QUARTERLY OF APPLIED MATHEMATICS
JANUARY 1979

ESTIMATES OF PLANAR REGIONS OF ASYMPTOTIC STABILITY*

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1. Introduction. In an earlier paper, the current author and Walter Leighton showed that in many general cases one could find weight functions $\theta(x, y)$ such that functions

$$V_\theta = \int_{h(x)}^{y} \theta(x, u) f(x, u)\, du - \int_0^x \theta(u, h(u)) g(u, h(u))\, du$$

are Liapunov functions for a differential system $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where $f(x, h(x)) \equiv 0$.

In the case of an isolated, asymptotically stable equilibrium point, an unsolved problem is the problem of finding a function θ so that the approximation by V_θ of the region of asymptotic stability is in some sense maximal among admissible values of θ.

It was shown in Theorem 3.1 of [1] that in a large class of cases the estimate of the region of asymptotic stability given by the curves $V_\theta = c$ could always be improved. The current author was able to show that under the hypotheses of Theorem 3.1 of [1], one can find a function $g(x)$ and a number $\epsilon > 0$ such that if $r(x, y, \epsilon) = p(x, y) + \epsilon g(x)$, then the estimate of the region of asymptotic stability given by V_r totally contains the estimate given by V_p. The argument is similar to that given in [1, pp 662–663] and is omitted here.

The above suggests that in at least a large number of cases there is no "best" choice of θ. Given this, is it possible to consider some subclass of admissible functions and to maximize estimates over this subclass? Can one take an estimate given by V_θ and *substantially* increase the estimate with a function V_α?

In what follows, it will be seen that in a special case the answer to the above questions is in the affirmative. Below, we take V to be V_θ, where $\theta = x^2a^{-2} + y^2b^{-2}$, and obtain an estimate that improves an estimate in [4] that is derived by considering level curves of $(2V_\theta)^{1/2}$, $\theta = x^2 + y^2$. (In the system considered below, the roles of f, g, x, y are reversed in the construction of $V_\theta = V$.)

2. An estimate for the van der Pol equation. It is well known that the system

$$\dot{x} = y - \epsilon\left(\frac{x^3}{3} - x\right) \quad (\epsilon < 0),$$

$$\dot{y} = -x \tag{1}$$

has a unique limit cycle Ω. In this section we give an estimate for the diameter of this limit cycle. An estimate of $2\sqrt{3}$ is developed in [4], and in the following we improve that estimate.

* Received November 11, 1977; revised version received January 30, 1978. The author would like to thank Prof. T. Lindstrom for preparing the numerical data used in sketching the curves in Sec. 3.

We define

$$V(x, y) = \frac{x^2y^2}{2b^2} + \frac{y^4}{4b^2} + \frac{x^4}{4a^2},$$

where a and b are positive constants to be determined. We shall assume throughout the sequel that $0 < a < 1$.

A little computation shows that

$$\dot{V} = x^2\epsilon \left[\frac{y^2}{b^2} + \left(\frac{1}{a^2\epsilon} - \frac{1}{b^2\epsilon}\right) xy + \frac{(1-a)}{a^2} x^2 + x^2\left(\frac{1}{a} - \frac{x^2}{3a^2} - \frac{y^2}{3b^2}\right)\right], \tag{2}$$

where $\dot{V}$ is the derivative of V taken along trajectories of the system (1).

For $x \neq 0$, $\dot{V}$ will be negative inside the ellipse

$$\frac{1}{a} = \frac{x^2}{3a^2} + \frac{y^2}{3b^2} \tag{3}$$

if the quadratic form

$$\frac{y^2}{b^2} + \left(\frac{1}{a^2\epsilon} - \frac{1}{b^2\epsilon}\right) xy + \frac{(1-a)x^2}{a^2}$$

is positive semidefinite—that is, if

$$\frac{1}{\epsilon^2}\left(\frac{b^2 - a^2}{a^2b^2}\right)^2 - \frac{4(1-a)}{a^2b^2} \leq 0.$$

This inequality is equivalent to the inequality $|b^2 - a^2| \leq 2\,|\epsilon|\, ab(1-a)^{1/2}$. We assume henceforth that $b > a$, and the latter inequality reduces to

$$0 < \frac{b}{a} - \frac{a}{b} \leq 2(1-a)^{1/2}\,|\epsilon|. \tag{4}$$

Now the curves $V = c$ ($c > 0$) will bound a region of stability of the origin when (4) holds and when the curves $V = c$ lie inside the region defined by (3), i.e. inside the region

$$x^2 = 3a^2\left(\frac{1}{a} - \frac{y^2}{3b^2}\right) = 3a - \frac{y^2a^2}{b^2}. \tag{5}$$

We can solve the equation $V = c$ for x^2. We obtain

$$x^2 = \frac{-y^2a^2}{b^2} + \frac{a^2}{b^2}\left(y^4\left(1 - \frac{b^2}{a^2}\right) + \frac{4cb^4}{a^2}\right)^{1/2}. \tag{6}$$

The region enclosed by (5) contains the region enclosed by (6) if

$$3a - \frac{y^2a^2}{b^2} \geq \frac{-y^2a^2}{b^2} + \frac{a^2}{b^2}\left(y^4\left(1 - \frac{b^2}{a^2}\right) + \frac{4cb^4}{a^2}\right)^{1/2},$$

which reduces to $b^4(9 - 4c) \geq 0$.

