THE TWO-VARIABLE TECHNIQUE FOR SINGULAR PARTIAL DIFFERENTIAL PROBLEMS AND ITS JUSTIFICATION*

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Abstract. Dirichlet problems with a small parameter in factor of the highest derivative are considered for bounded domains. A two-variable technique is formalized in order to carry out the study of the main boundary layer. A "secular" hypothesis is made, and a unique and uniformly valid asymptotic expansion is obtained. However, it is shown that the "secular" hypothesis may be weakened and this yields a whole set of expansions. Then the asymptotic validity of each expansion in the set can be proven for second-order operators by means of an extension of a theorem due to Eckhaus and Jager.

1. Introduction. Until recently, problems of singular perturbations have generally been treated by the method of matched asymptotic expansions whose origin goes back to the beginning of the century with the study of flows at large Reynolds number. This method, which successively looks for two or more local approximations of the solution (which are afterwards combined into a uniformly valid approximation), differs fundamentally from the multiple-scale method for which the uniformly valid approximation is obtained in a straightforward manner in the very process of building the approximation. The latter is mainly used in problems involving slowly varying coefficients and infinite domains (see, e.g., Whitham); an account of multiple-scale methods and their applications to several problems of this sort may be found in Cole or Nayfeh.

The method of matched asymptotic expansions is now firmly established (Van Dyke, Kaplun, Lagerstrom and Casten) and the mathematical proof of its validity through rigorous estimates of the error has been completed in a number of situations (Eckhaus, Mauss, Lions).

Despite the fact that this method is usually thought to be the best suited for problems involving a singular perturbation, the very fact that the ultimate goal is the search for some uniformly valid composition expansion suggests that a multiple-scale technique may be quite appropriate as well. As a matter of fact, Nayfeh, Erdelyi, Reiss, Smith, Wollkind have found uniformly valid approximations with the multiple-scale method in the case of differential equations.

Following along the line of the corresponding study of ordinary differential equations, we consider here problems with partial differential equations of the elliptic type. Such an example is provided by the equation treated by Comstock:

$$-\varepsilon^2 \nabla^4 f + a(x, y)\frac{\partial^2 f}{\partial x^2} + b(x, y)\frac{\partial^2 f}{\partial y^2} + c(x, y)\frac{\partial f}{\partial x} = 0,$$

in a bounded domain, but this work mainly is an extension of that on ordinary differential equations because of the way in which the variable x is split (x and $\tau = \varepsilon^{-1}u(x, y)$) with the variable y considered as a parameter. Thus, it is assumed that:

$$\frac{\partial}{\partial x} = \varepsilon^{-1}u_x\frac{\partial}{\partial \tau} + \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y}.$$

Moreover, from a mathematical point of view these formulas are in contradiction with the very definition of τ. These shortcomings are remedied in the present study (Sec. 2) and the "secular" hypothesis (H_1) leads to an approximation which seems similar to that found in the W.K.B. method (Levinson).

However, certain difficulties (which do not invalidate asymptotic validity) occur in the use of hypothesis (H_1). Moreover, the formalism cannot be applied in the case of a parabolic boundary layer. Thus an enlargement of the hypothesis (H_1) has to be envisaged. This is done in Sec. 3 where a whole set of asymptotic expansions is obtained. This is very close to a point raised by Erdelyi when dealing with an ordinary differential problem. The asymptotic validity of each asymptotic expansion in the set can be proven from a theorem due to Eckhaus and Jager.

2. The two-variable technique in the case of partial differential problems: the restricted hypothesis.

2.1. *The formalism.* Let Ω be an open bounded set of R^n, Γ its boundary, and $\mathbf{v}$ the inward normal unit vector to $\Gamma \cdot \varepsilon$ will be a parameter which may take very small values. We seek an asymptotic approximation of the real function $f(x, \varepsilon)$ solution of the Dirichlet problem:

$$(\varepsilon A + B)f = g, \qquad \forall \mathbf{x} \in \Omega,$$

$$f\Big|_\Gamma, \frac{\partial f}{\partial v}\Big|_\Gamma, \ldots, \frac{\partial^{k-1}f}{\partial v^{k-1}}\Big|_\Gamma \quad \text{prescribed.} \tag{1}$$

A is a partial differential operator: it is linear and strongly elliptic. Its coefficients are variable with $\mathbf{x}$ and its order is $2k$. B is a linear partial differential operator of order $m < 2k$, and g is a function of the variable $\mathbf{x} \in \bar{\Omega}$ (closure of Ω). In the following, we need to break down A and B into their homogeneous differential parts:

$$A = \sum_{j=0}^{2k} P_j(\mathbf{x}, \mathbf{D}), \qquad B = \sum_{j=0}^{m} Q_j(\mathbf{x}, \mathbf{D}),$$

where $P_j(\mathbf{x}, \cdot)$, $Q_j(\mathbf{x}, \cdot)$ represent homogeneous polynomials of degree j with coefficients depending upon the variable $\mathbf{x}$; $\mathbf{D}$ is the usual gradient operator ($\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$) so that $\mathbf{D}f$ will be the vector ($\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$).

Let the hypothesis of ellipticity be written in the form:

$$\exists \eta > 0, \forall \mathbf{p} \in R^n, \forall \mathbf{x} \in \bar{\Omega}: \ (-1)^k P_{2k}(\mathbf{x}, \mathbf{p}) \geq \eta|\mathbf{p}|^{2k}. \tag{2}$$

We require Ω to be a strictly B-convex set (see Hormander); this assumption is not absolutely necessary but it avoids free boundary layers and other parabolic boundary layers, which, for the time being, are not within the scope of this paper.

