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SOLUTIONS OF THE ELECTROMAGNETIC WAVE EQUATIONS FOR POINT DIPOLE SOURCES AND SPHERICAL BOUNDARIES*

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Abstract. Solutions of the electromagnetic wave equation are derived in systems containing spherical interfaces when the source field is that of a magnetic or electric point dipole. Piecewise constant electromagnetic parameters are assumed, but their values as well as the frequency of the source field are arbitrary. The solutions are obtained in terms of scalar and vector spherical harmonics. A sphere embedded in full space with a radial or transverse source dipole is considered explicitly.

1. Introduction. The solution of the electromagnetic wave equation is a central problem in many fields of applied physics and engineering, e.g., in radio science and geophysics. The solutions used in practice are often approximate ones. In calculations concerning radio waves infinite conductivities and high frequency limits are used, while in geophysics a quasistatic approximation is common practice. In scattering problems the incident field is usually assumed to be a plane wave even though the field may actually be dipolar.

Attempts to find solutions of the wave equation in analytic form usually involve separation of variables and looking for the solution as an infinite series. If cartesian coordinates are used, the general solution requires integration over the parameters of separation. The spherical coordinate system, on the other hand, allows the field vectors to be expanded in terms of a complete and discrete set of functions. The coefficients of the expansion can be uniquely determined from the boundary conditions.

In this article we consider the behavior of spherical waves emitted by magnetic or electric point dipoles in the presence of a spherical interface. We have in mind geophysical applications, but our method of solution is independent of the specific properties of the physical systems which our models represent. The fields are assumed to have harmonic time dependences, but no restrictions are imposed on the frequency. The electromagnetic properties of the medium are allowed to change only at the interfaces, but otherwise they are arbitrary. We first outline the method of solution. Then we proceed to derive the true wave solution for a spherical region embedded in full space.

2. Method of solution. The electromagnetic potentials are generally determined by the density $\rho(\mathbf{r}, t)$ of free charge and the total current density $\mathbf{j}(\mathbf{r}, t)$ through the coupled differential equations

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$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}, \tag{1}$$

$$\nabla^2\mathbf{A} - \varepsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{j} + \nabla\left(\mu\varepsilon\frac{\partial\Phi}{\partial t} + \nabla \cdot \mathbf{A}\right), \tag{2}$$

which are equivalent to Maxwell's equations for **E** and **B**. It is understood that **j** includes terms due to magnetization or electric polarization acting as sources of the field. The parameter μ is the magnetic permeability, and ε is the electric permittivity of the medium. The field vectors can be directly obtained from Φ and **A**:

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}. \tag{3, 4}$$

The total current density is the sum of the primary current density $\mathbf{j}_P$, which is imposed on the system by external means, and the conduction current density,

$$\mathbf{j} = \mathbf{j}_P + \sigma\mathbf{E} = \mathbf{j}_P - \sigma\left(\nabla\Phi + \frac{\partial\mathbf{A}}{\partial t}\right). \tag{5}$$

The conductivity is here assumed to be a scalar. By imposing the Lorentz condition

$$\nabla \cdot \mathbf{A} + \mu\varepsilon\frac{\partial\Phi}{\partial t} + \mu\sigma\Phi = 0 \tag{6}$$

on the potentials Φ and **A**, they can be decoupled to obtain the wave equations

$$\nabla^2\Phi - \mu\varepsilon\frac{\partial^2\Phi}{\partial t^2} - \mu\sigma\frac{\partial\Phi}{\partial t} = -\frac{\rho}{\varepsilon}, \tag{7}$$

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} = -\mu\mathbf{j}_P. \tag{8}$$

Assuming the time-dependence to be given by the factor $\exp(-i\omega t)$, we obtain for the space-dependent parts of the potentials the scalar and vector Helmholtz equations

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho}{\varepsilon}, \qquad \nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{j}_P, \tag{9, 10}$$

where the complex wave number k is defined by

$$k^2 = \mu\omega(\varepsilon\omega + i\sigma). \tag{11}$$

Thus we allow complex representations of the fields. The physical fields are conventionally obtained by taking the real parts of the complex fields. Our problem is essentially to solve Eq. (10) for the vector potential, from which **B** is directly obtained using Eq. (3). The electric field is given by Maxwell's equation

$$\mathbf{E} = \frac{i\omega}{k^2}\nabla \times \mathbf{B}. \tag{12}$$

The scalar potential Φ is not needed but can, of course, be obtained from Eq. (6). In particular, if there are no free charges and if $\nabla \cdot \mathbf{A} = 0$, then Φ can be chosen to be identically zero.