Accordingly, the curve $V = 9/4$ contains a region of stability of the origin if condition (4) holds. It is well known that any such region must lie inside Ω, and hence the curve $V = 9/4$ lies inside Ω if (4) holds. (One could also show that the set $\{X \mid \dot{V}(X) = 0\}$ contains no invariant set other than the origin and hence $V = 9/4$ encloses a region of asymptotic stability when condition (4) holds.)

We note that the curve $V = c$ is symmetric in the origin. Thus a lower estimate for the diameter of Ω is obtained by taking twice the distance between the origin and any point on the curve $V = c$. The square of the distance from any point on $V = c$ to the origin can be computed from (6). This squared distance is

$$y^2\left(1 - \frac{a^2}{b^2}\right) + \frac{a^2}{b^2}\left(y^4\left(1 - \frac{b^2}{a^2}\right) + \frac{4cb^4}{a^2}\right)^{1/2}.$$

If we put

$$P(y) = y^2\left(1 - \frac{a^2}{b^2}\right) + \frac{a^2}{b^2}\left(y^4\left(1 - \frac{b^2}{a^2}\right) + \frac{4cb^4}{a^2}\right)^{1/2} \qquad (0 \le y \le (4b^2c)^{1/4}),$$

then one can show that the maximum value of P occurs at the end point $y = (4b^2c)^{1/4}$. Thus, with our function V, an optimal estimate of the diameter of Ω is given by $2(4b^2c)^{1/4}$, which becomes $2(3b)^{1/2}$ when $c = 9/4$.

We now determine b so as to maximize $2(3b)^{1/2}$ above. First, let $a \in (0, 1)$ be fixed. The function $r(x) = xa^{-1} - ax^{-1}$ $(a \le x)$ is strictly increasing on $[a, \infty)$, and hence the largest value of b satisfying (4) is given by $ba^{-1} - ab^{-1} = 2(1 - a)^{1/2}|\epsilon|$, which reduces to $b^2 - 2(1 - a)^{1/2}ab|\epsilon| - a^2 = 0$. Solving, we obtain $b = a(|\epsilon|(1 - a)^{1/2} + (\epsilon^2(1 - a) + 1)^{1/2})$. Next, we find the largest value of the above expression for $a \in (0, 1)$. Put $F(a) = a(|\epsilon|(1 - a)^{1/2} + (\epsilon^2(1 - a) + 1)^{1/2})$, $0 < a < 1$, and let $a = 1 - t$. We obtain $\gamma(t) = F(1 - t) = |\epsilon|(t^{1/2} + (t + \epsilon^{-2})^{1/2})(1 - t)$. Note that $\lim_{t\to 0^+}\gamma(t) = 1$ and that $\gamma(1) = 0$. Now

$$\gamma'(t) = |\epsilon|\left[\frac{(1 - 3t)}{2t^{1/2}} + \frac{1}{2\left(t + \frac{1}{\epsilon^2}\right)^{1/2}}\left(1 - 3t - \frac{2}{\epsilon^2}\right)\right]$$

for $0 < t < 1$. Since $\lim_{t\to 0^+}\gamma'(t) = +\infty$, it follows that the maximum value of γ on $[0, 1]$ occurs in $(0, 1)$. Moreover, this maximum value is greater than one.

The only critical point of γ in $(0, 1)$ occurs at

$$t_0 = \left[-\left(2 + \frac{4}{\epsilon^2}\right) + \left(\left(2 + \frac{4}{\epsilon^2}\right)^2 + 12\right)^{1/2}\right] \Big/ 6,$$

and thus $\gamma(t_0)$ is the maximum value of γ on $[0, 1]$. The maximum value of F then occurs at $a_0 = 1 - t_0$. Letting

$$a_0 = 1 + \left[\left(2 + \frac{4}{\epsilon^2}\right) - \left(\left(2 + \frac{4}{\epsilon^2}\right)^2 + 12\right)^{1/2}\right] \Big/ 6$$

and

$$b_0 = a_0(|\epsilon|(1 - a_0)^{1/2} + (\epsilon^2(1 - a_0) + 1)^{1/2}),$$

we obtain the following result.