Let $(\cdot, \cdot)$ and $|\cdot|$ stand respectively for the scalar product and the norm in L^2. Let ϕ

be any smooth (that is, infinitely differentiable) function with compact support in Ω. We assume that the coercivity inequality:

$$\exists\, c > 0, \forall \phi: \ (\phi,\, B\phi) \geq c|\phi|^2, \tag{3}$$

holds good and that the coefficients of (1) are sufficiently regular. Then we can find an ε_0 such that the problem (1) has one and only one solution when ε belongs to $(0,\, \varepsilon_0]$ (for details, see Bouthier).

We do not suppose anything else about the form of the operator B: it could be elliptic, hyperbolic or of a mixed type. It will only be assumed that, when $\varepsilon = 0$, the limit of problem (1) has a unique solution.

We want to study the main boundary layer of f by means of a two-variable expansion technique. Thus, let us set:

$$\tau = \delta^{-1}\theta(\mathbf{x}), \tag{4}$$

where

$$\delta = \varepsilon^{1/(2k-m)},$$

while the function $\theta(\mathbf{x})$, which depends on the variable $\mathbf{x} \in R^n$, is unknown for the time being. The location Γ_c of the boundary layer is defined by the equation

$$\theta(\mathbf{x}) = 0.$$

Let us look for an asymptotic expansion of f as:

$$F(\mathbf{x},\, \tau,\, \varepsilon) = F^{(0)}(\mathbf{x},\, \tau) + \delta F^{(1)}(\mathbf{x},\, \tau) + \delta^2 F^{(2)} \cdots. \tag{5}$$

In order to derive this, the tool used is the chain rule for derivatives: the gradient of F is the vector:

$$\delta^{-1}\mathbf{p}(\mathbf{x})\frac{\partial F}{\partial \tau} + (\mathbf{D}F)\tau = \delta^{-1}\theta(\mathbf{x}). \tag{6}$$

where:

$$\mathbf{p}(\mathbf{x}) = \mathbf{D}\theta.$$

Note that this formalism (4)–(6) actually is multidimensional and that all directions in R^n-space are treated on an equal footing. For the sake of brevity, we shall henceforth omit the index $\tau = \delta^{-1}\theta(\mathbf{x})$; as a matter of fact the introduction of τ as a new independent variable leads merely to performing the substitution

$$\mathbf{D} \rightarrow \delta^{-1}\mathbf{p}(\mathbf{x})\frac{\partial}{\partial \tau} + \mathbf{D}, \tag{7}$$

within (1). Computing higher derivatives of F, we find that A and B transform according to:

$$A \rightarrow \sum_{s=0}^{2k} \delta^{-2k+s} S_s(\mathbf{D})\frac{\partial^{2k-s}}{\partial \tau^{2k-s}}, \qquad B \rightarrow \sum_{s=0}^{m} \delta^{-m+s} T_s(\mathbf{D})\frac{\partial^{m-s}}{\partial \tau^{m-s}}, \tag{8}$$

where $S_s(\mathbf{D})$ and $T_s(\mathbf{D})$ denote differential polynomials of the sth degree with coefficients depending on $\mathbf{x}$ and the s first partial derivatives of θ. In particular, we have:

$$S_0(\mathbf{D}) = P_{2k}(\mathbf{p}), \qquad T_0(\mathbf{D}) = Q_m(\mathbf{p}).$$

Now, we require that F satisfy:

$$\left[P_{2k}(\mathbf{p})\frac{\partial^{2k}}{\partial\tau^{2k}} + Q_m(\mathbf{p})\frac{\partial^m}{\partial\tau^m}\right]F + \sum_{s=1}^{2k}\delta^s\left[S_s(\mathbf{D})\frac{\partial^{2k-s}}{\partial\tau^{2k-s}} + T_s(\mathbf{D})\frac{\partial^{m-s}}{\partial\tau^{m-s}}\right]F = \delta^m g, \qquad (9)$$

where we impose the convention that $T_s(\mathbf{D}) = 0$ if $s > m$.

The first boundary condition of (1) defines the value of $F(\mathbf{x}, \tau, \varepsilon)$ when $\mathbf{x}$ belongs to Γ and $\tau = 0$. It will be written as[1]:

$$F\bigg|_{\Gamma} \quad \text{prescribed.}$$

The second boundary condition will be:

$$\delta^{-1}\mathbf{p}\cdot\mathbf{v}\frac{\partial F}{\partial\tau} + \left(\frac{\partial F}{\partial v}\right)_{\tau = \delta^{-1}\theta(\mathbf{x})} \quad \text{prescribed, when } \mathbf{x} \in \Gamma, \tau = 0.$$

Again, we drop the index $\tau = \delta^{-1}\theta(\mathbf{x})$ and simply write this as:

$$\left(\delta^{-1}\mathbf{p}\cdot\mathbf{v}\frac{\partial F}{\partial\tau} + \frac{\partial F}{\partial v}\right)\bigg|_{\Gamma} \quad \text{prescribed.} \qquad (11)$$

More generally, the k boundary conditions for F will be:

$$\left[\sum_{s=0}^{h}\delta^{-h+s}U_s^{(h)}\left(\frac{\partial}{\partial v}\right)\frac{\partial^{h-s}F}{\partial\tau^{h-s}}\right]\bigg|_{\Gamma} \quad \text{prescribed, } 0 \le h \le k-1. \qquad (12)$$

Here $U_s^{(h)}$ denotes differential polynomials having $\partial/\partial v$ as their argument. Their degrees are s and their coefficients are functions of $\mathbf{x}$ (because of the occurrence of $\mathbf{v}(\mathbf{x})$) and the $(s + 1)$ first derivatives of θ.