The conditions to be satisfied by $\mathbf{E}$ and $\mathbf{B}$ throughout the boundary between any two regions 1 and 2 are

$$\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0; \qquad \mathbf{n} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = 0, \tag{13}$$

where $\mathbf{n}$ is a vector normal to the boundary. If the boundary is spherical about the origin, then we can simply take $\mathbf{n} = \mathbf{e}_r$, and the conditions are $E_{1\theta} = E_{2\theta}$, $E_{1\varphi} = E_{2\varphi}$, $H_{1\theta} = H_{2\theta}$, and $H_{1\varphi} = H_{2\varphi}$.

The general solution of Eq. (10) is obtained by adding a particular solution to the general solution $\mathbf{A}_s$ of the homogeneous equation. The latter can be given in terms of three sets of vector functions,

$$\mathbf{A}_s = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \{A_n^m \mathbf{L}_n^m(\mathbf{r}) + D_n^m \mathbf{M}_n^m(\mathbf{r}) + G_n^m \mathbf{N}_n^m(\mathbf{r})\}, \tag{14}$$

where $\nabla \times \mathbf{L}_n^m = 0$, $\nabla \cdot \mathbf{M}_n^m = \nabla \cdot \mathbf{N}_n^m = 0$, and

$$\mathbf{N}_n^m = \frac{1}{k} \nabla \times \mathbf{M}_n^m; \qquad \mathbf{M}_n^m = \frac{1}{k} \nabla \times \mathbf{N}_n^m. \tag{15}$$

We note that $\mathbf{A}_s$ can always be chosen to have zero divergence, since $\mathbf{B}_s = \nabla \times \mathbf{A}_s$ is independent of $\mathbf{L}_n^m$. The vector functions can be constructed in many ways from solutions of the scalar Helmholtz equation. In spherical coordinates the spherical harmonics

$$Y_n^m(\theta, \varphi) = (-1)^m \left\{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}\right\}^{1/2} P_n^m(\cos\theta) e^{im\varphi}, \tag{16}$$

where P_n^m is the associated Legendre function, and the spherical Bessel functions can be used. We adopt the vector spherical harmonics

$$\begin{aligned}
\mathbf{P}_n^m &= \mathbf{e}_r P_n^m e^{im\varphi},\\
\mathbf{B}_n^m &= \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta} \left\{\mathbf{e}_\theta \left(\frac{n-m+1}{n+1} P_{n+1}^m - \frac{n+m}{n} P_{n-1}^m\right) + \mathbf{e}_\varphi \frac{m(2n+1)}{n(n+1)} iP_n^m\right\} e^{im\varphi},\\
\mathbf{C}_n^m &= \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta} \left\{\mathbf{e}_\theta \frac{m(2n+1)}{n(n+1)} iP_n^m - \mathbf{e}_\varphi \left(\frac{n-m+1}{n+1} P_{n+1}^m - \frac{n+m}{n} P_{n-1}^m\right)\right\} e^{im\varphi}
\end{aligned} \tag{17}$$

given by Morse and Feshbach [1], and the corresponding vector functions

$$\begin{aligned}
\mathbf{L}_n^m &= j_n' \mathbf{P}_n^m + \sqrt{n(n+1)} \frac{j_n}{kr} \mathbf{B}_n^m,\\
\mathbf{M}_n^m &= \sqrt{n(n+1)} j_n \mathbf{C}_n^m,\\
\mathbf{N}_n^m &= n(n+1) \frac{j_n}{kr} \mathbf{P}_n^m + \sqrt{n(n+1)} \left(\frac{j_n}{kr} + j_n'\right) \mathbf{B}_n^m,
\end{aligned} \tag{18}$$

where j_n stands for any of the spherical Bessel functions of order n and the prime indicates the derivative with respect to kr. The vector spherical harmonics for $m = 0$ (the zonal harmonics) have no φ-dependence, and they have the special forms

$$\mathbf{P}_n^0 = \mathbf{e}_r P_n; \qquad \mathbf{B}_n^0 = -\mathbf{e}_\theta \frac{P_n^1}{\sqrt{n(n+1)}}; \qquad \mathbf{C}_n^0 = \mathbf{e}_\varphi \frac{P_n^1}{\sqrt{n(n+1)}}, \tag{19}$$