THEOREM: For all $\epsilon < 0$, the curve

$$\frac{x^2y^2}{2b_0^2} + \frac{y^4}{4b_0^2} + \frac{x^4}{4a_0^2} = \frac{9}{4}$$

bounds a region of asymptotic stability of the origin for the system (1). The diameter of this region is a function G of ϵ given by

$$G(\epsilon) = 2\sqrt{3}\cdot\sqrt{a_0}\,(|\epsilon|(1 - a_0)^{1/2} + (\epsilon^2(1 - a_0) + 1)^{1/2})^{1/2}$$

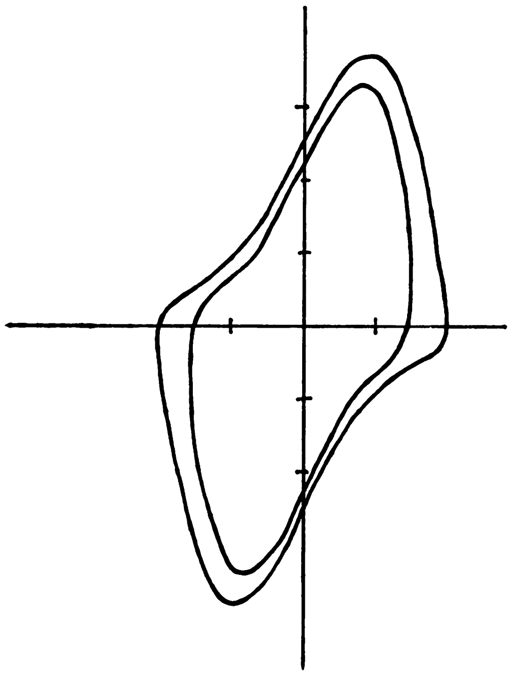

FIG. 1. Outer curve is the periodic orbit for Eq. (9) and the inner curve is (8), $\epsilon = -2$.

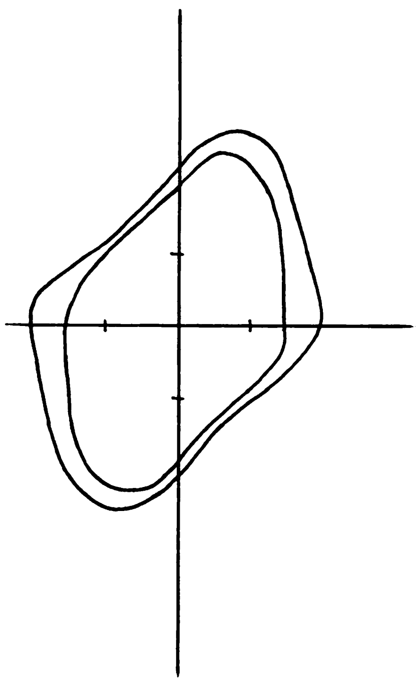

FIG. 2. Outer curve is the periodic orbit for Eq. (9) and the inner curve is (8), $\epsilon = -1$.

and satisfies

$$(a) \qquad G(\epsilon) > 2\sqrt{3} \text{ for all } \epsilon < 0,$$

$$(b) \qquad \lim_{|\epsilon| \to \infty} \frac{G(\epsilon)}{\sqrt{|\epsilon|}} = \frac{4}{3^{1/4}}.$$

It is clear that the above estimate generalizes the estimate $2\sqrt{3}$ given in [4]. Moreover, we have derived explicitly a set of curves that approximate the region within Ω. These curves enlarge as $|\epsilon|$ grows, but the rate of growth is not sharp—i.e., the estimate obtained for the diameter $D(\epsilon)$ of Ω is not good for $|\epsilon|$ large. Indeed, it is known (see [2]) that $D(\epsilon)$ satisfies $D(\epsilon) \sim 4/3\ |\epsilon|$ as $|\epsilon| \to \infty$.

In [3], the author provides a method for constructing an annular region containing Ω by constructing closed curves interior and exterior to Ω. This region can be modified by using the curve in our theorem as the interior curve rather than the circle of fixed radius $\sqrt{3}$ given in [3].

Finally, we point out that the trajectories of the system (1) are the same as those of the van der Pol equation except that their orientation is reversed (see [4]).

3. An estimate for an equivalent system. The system of differential equations considered above is, of course, a system associated with the van der Pol equation

$$\ddot{x} = \epsilon(1 - x^2)\dot{x} - x \qquad (\epsilon > 0). \tag{7}$$

In the system considered in Sec. 2 we note that $\dot{x} = y - \epsilon(x^3/3 - x)$, where x is a solution of (7). Then

$$y = \dot{x} + \epsilon\left(\frac{x^3}{3} - x\right),$$

and the curve in our theorem becomes

$$\frac{9}{4} = \frac{x^2\left[\dot{x} + \epsilon\left(\frac{x^3}{3} - x\right)\right]^2}{2b_0^{\,2}} + \frac{\left[\dot{x} + \epsilon\left(\frac{x^3}{3} - x\right)\right]^4}{4b_0^{\,2}} + \frac{x^4}{4a_0^{\,2}}. \tag{8}$$

If we replace $\epsilon > 0$ by $\epsilon < 0$ (see [4]) in (7), the preceding curve then encloses a region of asymptotic stability for (7) (with $\epsilon < 0$) relative to the system

$$\dot{x} = y, \quad \dot{y} = \epsilon(1 - x^2)y - x, \tag{9}$$

i.e., the phase space of (7).

The curve in (8) is illustrated in Figs. 1 and 2 in two separate cases. In each case the outer curve is the limit cycle for (9) and the inner curve is the curve (8).

References

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