2.2. *The restricted "secular" hypothesis.* 2.2.1. We can now use the expansion (5) to obtain a hierarchy of problems for the successive approximations $F^{(0)}$, $F^{(1)}$, ...; but we must previously define what will be considered as a "secular" term in the expansion (5); hence the definition/hypothesis:

> A term of $F^{(r)}(\mathbf{x}, \tau)$ is said to be secular and has to be cancelled as non-uniformly valid, if either the ratio of $F^{(r)}$ to $F^{(r-1)}$, or the ratio of a derivative of $F^{(r)}$ (with respect to $\mathbf{x}$ or τ) to the homologous derivative of $F^{(r-1)}$, is not uniformly bounded on the set $\bar{\Omega} \times [0, +\infty)^2$. $\qquad$ (H$_1$)

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[1] This can be mathematically justified if we consider the space $\{\mathbf{x}; \mathbf{x} \in R^n\}$ as a subspace of $\{(\mathbf{x}, \tau); \mathbf{x} \in R^n, \tau \in [0, +\infty)\}$. Then Γ is a $(n - 1)$-dimensional manifold of this larger space.

[2] As we shall see hereafter, the ratios are only uniformly bounded on $\{\Omega - v(\gamma)\} \times [0, +\infty)$, whatever the neighborhood $v(\gamma)$ of γ where γ is given by (19).

Moreover, we look for approximations of f which are $2k$-continuously differentiable.

2.2.2. *Study of the first-order term.* $F^{(0)}$ has to satisfy an ordinary differential problem with respect to τ:

$$\left[P_{2k}(\mathbf{x}, \mathbf{p})\frac{\partial^{2k}}{\partial\tau^{2k}} + Q_m(\mathbf{x}, \mathbf{p})\frac{\partial^m}{\partial\tau^m}\right]F^{(0)} = \begin{cases} 0, & \text{if } m > 0, \\ g, & \text{if } m = 0, \end{cases}$$

$$F^{(0)}\Big|_{\Gamma} \quad \text{prescribed,} \tag{13}$$

$$\frac{\partial^h F^{(0)}}{\partial\tau^h}\Big|_{\Gamma} = 0, \qquad 1 \leq h \leq k - 1.$$

The bounded solutions have the form:

$$a^{(0)}(\mathbf{x}) + \sum_{i=1}^{2k-m} b_i^{(0)}(\mathbf{x})e^{\lambda_i(\mathbf{x})\tau}, \tag{14}$$

where $\lambda_i(\mathbf{x})$ $(1 \leq i \leq 2k - m)$ are the $(2k - m)$ roots of:

$$\lambda^{2k-m} = \Lambda(\mathbf{x}) = -\frac{Q_m(\mathbf{x}, \mathbf{p})}{P_{2k}(\mathbf{x}, \mathbf{p})}, \tag{15}$$

and $a^{(0)}(\mathbf{x})$ and $b_i^{(0)}(\mathbf{x})$ are indetermined functions of $\mathbf{x}$. Note that, because of (2), $\lambda_i(\mathbf{x})$ is bounded on the set where the new variable τ is well-defined ($\mathbf{p} \neq 0$). However, an exponential term $\exp[\lambda_i(\mathbf{x})\tau]$ is bounded only if the real part of λ_i is negative. Close to Γ_c, $\text{Re}(\lambda_i\theta)$ has the same sign as $\text{Re}(\lambda_i\partial\theta/\partial\nu|_{\Gamma_c})$, ν being the unit vector normal to Γ_c, pointing towards the side where the boundary layer lies (that is to say, the interior of Ω). Hence, we have to study the roots of:

$$q^{2k-m} = \left\{-\frac{Q_m(\mathbf{x}, \mathbf{p})}{P_{2k}(\mathbf{x}, \mathbf{p})}\left(\frac{\partial\theta}{\partial\nu}\right)^{2k-m}\right\}\Big|_{\Gamma_c}. \tag{16}$$

First, θ vanishes on the manifold Γ_c and:

$$\mathbf{p}\Big|_{\Gamma_c} = \frac{\partial\theta}{\partial\nu}\nu. \tag{17}$$

Secondly, Q_m and P_{2k} are homogeneous with respect to $\mathbf{p}$, so that (16) may be written as:

$$q^{2k-m} = -\frac{Q_m(\mathbf{x}, \nu)}{P_{2k}(\mathbf{x}, \nu)}\Big|_{\Gamma_c}. \tag{18}$$

Now we have to split Γ into three sets:

$$\Gamma^+ = \{\mathbf{x} \in \Gamma; (-1)^{[(m+1)/2]}Q_m(\mathbf{x}, \nu) > 0\},$$

$$\Gamma^- = \{\mathbf{x} \in \Gamma; (-1)^{[(m+1)/2]}Q_m(\mathbf{x}, \nu) < 0\}, \tag{19}$$

$$\gamma = \{\mathbf{x} \in \Gamma; Q_m(\mathbf{x}, \nu) = 0\},$$

where $[(m + 1)/2]$ stands for the integer part of $(m + 1)/2$. It can be shown that only $i^+ = [(2k - m + 1)/2]$ (respectively $i^- = [(2k - m - 1)/2]$) exponentials of (14) are

bounded on Γ^+ (respectively on Γ^-). Hence $F^{(0)}(\mathbf{x}, \tau)$ can be given in the form:

$$F^{(0)} = a^{(0)}(\mathbf{x}) + \sum_{i=1}^{i^+} b_i^{(0)}(\mathbf{x})e^{\lambda_i(\mathbf{x})\tau} + \sum_{j=1}^{i^-} b_{-j}^{(0)}(\mathbf{x})e^{\lambda_{-j}(\mathbf{x})\tau}. \tag{20}$$

Here $b_i^{(0)}(\mathbf{x})$, $1 \le i \le i^+$ (respectively $b_{-j}^{(0)}(\mathbf{x})$, $1 \le j \le i^-$) has to vanish on Γ^- (respectively on Γ^+). We can easily recognize that (20) has the structure of a composite approximation according to the W.K.B. method: $a^{(0)}$ is the regular approximation and there are two boundary layers. Let us note that the location of the boundary layer (called hereafter Γ_c; Γ_c may stand for either Γ^+ or Γ^-) has been defined before we obtain any exact expression of the new variable τ.