where P_n is the Legendre polynomial. Thus the solution of the homogeneous vector wave equation can be written in spherical polar coordinates as

$$A_{sr} = \sum_0^\infty \left\{A_n^0 j_n' + G_n^0 n(n+1)\frac{j_n}{kr}\right\} P_n + \sum_1^\infty \sum_{-n}^{n}{}' \left\{A_n^m j_n' + G_n^m n(n+1)\frac{j_n}{kr}\right\} P_n^m e^{im\varphi},$$

$$\begin{aligned} A_{s\theta} = &-\sum_1^\infty \left\{A_n^0 \frac{j_n}{kr} + G_n^0\left(\frac{j_n}{kr} + j_n'\right)\right\} P_n^1 + \sum_1^\infty \sum_{-n}^{n}{}' \frac{n(n+1)e^{im\varphi}}{(2n+1)\sin\theta} \\ &\cdot \left\{\left\{A_n^m \frac{j_n}{kr} + G_n^m\left(\frac{j_n}{kr} + j_n'\right)\right\}\left(\frac{n-m+1}{n+1} P_{n+1}^m - \frac{n+m}{n} P_{n-1}^m\right)\right. \\ &\left. + D_n^m j_n \frac{m(2n+1)}{n(n+1)} iP_n^m\right\}, \end{aligned} \tag{20}$$

$$\begin{aligned} A_{s\varphi} = &\sum_1^\infty D_n^0 j_n P_n^1 + \sum_1^\infty \sum_{-n}^{n}{}' \frac{n(n+1)e^{im\varphi}}{(2n+1)\sin\theta} \cdot \left\{\left\{A_n^m \frac{j_n}{kr} + G_n^m\left(\frac{j_n}{kr} + j_n'\right)\right\} \frac{m(2n+1)}{n(n+1)} iP_n^m\right. \\ &\left. - D_n^m j_n\left(\frac{n-m+1}{n+1} P_{n+1}^m - \frac{n+m}{n} P_{n-1}^m\right)\right\}, \end{aligned}$$

where the prime on the summation indicates the omission of the terms with $m = 0$. The terms with A_n^m are not actually needed, since we may choose $\nabla \cdot \mathbf{A}_s = 0$. Finally, to obtain the most general solution we must take a linear combination of two expressions like Eq. (20) having linearly independent Bessel functions.

In the important case of azimuthal symmetry the general solution (20) reduces to the terms with $m = 0$.

The particular solution of Eq. (10) reflects the nature of the primary potential $\mathbf{A}_P$, which is the potential due to $\mathbf{j}_P$ in a homogeneous space. We consider explicitly two kinds of sources, a point magnetic dipole $\mathbf{m}_d e^{-i\omega t}$, and a point current dipole $Id\mathbf{l}e^{-i\omega t}$. The vector potential of a magnetic dipole at $\mathbf{r}_0$ in a space with wave number k_0 is

$$\mathbf{A}_P = \frac{\mu_0}{4\pi}\left(\frac{ik_0}{|\mathbf{r}-\mathbf{r}_0|^2} - \frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}\right) e^{ik_0|\mathbf{r}-\mathbf{r}_0|}(\mathbf{r}-\mathbf{r}_0) \times \mathbf{m}_d e^{-i\omega t} \tag{21}$$

and that of a current dipole is

$$\mathbf{A}_P = \frac{\mu_0 I}{4\pi} \, d\mathbf{l} \, \frac{e^{ik_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} e^{-i\omega t}. \tag{22}$$

By definition, $\mathbf{A}_P$ satisfies Eq. (10) in the region where the dipole is situated; hence in this region $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_P$, while in all other regions $\mathbf{A} = \mathbf{A}_s$. We note that for the magnetic dipole $\nabla \cdot \mathbf{A}_P = 0$, but for the current dipole $\nabla \cdot \mathbf{A}_P \neq 0$.

To be able to determine the coefficients in the expression of $\mathbf{A}_s$ (Eq. (20)) we must expand $\mathbf{A}_P$ in terms of the spherical harmonics and Bessel functions. A procedure of finding the expansions of Eq. (22) and Eq. (21) is given in the Appendix. In the following sections we apply the general method outlined to specific problems.

