problem is

$$\rho_a(n + k_xU_a)^2\,\frac{\coth(m_ah_a)}{m_a} - \rho_b(n + k_xU_b)^2\,\frac{\coth(kh_b)}{k} - (\rho_bg - \rho_ag + T_sk^2) = 0. \tag{3.21}$$

3.1 *Unbounded flow.* If the upper rigid boundary is moved to infinity, Eq. (3.21) becomes

$$\frac{\rho_a(n + k_xU_a)^2}{(k^2 - (n + k_xU_a)^2/a_a^2)^{1/2}} - \frac{\rho_a(n + k_xU_b)^2\coth(kh_b)}{k} - (\rho_bg - \rho_ag + T_sk^2) = 0. \tag{3.22}$$

We apply here a graphical method [1, 7], first transforming the above equation into a pair of simultaneous equations

$$\frac{\sigma\xi^2}{(1 - \xi^2/\alpha^2)^{1/2}} + Q\eta^2 - \eta_0^2 = 0 \quad (p\ \text{branch}) \tag{3.23}$$

$$\xi - \eta = \bar{U}_a - \bar{U}_b\,, \tag{3.24}$$

where

$$\xi = \frac{n + k_xU_a}{(kg)^{1/2}},\ Q = \coth(-kh_b),\ \eta = \frac{n + k_xU_b}{(kg)^{1/2}},$$

$$\alpha = (k/g)^{1/2}a_a\,,\ \bar{U}_a = \frac{k_xU_a}{(kg)^{1/2}},\ \bar{U}_b = \frac{k_xU_b}{(kg)^{1/2}},\ \eta_0 = (1 - \sigma + \tau)^{1/2}.$$

By the normal mode defined previously, we know that for a stable system the imaginary part of n must not be negative. Hence, it is immediately seen that for supersonic flow ($\alpha < \xi$), with ξ and η real, (3.23) cannot be satisfied. Therefore, in this case ξ and η are complex, and the system is always unstable.

The square of Eq. (3) is

$$\frac{\sigma^2\xi^4}{(1 - \xi^2/\alpha^2)} - (\eta_0{}^2 - Q\eta^2)^2 = 0. \tag{3.25}$$

This equation, together with its auxiliary equation (3.24), is equivalent to a sixth-degree polynomial in n. Notice that in squaring (3.23), a new equation

$$\frac{\sigma\xi^2}{-(1 - \xi^2/\alpha^2)^{1/2}} + Q\eta^2 - \eta_0{}^2 = 0 \quad (s \text{ branch}) \tag{3.26}$$

is introduced in (3.25).

For subsonic flow, α^2 is generally much greater than ξ^2, and Eq. (3.25) defines a real locus in the (ξ, η) plane.

The locus of Eq. (3.25) shown in Fig. 7 consists of two branches: the p-branch, plotted in solid line, represents the locus of Eq. (3.23), while the s-branch, plotted in dotted line, represents Eq. (3.26).

For a given $(\bar{U}_a - \bar{U}_b)$, Eq. (3.24) is a straight line inclined to the ξ-axis by 45°, as line A in the diagram. The effect of varying $(\bar{U}_a - \bar{U}_b)$ can be found by drawing a series of lines parallel to line A.

The number of intersections of Eqs. (3.24) and (3.25) represents the real roots of the simultaneous equations. The number of roots of the p-branch plus those of the s-branch must be exactly six for a sixth degree polynomial. Since the complex roots appear in conjugate pairs, the system is stable only if all the roots of the simultaneous equations are real.

From the diagram, we can see that there are always six intersections as long as Eq. (3.24) intersects the p-branch. We can also see that to the right of line B, which is a line tangent to the p-branch at point T and parallel to line A, the total number of intersections

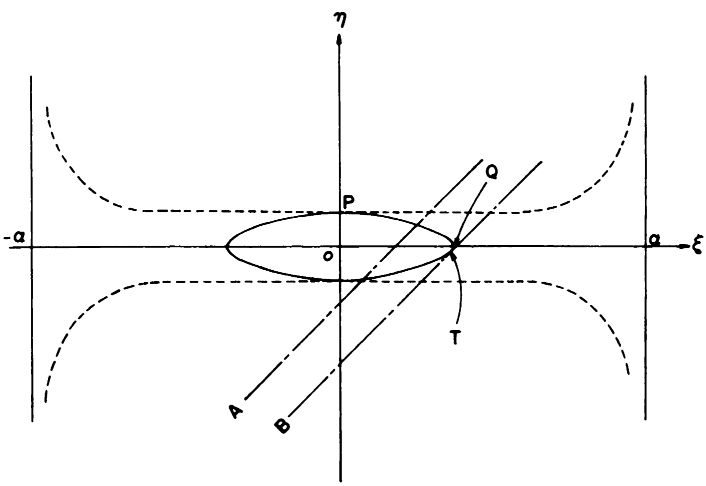

FIG 7. Graphical method. —— p branch, - - - - - s branch.