For the time being we are able only to obtain boundary conditions on Γ_c for θ, $a^{(0)}$, $b_i^{(0)}$ (for details, see Bouthier); as a consequence we need further information for their complete determination; this will be given by the consideration of higher order terms through the "secular" hypothesis.

2.2.3. *Study of the second-order term.* As usual, the problem for $F^{(1)}$ appears to be simply the same as for $F^{(0)}$, but with nonhomogeneous parts:

$$P_{2k}(\mathbf{x}, \mathbf{p})\left[\frac{\partial^{2k}}{\partial\tau^{2k}} - \Lambda(\mathbf{x})\frac{\partial^m}{\partial\tau^m}\right]F^{(1)}$$

$$+ \left[S_1(\mathbf{D})\frac{\partial^{2k-1}}{\partial\tau^{2k-1}} + T_1(\mathbf{D})\frac{\partial^{m-1}}{\partial\tau^{m-1}}\right]F^{(0)} = \begin{cases} 0, & \text{if } m \ne 1, \\ g, & \text{if } m = 1. \end{cases} \tag{21}$$

In accordance with the principle (H_1), we have to cancel three types of terms:

(i) Terms independent of τ: They exist only in the case $m = 1$ and they are:

$$g - T_1(\mathbf{D})a^{(0)} = g - Ba^{(0)}. \tag{22}$$

(ii) Exponential terms such as $\tau e^{\lambda_i(\mathbf{x})\tau}$: These yield $(2k - m)$ equations which are equivalent to one:

$$[\Lambda \nabla P_{2k}(\mathbf{p}) + \nabla Q_m(\mathbf{p})]\mathbf{D}\Lambda = 0, \tag{23}$$

where $\nabla P_{2k}(\mathbf{p})$ and $\nabla Q_m(\mathbf{p})$ stand for the gradients of P_{2k} and Q_m with respect to $\mathbf{p}$. From (23) we may know $\theta(\mathbf{x})$, whence the variable τ.

As a matter of fact, (23) is rather complicated, but let us consider the particular solution:

$$\Lambda = \text{Const} \leftrightarrow \lambda_i = \text{Const}, \forall i. \tag{24}$$

It can be shown that this is the only possible solution which agrees with the hypothesis (H_1) (see Bouthier). Thus θ is now defined by means of the Cauchy problem:

$$\Lambda P_{2k}(\mathbf{x}, \mathbf{D}\theta) + Q_m(\mathbf{x}, \mathbf{D}\theta) = 0,$$
$$\theta_{|\Gamma_c} = 0. \tag{25}$$

(iii) Exponential terms such as $e^{\lambda_i(\mathbf{x})\tau}$: These terms vanish if each $b_i^{(0)}$ satisfies the equation:

$$[\Lambda S_1(\mathbf{D}) + T_1(\mathbf{D})]b_i^{(0)} = 0, \qquad \forall i. \tag{26}$$

Adding to this equation the boundary conditions of (13), we get a problem of the Cauchy type for $b_i^{(0)}$.

Now, let us come back to the problem (25) for $\theta(\mathbf{x})$ and let us set:

$$u_i(\mathbf{x}) = \lambda_i \theta(\mathbf{x}), \qquad \forall i. \tag{27}$$

Since P_{2k} and Q_m are homogeneous polynomials, it follows that the functions $u_i(x)$ are solutions of:

$$P_{2k}(\mathbf{D}u) + Q_m(\mathbf{D}u) = 0,$$
$$u_{|\Gamma_c} = 0. \tag{28}$$

This problem can be solved with the method of bicharacteristics and it can be shown that $u_i(\mathbf{x})$ is well-defined in a neighborhood of Γ_c. We can also show that (26) are indeed ordinary differential equations with respect to the variable θ (for details, see Bouthier).

So the first approximation $F^{(0)}$ is completely and uniquely determined (if $m = 1$), and the second approximation $F^{(1)}$ takes the form:

$$F^{(1)} = a^{(1)}(\mathbf{x}) + \sum_{i=1}^{i^+} b_i^{(1)}(\mathbf{x})e^{\lambda_i \tau} + \sum_{j=1}^{i^-} b_{-j}^{(1)}(\mathbf{x})e^{\lambda_{-j}\tau}. \tag{29}$$

Here $a^{(1)}(\mathbf{x})$, $b_i^{(1)}(\mathbf{x})$ and $b_{-j}^{(1)}(\mathbf{x})$ are unknown functions of $\mathbf{x}$; at this stage, their values are known only on the boundary Γ_c.

2.2.4. It is not difficult to extend this formalism to the higher-order terms and to set out the "secular" conditions issuing from the hypothesis (H_1). In this way it is seen that:

(i) The general structure of $F^{(r)}$ is the same as for $F^{(0)}$ and $F^{(1)}$,

(ii) The principle (H_1) leads to a unique definition of the regular parts $a^{(r)}(\mathbf{x})$ and of the exponential terms corresponding to the boundary layer (at least in a neighborhood of Γ_c).

Consequently, the two-variable technique relying on the hypothesis (H_1) may be considered as self-consistent.