3. Radial magnetic dipole and sphere. The geometry of the problem is shown in Fig. 1. The spherical region has the wave number k_1, and the rest of the space has the wave number k_0. The model is a rather elementary one, but represents the first step in the

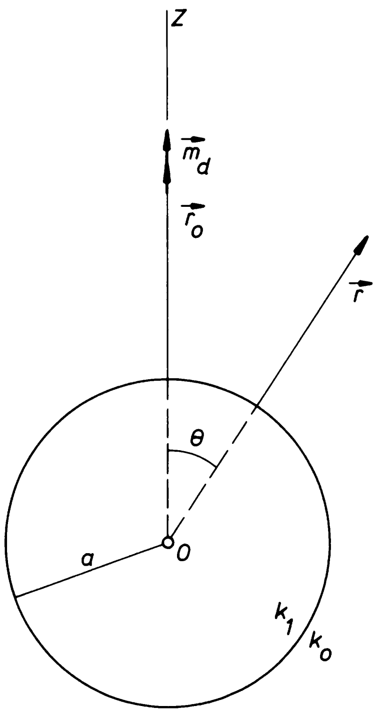

FIG. 1. A radial magnetic dipole outside a sphere with electromagnetic parameters different from those of the surroundings. The vector **r** represents the general point where the field is to be evaluated.

interpretation of e.g., double-dipole measurements. Without loss of generality we may choose the origin to be at the center of the sphere and the dipole to lie on the polar axis, i.e., $\mathbf{r}_0 = (r_0, 0, \text{arbitrary})$ and $\mathbf{m}_d = (m_d, 0, \text{arbitrary})$.

The expansion of the primary potential is

$$\mathbf{A}_P = \mathbf{e}_\varphi \frac{i\mu_0 k_0 m_d}{4\pi r_0} \sum_{n=1}^{\infty} (2n+1) j_n h_n P_n^1, \tag{23}$$

where the r-dependence is j_n for $r < r_0$ and h_n for $r > r_0$. The primary field and therefore the total field has azimuthal symmetry (no dependence on φ except for the unit vectors). The general solution $\mathbf{A}_s$ must be finite everywhere; hence inside the sphere its r-dependence must be given by j_n alone. Outside the sphere and far from the source the field must behave like an outgoing wave, since there is no other interface to scatter it. Therefore the r-dependence is given by the spherical Bessel function of the third kind, h_n. Thus the total vector potentials in the two regions are

$$\mathbf{A}_0 = \mathbf{e}_\varphi \sum_{n=1}^{\infty} D_n^0 h_n P_n^1 + \mathbf{e}_\varphi \frac{i\mu_0 k_0 m_d}{4\pi r_0} \sum_{n=1}^{\infty} (2n+1) j_n h_n P_n^1, \tag{24}$$

$$\mathbf{A}_1 = \mathbf{e}_\varphi \sum_{n=1}^{\infty} D_n^1 j_n P_n^1, \tag{25}$$

where the D-coefficients are to be determined from the boundary conditions at $r = a$. Since in this case $\mathbf{E} = i\omega\mathbf{A}$, one of the boundary conditions (13) is $A_{0\varphi} \equiv A_{1\varphi}$. Evaluating $\mathbf{B}$ and applying the second boundary condition easily gives for the coefficients ($j_n(k_0 a) = j_{n0}$, etc.)

$$D_n^0 = \frac{i\mu_0 k_0 m_d}{4\pi r_0}(2n+1)h_n \frac{\mu_0 j_{n0}(j_{n1} + k_1 a j'_{n1}) - \mu_1 j_{n1}(j_{n0} + k_0 a j'_{n0})}{\mu_1 j_{n1}(h_{n0} + k_0 a h'_{n0}) - \mu_0 h_{n0}(j_{n1} + k_1 a j'_{n1})},$$

$$D_n^1 = \frac{i\mu_0 k_0 m_d}{4\pi r_0}(2n+1)h_n \frac{\mu_1 j_{n0}(h_{n0} + k_0 a h'_{n0}) - \mu_1 h_{n0}(j_{n0} + k_0 a j'_{n0})}{\mu_1 j_{n1}(h_{n0} + k_0 a h'_{n0}) - \mu_0 h_{n0}(j_{n1} + k_1 a j'_{n1})}. \tag{26}$$

These expressions are enough to calculate the "homogeneous" part of the field at any point in space to the accuracy desired. The primary field can, of course, be obtained from the exact expression (21).