reduces to four or two, all these intersections belong to the s-branch, and the p-branch does not have any intersections; therefore, the system is unstable. Hence, by determining the corresponding value of $(\bar{U}_a - \bar{U}_b)$ of line B, the critical value of instability is determined.

Referring to the diagram and Eq. (3.23), we find that the ratio of the two intercepts, point Q and point P, is

$$\frac{\xi_0^2}{\eta_0^2/Q} = \frac{Q}{2\sigma}\left[\left(\frac{\eta_0^4}{\sigma^2\alpha^4} + 4\right)^{1/2} - \frac{\eta_0^2}{\sigma\alpha^2}\right]$$

where

$$\xi_0 = \frac{\eta_0^2}{2\sigma}\left[\left(\frac{\eta_0^4}{\sigma^2\alpha^4} + 4\right)^{1/2} - \frac{\eta_0^2}{\sigma\alpha^2}\right]$$

For a gas over a liquid σ is very small; hence the shape of the p-branch is long and thin, and point Q is a very good approximation for point T. Eq. (3.24) gives

$$(\bar{U}_a - \bar{U}_b)_T^2 \simeq \xi_0^2. \tag{3.27}$$

Returning the physical variables, the above analysis concludes that the system is stable if

$$(U_a - U_b)^2 \leq \frac{(\rho_b g - \rho_a g + T_s k^2)}{k_*^2 k \rho_a} G$$

where $k_* = k_x/k$ and

$$G = \frac{1}{2}\left[\left(\frac{(\rho_b g - \rho_a g + T_s k^2)^2}{k^2 a_a^4 \rho_a^2} + 4\right)^{1/2} - \frac{(\rho_b g - \rho_a g + T_s k^2)}{k a_a^2 \rho_a}\right]$$

Note that $a_a = \infty$ and thus $G = 1$ for incompressible flow. To consider some finite a_a, note that G is of the form

$$G = \tfrac{1}{2}(x^2 + 4)^{1/2} - x < \tfrac{1}{2}(x + 2) - x = 1 - x/2$$

for $x > 0$. That is, for a finite a_a (and thus finite x), $G < 1$. Introducing compressibility thus lowers the stability.

Furthermore, $k_* = 1$ gives the most stringent requirement for stability. If the system is stable in the x-direction, then it is stable in all other directions. In the following work, let us assume $k_* = 1$ $(k_x = k)$.

Eq. (3.27) indicates that at the neutral instability (point T) $\eta \simeq 0$: namely, n and kU_b are of the same order of magnitude. Since, in general, U_a is much greater than U_b, we have $kU_a >> |n|$ and Eq. (3.22) becomes

$$\frac{\rho_a k U_a^2}{(1 - M^2)^{1/2}} + \frac{\rho_b(n + kU_b)^2 \coth(-kh_b)}{k} - (\rho_b g - \rho_a g + T_s k^2) = 0$$

where $M = U_a/a_a$ is the Mach number. Hence the eigenvalue can be obtained as

$$\frac{n}{k} = -U_b \pm \left\{\frac{1}{k\rho_b \coth(-kh_b)}\left[(\rho_b g - \rho_a g + T_s k^2) - \frac{\rho_a k U_a^2}{(1 - M^2)^{1/2}}\right]\right\}^{1/2}.$$

Obviously, for a stable subsonic flow

$$U_a^2 \leq \frac{(1 - M^2)^{1/2}}{k\rho_a}(\rho_b g - \rho_a g + T_s k^2). \tag{3.28}$$

The critical velocity (the minimum of Eq. (3.28)) and the corresponding critical wave number are

$$V_c^2, \text{ comp.} = \frac{2(1 - M^2)^{1/2}}{\rho_a} [T_s(\rho_b g - \rho_a g)]^{1/2},$$

$$k_c = \frac{((\rho_b - \rho_a)g)^{1/2}}{T_s} \quad .$$

Compare this with the incompressible case [6]:

$$V_c^2, \text{ incomp.} = \frac{2(\rho_a + \rho_b)}{\rho_a \rho_b} [T_s(\rho_b g - \rho_a g)]^{1/2};$$

the critical velocity of a compressible subsonic flow is reduced by a factor of $(1 - M^2)^{1/2}$ if $\rho_a \ll \rho_b$.