While the process just completed defines the boundary layer parts of $F^{(0)}$ or $F^{(r)}$ only in a neighborhood of Γ_c, it may be desired to have $F^{(0)}$ and $F^{(r)}$ defined on the whole set $\bar{\Omega}$. This extension may be achieved in the following way: let us introduce smooth functions $\chi^{(+r)}(\mathbf{x})$ (resp. $\chi^{(-r)}(\mathbf{x})$) equal to one in a neighborhood of Γ^+ (resp. Γ^-) and zero outside the domain of definition of $\theta(\mathbf{x})$, $b_i(\mathbf{x})$. Whatever the conditions (25)–(26), the following approximations may be used:

$$F^{(r)} = a^{(r)}(\mathbf{x}) + \sum_{i=1}^{i^+} \chi^{(+r)}(\mathbf{x})b_i^{(r)}(\mathbf{x})e^{\lambda_i \tau} + \sum_{j=1}^{i^-} \chi^{(-r)}(\mathbf{x})b_{-j}^{(r)}(\mathbf{x})e^{\lambda_i \tau}. \tag{30}$$

Here we may use any extension of $\theta(x)$ provided that:

$$\mathrm{Re}(\lambda_i \theta) < 0, \qquad \mathbf{x} \in \Omega \cap \sup \chi^{(r)} b_i^{(r)}, \qquad \forall i.$$

2.3. *The asymptotic validity of the expansion.* In order to justify (H_1), it must be proven that $F^{(0)}$ (then $F^{(0)} + \delta F^{(1)}$, $F^{(0)} + \delta F^{(1)} + \delta^2 F^{(2)}$, ...) actually is an asymptotic approximation of f. To do this, we shall restrict the analysis to the case of a second-order operator A. The main tool will be the following theorem (Eckhaus and Jager) which can be demonstrated by the maximum principle (Protter and Weinberger).

Let $v_0(\gamma)$ be an open neighborhood of γ, bounded by a subcharacteristic manifold (when B is a 0-order operator, $v_0(\gamma)$ is the empty set) and $R(\mathbf{x}, \varepsilon)$ be a function which is twice continuously differentiable and uniformly bounded in $\Omega - v_0(\gamma)$ when ε goes to

zero. If there exists $\delta = \varepsilon^\alpha$, $\alpha \geq 0$, such that:

$$(\varepsilon A + B)R = O(\delta), \qquad \forall \mathbf{x} \in \bar{\Omega} - \upsilon_0(\gamma),$$
$$R = O(\delta), \qquad \forall \mathbf{x} \in \Gamma - \upsilon_0(\gamma), \tag{32}$$

then:

$$R = O(\delta), \qquad \forall \mathbf{x} \in \bar{\Omega} - \upsilon(\overline{\upsilon_0(\gamma)}),$$

for any neighborhood $\upsilon(\overline{\upsilon_0(\gamma)})$ of $\overline{\upsilon_0(\gamma)}$.

Let us now give the proof of the asymptotic validity of the first approximations $F^{(0)}$ which, in this case, must be written as:

$$F^{(0)}(\mathbf{x}, \varepsilon) = a^{(0)}(\mathbf{x}) + \chi^{(0)}(\mathbf{x})b^{(0)}(\mathbf{x})e^{\delta^{-1}u(\mathbf{x})}, \tag{33}$$

where:

$$\delta = \varepsilon^{1/(2-m)}, \qquad u(\mathbf{x}) = \lambda\theta(\mathbf{x}). \tag{34}$$

In a natural way, we set:

$$R = f(\mathbf{x}, \varepsilon) - F^{(0)}(\mathbf{x}, \varepsilon)$$

for the asymptotic error; it satisfies the regularity conditions of the last theorem.

First (excluding the singularities of $F^{(0)}$ on γ) the boundary condition:

$$R(\mathbf{x}, \varepsilon) = 0, \qquad \forall \mathbf{x} \in \Gamma - \upsilon(\gamma),$$

holds good whatever the neighborhood $\upsilon(\gamma)$ of γ.

Secondly, from the very definition of $F^{(0)}$:

$$(\varepsilon A + B)R = -\varepsilon[Aa^{(0)} + e^{\delta^{-1}u}Ab^{(0)}],$$

when $\chi^{(0)} = 1$. Moreover, when $0 \leq \chi^{(0)} < 1$:

$$(\varepsilon A + B)R = -\varepsilon Aa^{(0)} + \delta^{-m}e^{\delta^{-1}u}Z(\delta),$$

where Z stands for a polynomial in the variable δ, whose coefficients are functions of $\mathbf{x}$ alone. Because of (31) and the regularity of $a^{(0)}$, $b^{(0)}$, $\chi^{(0)}$ in $\bar{\Omega} - \upsilon(\gamma)$, R satisfies all the conditions (32). Hence:

THEOREM. The function:

$$F^{(0)} = a^{(0)}(\mathbf{x}) + \chi^{(0)}(\mathbf{x})b^{(0)}(\mathbf{x})e^{\lambda\tau},$$
$$\tau = \varepsilon^{1/(2-m)}\theta(\mathbf{x}),$$

determined by the formalism of Sec. 2.1 and the hypothesis (H_1) is a valid asymptotic approximation of the solution of (1) in the sense that:

$$F^{(0)} = f + O(\delta), \qquad \forall \mathbf{x} \in \bar{\Omega} - \upsilon(\gamma), \tag{35}$$

for any neighborhood $\upsilon(\gamma)$ of γ.

A similar theorem holds true for the higher approximations. Its proof, which is slightly more complicated than but very similar to the preceding one for $F^{(0)}$, will not be given here (refer to Bouthier). Analogously to (35) we state it as:

$$\sum_{s=0}^{r} \delta^s F^{(s)} = f + O(\delta^{r+1}), \qquad \forall \mathbf{x} \in \bar{\Omega} - \upsilon(\gamma). \tag{36}$$

As a matter of fact, the proof of these results is actually simpler than for those obtained through matched asymptotic expansions (Eckhaus and Jager): in the latter case $(\varepsilon A + B)Z$ is not uniformly $O(\delta)$ in $\bar{\Omega} - v(\gamma)$. This means that the boundary layer approximation found here is very good indeed (this can be shown when the regular part of F reduces to a constant).