4. Transverse magnetic dipole and sphere. The model is the same as that of Fig. 1 except that now the dipole is perpendicular to the polar axis, $\mathbf{m}_d = (m_d, \pi/2, 0)$. The field is no longer azimuthally symmetric. The expansion of $\mathbf{A}_P$ can be calculated as outlined in the Appendix, and its spherical components can be written as

$$A_{Pr} = iC_0 r_0 \frac{\sin\varphi}{r} \sum_1^\infty (2n+1) j_n h_n P_n^1,$$

$$A_{P\theta} = iC_0 \frac{\sin\varphi}{r\sin\theta} \sum_1^\infty j_n h_n \{n r_0 P_{n+1}^1 - (2n+1) r P_n^1 + (n+1) r_0 P_{n-1}^1\}, \tag{27}$$

$$A_{P\varphi} = -iC_0 \frac{\cos\varphi}{r\sin\theta} \sum_1^\infty j_n h_n \{n r P_{n+1}^1 - (2n+1) r_0 P_n^1 + (n+1) r P_{n-1}^1\},$$

where $C_0 = \mu_0 k_0 m_d/4\pi r_0$. We shall also need the θ- and φ- components of $\mathbf{B}_P = \nabla \times \mathbf{A}_P$,

$$B_{P\theta} = iC_0 \frac{\cos\varphi}{r\sin\theta} \sum_1^\infty h_n\{n(2j_n + k_0 r j'_n) P_{n+1}^1$$
$$- k_0 r_0 (2n+1) j'_n P_n^1 + (n+1)(2j_n + k_0 r j'_n) P_{n-1}^1\},$$

$$B_{P\varphi} = -iC_0 \frac{\sin\varphi}{r^2 \sin\theta} \sum_1^\infty h_n\{n r_0 (n j_n - j_n - k_0 r j'_n) P_{n+1}^1 \tag{28}$$
$$+ r(2n+1)(2j_n + k_0 r j'_n) P_n^1 - r_0(n+1)(n j_n + 2j_n + k_0 r j'_n) P_{n-1}^1\}.$$

Since the φ-dependence is linear in $\cos\varphi$ or $\sin\varphi$, it is clear that it is enough to take the terms with $m = 1$ in the expansion of the homogeneous field, Eq. (20). Furthermore, instead of the factor $\exp(i\varphi)$ we must take $i \sin\varphi$ in A_{sr} and $A_{s\theta}$, and $\cos\varphi$ in $A_{s\varphi}$ to be able to satisfy the boundary conditions. Thus we arrive at the following expressions:

$$A_{sr} = i \sum_1^\infty n(n+1) G_n \frac{j_n}{kr} P_n^1 \sin\varphi,$$

$$A_{s\theta} = i \frac{\sin\varphi}{\sin\theta} \sum_1^\infty \left\{\left(\frac{j_n}{kr} + j'_n\right) \frac{G_n}{2n+1} \{n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1\} + i j_n D_n P_n^1\right\}, \tag{29}$$

$$A_{s\varphi} = -\frac{\cos\varphi}{\sin\theta} \sum_1^\infty \left\{ i j_n \frac{D_n}{2n+1} \{n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1\} + \left(\frac{j_n}{kr} + j'_n\right) G_n P_n^1\right\},$$

$$B_{sr} = k \sum_{1}^{\infty} n(n+1) D_n \frac{j_n}{kr} P_n^1 \cos\varphi,$$

$$B_{s\theta} = k \frac{\cos\varphi}{\sin\theta} \sum_{1}^{\infty} \left\{ \left(\frac{j_n}{kr} + j_n' \right) \frac{D_n}{2n+1} \{ n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1 \} + i j_n G_n P_n^1 \right\}, \tag{30}$$

$$B_{s\varphi} = -ik \frac{\sin\varphi}{\sin\theta} \sum_{1}^{\infty} \left\{ i j_n \frac{G_n}{2n+1} \{ n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1 \} + \left(\frac{j_n}{kr} + j_n' \right) D_n P_n^1 \right\}.$$

The boundary conditions give a set of equations for the coefficients which may be solved recursively. All the first-order coefficients vanish, and the second-order coefficients are

$$D_2^0 = -\frac{5C_0}{3a} \frac{i j_{21} \{ k_0 r_0 h_1 j_{10}' - h_2 (2 j_2 + k_0 a j_2') \} + k_1 \dfrac{\mu_0}{\mu_1} \left(\dfrac{j_2}{k_1 a} + j_2' \right) (a h_2 j_{20} - r_0 h_1 j_{10})}{k_0 j_{21} \left(\dfrac{h_2}{k_0 a} + h_2' \right) - k_1 \dfrac{\mu_0}{\mu_1} h_{20} \left(\dfrac{j_2}{k_1 a} + j_2' \right)},$$