3.2 *Bounded flow.* By applying the assumption $kU_b \gg |n|$, Eq. 3.21 becomes

$$\rho_a U_a^2 \frac{\coth [(1 - M^2)^{1/2} k h_a]}{(1 - M^2)^{1/2}} - \rho_b \left(\frac{n}{k} + U_b\right)^2 \coth (k h_b)$$

$$- \frac{1}{k} (\rho_b g - \rho_a g + T_s k^2) = 0. \qquad (3.29)$$

For a supersonic flow Eq. (3.29) can be written as

$$\rho_a U_a^2 \frac{\cot [(M^2 - 1)^{1/2} k h_a]}{(M^2 - 1)^{1/2}} - \rho_b \left(\frac{n}{k} + U_b\right)^2 \coth (k h_b)$$

$$- \frac{1}{k} (\rho_b g - \rho_a g + T_s k^2) = 0; \qquad (3.30)$$

hence

$$\frac{n}{k} = - U_b \pm \left\{ \frac{1}{\rho_b \coth (-k h_b)} \left[\frac{\rho_b g - \rho_a g + T_s k^2}{k} + \rho_a U_a^2 \frac{\cot ((M^2 - 1)^{1/2} k h_a)}{(M^2 - 1)^{1/2}} \right] \right\}^{1/2} .$$

It is seen that the system is always stable as long as $\cot [(M^2 - 1)^{1/2} k h_a] > 0$. On the other hand, if $\cot [(M^2 - 1)^{1/2} k h_a] < 0$ the system is stable only for

$$U_a^2 \leq \frac{(M^2 - 1)^{1/2}}{k \rho_a \mid \cot ((M^2 - 1)^{1/2} k h_a) \mid} (\rho_b g - \rho_a g + T_s k^2).$$

In other words,

$$k = N\pi / (M^2 - 1)^{1/2} h_a \qquad (N = 1, 2, 3, \cdots)$$

are the "dangerous" wave numbers.

Since a stable system is a system which is stable with respect to every possible disturbance to which it is *or* can be subjected, we conclude that a system with a supersonic upper flow is always unstable.

For a subsonic flow, Eq. (3.29) gives

$$\frac{n}{k} = - U_b \pm \left\{ \frac{1}{\rho_b \coth (-k h_b)} \left[\frac{\rho_b g - \rho_a g + T_s k^2}{k} - \rho_a U_a^2 \frac{\coth ((1 - M^2)^{1/2} k h_a)}{(1 - M^2)^{1/2}} \right] \right\}^{1/2} ; \qquad (3.31)$$

therefore, the system is stable if

$$U_a^2 \leq \frac{(1 - M^2)^{1/2}}{k\rho_a \coth[(1 - M^2)^{1/2}kh_a]}(\rho_b g - \rho_a g + T_s k^2). \tag{3.32}$$

Eq. (3.32) reduces to Eq. (3.28) as $h_a \to \infty$.

From Eq. (3.31), we find that $kU_a \gg |n|$ if

(1) $U_b \ll U_a$;

(2) $\rho_a \ll \rho_b$;

(3) The gas flow is not a transonic flow;

(4) h_a is small.

Under the above conditions, to neglect n/k compared with U_a in Eq. (3.21) is justified.

4. Conclusion. It has been shown that, for the unbounded idealized atmosphere, due to the stratification effect, a system becomes unstable at a low critical velocity which can be found from Eq. (2.46), i.e.

$$V_c^2 = \frac{2((\rho_b - \rho_a)g\,T_s)^{1/2}}{\rho_b}.$$

For small h_a there is no distinction between initial instability and gross instability.

As for the perfect gas, because of the compressibility the critical velocity decreases by a factor of $(1 - M^2)^{1/2}$ for the subsonic cases, while the supersonic flow is always unstable.

The effect of surface tension and gravity force on stability can also be found by studying (2.44), (2.45) and (2.32): surface tension is effective in stabilizing large wave-number disturbances, while gravity force is effective in stabilizing small wave-number disturbances.

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