Now it appears that the principle (H_1) is completely justified; strictly speaking, however, it holds only in a neighborhood of Γ_c.

3. Enlargement of the "secular" hypothesis.

3.1. Definition of the new "secular" hypothesis. The two-variable technique used in Secs. 2.1–2.2 led to a valid asymptotic expansion of f, but the functions $\chi^{(-r)}(x)$ had to be introduced, and it appeared that the hypothesis (H_1) is not entirely satisfied in the whole of $\bar{\Omega} - v(\gamma)$. More generally, it is known that a boundary layer approximation can be multiplied by any function $\chi(x)$ without losing its validity, if $\chi(x) = 1$ on Γ_c. Otherwise Erdelyi has remarked that, in the ordinary differential case, there is no need to impose the condition corresponding to (26). Finally, if we try to apply the formalism to parabolic boundary layers, serious difficulties arise with regard to the application of the principle (H_1).

All this means that it may be possible to enlarge the "secular" hypothesis. To this end we consider the following definition:

> A term of $F^{(r)}(x, \tau)$ is said to be "secular" and has to be cancelled as non-uniformly valid if $F^{(r)}$ or one of its derivatives (with respect to either x $\qquad$ (H_2) or τ) is not uniformly bounded on $\bar{\Omega} \times [0, +\infty)$.[3]

Now the algorithm used in Secs. 2.1–2.2 remains unchanged up to the second order. $F^{(0)}$ again takes the form (20). $F^{(1)}$ must satisfy the problem (21), but the condition (22)–(23)–(26) reduce to only one condition: $F^{(1)}$ must not contain terms linear with respect to τ. Thus, again, $a^{(0)}$ is defined by the problem (22), but we allow exponential polynomial terms (such as $\tau^2 e^{\lambda_i(x)\tau}$, $\tau e^{\lambda_i(x)\tau}$, ...) to be found in $F^{(1)}$. Eqs. (23)–(26) no longer hold, and the functions $\theta(x)$, $b_i^{(0)}(x)$ remain undetermined to the same extent as for the first order. Only their boundary values on Γ_c are known (because of the boundary conditions of (13)). Of course there is no particular reason to choose the λ_i constant. This lack of unique determination must not be considered as a failure but as a sign of greater generality: we get a set of approximations $F^{(0)}(x, \tau)$ which include the previous situation as a special case.

Now the unchanged problem (21) requires that for each $F^{(0)}$, $F^{(1)}$ must be written as:

$$F^{(1)}(x, \tau) = a^{(1)}(x) + \sum_i b_i^{(1)}(x, \tau) e^{\lambda_i(x)\tau}. \qquad (37)$$

Here, $b_i^{(1)}(x, \tau)$ is a polynomial of second degree in τ. The coefficients of τ and τ^2 in it are uniquely defined from the functions $\theta(x)$, $b_i^{(0)}(x)$ but the term independent of τ is undetermined (except on Γ_c).

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[3] As a matter of fact, this will be applied on $\{\bar{\Omega} - v(\gamma)\} \times [0, +\infty)$.

It can be shown that the work can be extended similarly for the higher orders: this yields a sequence of problems and their solutions $F^{(r)}$ take the form:

$$F^{(r)} = a^{(r)}(\mathbf{x}) + \sum_i b_i^{(r)}(\mathbf{x}, \tau)e^{\lambda_i(\mathbf{x})\tau}, \tag{38}$$

where $b_i^{(r)}(\mathbf{x}, \tau)$ is a polynomial in τ (its degree is equal to or less than $2r$).

The final expansion $F(\mathbf{x}, \tau)$ depends upon the choice which has been made at each order for the term independent of τ in $b_i^{(r)}$. Notice that the boundary value on Γ_c of this term is imposed.

The new hypothesis (H_2) avoids solving the problems (25)–(26) and, incidentally, makes the process as easy to apply as the matched asymptotic expansions method; indeed it is also no longer necessary to introduce the functions $\chi(\mathbf{x})$. However, we must require that the exponential terms are small. Consequently:

$$\operatorname{Re} \lambda_i(\mathbf{x})\theta(\mathbf{x}) < 0, \quad \text{when} \quad \mathbf{x} \in \Omega \cap \sup b_i^{(r)}, \ \forall r, \ \forall i. \tag{39}$$

3.2. *Asymptotic validity.* 3.2.1. As may be expected, the asymptotic validity of the expansions now is more difficult to prove than in the previous case (Sec. 2.3).

First, let us remark that the exponents, $u_i(\mathbf{x}) = \lambda_i(\mathbf{x})\theta(\mathbf{x})$, have gradients which satisfy (28) on Γ_c. Although λ_i is not constant, we may write:

$$\mathbf{D}u_{i|\Gamma_c} = \lambda_{i|\Gamma_c} \cdot \mathbf{D}\theta_{|\Gamma_c}.$$

By the very definition of λ_i (15), it follows that:

$$P_{2k}(\mathbf{D}u_{i|\Gamma_c}) + Q_m(\mathbf{D}u_{i|\Gamma_c}) = 0. \tag{40}$$

Hence, whatever choice is made for the function $\theta(\mathbf{x})$, the $\mathbf{D}u_{i|\Gamma_c}$ are roots of (40).

Let us now define the asymptotic error by:

$$R = F^{(0)} - f.$$

As before, and for the same reason, the case of a second-order operator A only will be considered: the boundary layer will contain only one exponential term (see (33)) and if we use the variable $\delta^{-1}u(\mathbf{x})$ instead of $\delta^{-1}\theta(\mathbf{x})$, the algebra of Secs. 2.1–2.2 yields:

$$(\varepsilon A + B)R = -\varepsilon Aa^{(0)} - \delta^{-m}e^{\delta^{-1}u}[P_2(\mathbf{D}u) + Q_m(\mathbf{D}u)]b^{(0)}$$
$$- \delta^{1-m}e^{\delta^{-1}u}[S_1(u, \mathbf{D}) + T_1(u, \mathbf{D})]b^{(0)} - \varepsilon e^{\delta^{-1}u}Ab^{(0)}.$$

Here the definitions of $S_1(u, \mathbf{D})$ and $T_1(u, \mathbf{D})$ are the same as in (8), with the function $u(\mathbf{x})$ replacing $\theta(\mathbf{x})$. (40) tells us that there exists a constant $M > 0$ such that (excluding a neighborhood of γ)[4]:

$$\left| \delta^{-m}e^{\delta^{-1}u}[P_2(\mathbf{D}u) + Q_m(\mathbf{D}u)]b^{(0)} \right| \leq M\delta^{-m}|u|e^{\delta^{-1}u}$$

holds uniformly. Hence, we get:

$$(\varepsilon A + B)R = O(\delta^{-m}|u|e^{\delta^{-1}u}) + O(\delta^{1-m}e^{\delta^{-1}u}) + O(\delta). \tag{41}$$

3.2.2. The left-hand side of this equation does not appear to be uniformly $O(\delta)$: owing to terms being $O(1)$ in the boundary layer the theorem of Eckhaus and Jager can no longer be applied and we need the following stronger theorem.

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[4] This constant M depends on the chosen neighborhood of γ.

THEOREM. Let $v_0(\gamma)$ be any open neighborhood of γ, bounded by a subcharacteristic manifold (when B is an O-order operator $v_0(\gamma)$ will be the empty set), $v(\mathbf{x})$ be the distance from the point $\mathbf{x}$ to Γ_c (close to Γ_c, $v(\mathbf{x})$ is the abscissa along the normal to Γ_c) and $R(\mathbf{x}, \varepsilon)$ be a twice continuously differentiable function. If there exists a constant $\alpha > 0$ such that:

$$(\varepsilon A + B)R = g_1(\mathbf{x}, \varepsilon) + g_2(\mathbf{x}, \varepsilon) \quad \text{in} \quad \bar{\Omega} - v_0(\gamma),$$

$$R\Big|_{\Gamma_c} = O(\delta), \tag{42}$$

R uniformly bounded in $\bar{\Omega} - v_0(\gamma)$,

with:

$$\begin{aligned} g_1(\mathbf{x}, \varepsilon) &= O(\delta^{1-m} e^{-\delta^{-1}\alpha v}), \\ g_2(\mathbf{x}, \varepsilon) &= O(\delta), \end{aligned} \tag{43}$$

then

$$R = O(\delta) \quad \text{in} \quad \bar{\Omega} - v(\overline{v_0(\gamma)}),$$

for any neighborhood $v(\overline{v_0(\gamma)})$ of $\overline{v_0(\gamma)}$.

In order to prove this, we shall use the technique of barrier functions, and introduce a function w defined in a neighborhood $v_1(\Gamma_c)$ of Γ_c and such that[5]:

(i) $\qquad w = 0, \qquad \partial w / \partial v \neq 0, \qquad$ on $\quad \Gamma_c - v_0(\gamma),$

(ii) $\qquad -\alpha v \leq w < 0, \qquad$ in $\quad v_1(\Gamma_c) - v_0(\gamma),$ $\qquad$ (44)

(iii) $\qquad \exists K > 0, \qquad P_2(\mathbf{D}w) + Q_m(\mathbf{D}w) \geq K, \qquad$ in $\quad v_1(\Gamma_c) - v_0(\gamma).$

Let u_0 be the unique negative solution[6] of:

$$P_2(\mathbf{D}u_0) + Q_m(\mathbf{D}u_0) = 0, \qquad u_{0|\Gamma_c} = 0,$$

and ρ be a positive constant such that:

$$0 < \rho < \text{Min} \left\{ 1, \ \underset{\mathbf{x} \in \Gamma_c - v_0(\gamma)}{\text{Inf}} - \left(\frac{\partial u_0}{\partial v} \right)^{-1} \right\}.$$

Then the function $w = \rho u_0$ satisfies the requirement (i) and is negative. When $\mathbf{x}$ tends toward a point of Γ_c, the limit of $-w/\alpha v$ is:

$$-\rho \frac{\partial u_0}{\partial v} \alpha^{-1} < 1,$$

and the condition $-\alpha v \leq w$ holds by continuity in a neighborhood of $\Gamma_c - v_0(\gamma)$.

At last, we may write:

$$P_2(\mathbf{D}w) + Q_m(\mathbf{D}w) = (\rho^2 - \rho^m)P_2(\mathbf{D}u_0),$$

---

[5] The function w and the neighborhood $v_1(\Gamma_c)$ depend on the neighborhood $v_0(\gamma)$.

[6] It is well-defined only in a neighborhood of Γ_c.

and the constant K occurring in (iii) can be chosen as:

$$K = -(\rho^2 - \rho^m)\eta \operatorname*{Inf}_{\mathbf{x} \in \Gamma_c - v_0(\gamma)} \left(\frac{\partial u_0}{\partial v}\right)^2 > 0.$$

Let us now consider the following barrier function:

$$Y = \delta\chi(\mathbf{x})e^{\delta^{-1}w(\mathbf{x})},$$

where χ is an infinitely differentiable function such that:
  (i) it is equal to one in a neighborhood of Γ_c,
  (ii) its support is contained in $v_1(\Gamma_c)$.
Through straightforward manipulations we get:

$$(\varepsilon A + B)Y = \delta^{1-m}e^{\delta^{-1}w}[P_2(\mathbf{D}w) + Q_m(\mathbf{D}w)]\chi + \delta^{2-m}g_3(\mathbf{x}, \varepsilon),$$

where:

$$g_3(\mathbf{x}, \varepsilon) = e^{\delta^{-1}w}[S_1(w, \mathbf{D}) + T_1(w, \mathbf{D})]\chi + \delta e^{\delta^{-1}w}A\chi.$$

Owing to (44), for any positive constant c_1, we have:

$$(\varepsilon A + B)c_1 Y - |(\varepsilon A + B)R| \geq (c_1 K e^{\delta^{-1}w} - |g_1|)\chi - (1 - \chi)|g_1| - |g_2| + \varepsilon c_1 g_3.$$

Now the assumption (43) about the magnitude of $|g_1|$ means that the constant c_1 can be chosen large enough to get

$$(\varepsilon A + B)c_1 Y - |(\varepsilon A + B)R| \geq -(1 - \chi)|g_1| - |g_2| + \varepsilon c_1 g_3.$$

But when ε is small,

$$(1 - \chi)|g_1| + |g_2| = O(\delta),$$

so we introduce a last positive constant c_2 which allows us to verify simultaneously:

$$(\varepsilon A + B)(c_1 Y + c_2 \delta) - |(\varepsilon A + B)R| \geq 0,$$

and:

$$(c_1 Y + c_2 \delta)_{|\Gamma_c - v_0(\gamma)} = c_1 + c_2 \delta \geq |R|_{|\Gamma_c - v_0(\gamma)},$$

where the first inequality holds provided that:

$$(\varepsilon P_0 + Q_0)c_2 \delta \geq -\varepsilon c_1 g_3 + (1 - \chi)|g_1| + |g_2|.$$

This is indeed possible since g_3 is bounded and $Q_0 > 0$.

Thus $c_1 Y + c_2 \delta$ actually appears to be a barrier function for R if on the boundary of $v_0(\gamma)$ it is greater than $|R|$. To do this, we can use the method of Eckhaus and Jager, and this yields:

$$R = O(Y) + O(\delta) = O(\delta),$$

uniformly in $\bar{\Omega} - v(\overline{v_0(\gamma)})$.

3.2.3. Our aim now is to apply the previous theorem to the case of (41). With a view to do this, it is useful to notice that the function u has the following main properties:

  (i) $u = 0 \leftrightarrow \mathbf{x} \in \Gamma_c,$

(ii)
$$\left.\frac{\partial u}{\partial v}\right|_{\Gamma_c} \neq 0, \qquad \left(\left.\frac{\partial u}{\partial v}\right|_{\Gamma_c} < 0\right), \tag{45}$$

(iii)
$$u < 0 \quad \text{in} \quad \Omega.$$

From these properties it follows that the asymptotic evaluations:

$$\delta^{1-m}e^{\delta^{-1}u} = O(\delta^{1-m}e^{-\delta^{-1}\alpha v}) + O(\delta^r), \qquad \forall r > 0, \tag{46}$$

and

$$\delta^{-m}|u|e^{\delta^{-1}u} = O(\delta^{1-m}e^{-\delta^{-1}\alpha v}) + O(\delta^r), \qquad \forall r > 0, \tag{47}$$

hold provided the positive constant α is small enough:

$$0 < \alpha < \inf_{\mathbf{x} \in \Gamma_c - v_0(\gamma)} \left(-\frac{\partial u}{\partial v}\right).$$

As a matter of fact, by continuity $\alpha v u^{-1}$ can be set equal to $\alpha(\partial u/\partial v)^{-1}$ on Γ_c, so that we obtain the statement:

$$u < -\alpha v < 0,$$

in the interior of a neighborhood of $\Gamma_c - v_0(\gamma)$, say $0 < v < V$. In that neighborhood, (46) is readily verified, while

$$-\delta^{-1}u e^{\delta^{-1}u}/e^{-\delta^{-1}\alpha v} = \frac{-u}{u + \alpha v}\delta^{-1}(u + \alpha v)e^{\delta^{-1}(u + \alpha v)},$$

is bounded in view of the fact that:

(i) The maximum of the function te^{-t} on $[0, +\infty)$ is finite,

(ii) $u(u + \alpha v)^{-1}$ can be extended into a continuous function (with value of Γ_c given as $\partial u/\partial v(\partial u/\partial v + \alpha)^{-1}$.

Thus:

$$\delta^{-1}|u|e^{\delta^{-1}u} = O(e^{-\delta^{-1}\alpha v})$$

holds if $0 < v < V$. Now in the remaining domain of $\bar{\Omega} - v_0(\gamma)$, all the exponentials of (46), (47) are smaller than $O(\delta^r)$, for any positive r, since

$$v \geq V > 0,$$

and

$$u \leq -Cst < 0$$

(from (45)). Using (41), (46), (47) and the theorem proven above, we may state the theorem:

THEOREM. The functions

$$F^{(0)} = a^{(0)}(\mathbf{x}) + b^{(0)}(\mathbf{x})e^{\delta^{-1}u(\mathbf{x})},$$

determined by the formalism of Sec. 2.1 and the hypothesis (H_2) are all valid asymptotic approximations of the solution of (1). The asymptotic error is:

$$F^{(0)} = f + O(\delta), \qquad \forall \mathbf{x} \in \overline{\Omega - v(\gamma)},$$

whatever the neighborhood $v(\gamma)$ of γ.

4. Conclusion. It has been shown that the two-variable technique can be used in the field of partial differential problems. This may be done in two natural ways: the first one follows from the principle (H_1)—the asymptotic expansion obtained is uniquely defined and takes the form of a W.K.B. expansion, while the second one, which includes the former as a particular case, leads to the enlargement (H_2) of the "secular" hypothesis and to a whole set of asymptotic expansions. As occurs in ordinary differential equations (Erdelyi), each expansion of this set is valid. The asymptotic validity has been proven for second-order operators by means of an extension of a theorem due to Eckhaus and Jager.

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