$$D_2^1 = -\frac{5C_0}{3a} \frac{k_0 \left(\dfrac{h_2}{k_0 a} + h_2' \right) (a h_2 j_{20} - r_0 h_1 j_{10}) + i h_{20} \{ k_0 r_0 h_1 j_{10}' - h_2 (2 j_2 + k_0 a j_2') \}}{k_0 j_{21} \left(\dfrac{h_2}{k_0 a} + h_2' \right) - k_1 \dfrac{\mu_0}{\mu_1} h_{20} \left(\dfrac{j_2}{k_1 a} + j_2' \right)}, \tag{31}$$

$$G_2^0 = -\frac{5C_0}{3aF} \left\{ k_1 \frac{\mu_0}{\mu_1} j_{21} (a h_1 j_{10} - r_0 h_2 j_{20}) - \frac{i}{a} \left(\frac{j_2}{k_1 a} + j_2' \right) \{ a h_1 (2 j_1 + k_0 a j_1') - r_0 h_2 (4 j_2 + k_0 a j_2') \} \right\},$$

$$G_2^1 = -\frac{5C_0}{3aF} \left\{ k_0 h_{20} (a h_1 j_{10} - r_0 h_2 j_{20}) - \frac{i}{a} \left(\frac{h_2}{k_0 a} + h_2' \right) \{ a h_1 (2 j_1 + k_0 a j_1') - r_0 h_2 (4 j_2 + k_0 a j_2') \} \right\},$$

where

$$F = k_1 \frac{\mu_0}{\mu_1} j_{21} \left(\frac{h_2}{k_0 a} + h_2' \right) - k_0 h_{20} \left(\frac{j_2}{k_1 a} + j_2' \right).$$

The electromagnetic field due to a general magnetic dipole in the presence of a sphere can, of course, be calculated by resolving the dipole into a radial and a transverse component.

5. Radial current dipole and sphere. The current dipole $I d\mathbf{l} = -I dl \mathbf{e}_z = -I dl(\mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta)$ is placed inside the sphere at $\mathbf{r}_0 = (r_0, 0, \text{arbitrary})$, as shown in Fig. 2. The primary field has azimuthal symmetry, and its expansion is obtained from Eqs. (22) and (A2),

$$\mathbf{A}_P = C_1 (\mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta) \sum_{0}^{\infty} (2n+1) j_n h_n P_n, \tag{32}$$

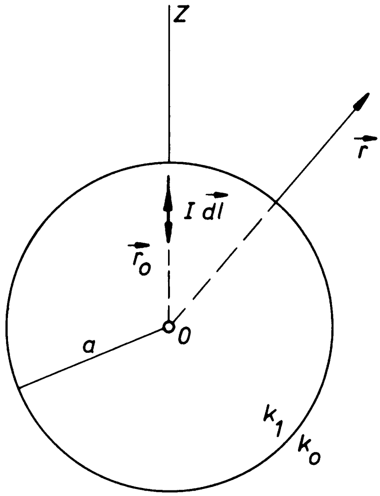

FIG. 2. A radial current dipole inside a sphere.

where $C_1 = -i\mu_1 k_1 Id1/4\pi$. Since $\nabla \cdot \mathbf{A}_P \neq 0$, we must use Eq. (12) to obtain $\mathbf{E}_P$,

$$\mathbf{B}_P = \mathbf{e}_\varphi C_1 \sum_1^\infty (2n+1) j_n \frac{h_n}{r} P_n^1, \tag{33}$$

$$\mathbf{E}_P = \mathbf{e}_r \frac{i\omega}{k_1^2} C_1 \sum_1^\infty (2n+1) n(n+1) j_n \frac{h_n}{r^2} P_n - \mathbf{e}_\theta \frac{i\omega}{k_1} C_1 \sum_1^\infty (2n+1) j_n \frac{h_n'}{r} P_n^1. \tag{34}$$

The homogeneous field is evidently

$$\mathbf{A}_s = \mathbf{e}_r \sum_0^\infty G_n n(n+1) \frac{j_n}{kr} P_n - \mathbf{e}_\theta \sum_0^\infty G_n \left(\frac{j_n}{kr} + j_n'\right) P_n^1, \tag{35}$$

$$\mathbf{B}_s = \mathbf{e}_\varphi \sum_0^\infty k G_n j_n P_n^1, \tag{36}$$

$$\mathbf{E}_s = \mathbf{e}_r i\omega \sum_0^\infty G_n n(n+1) \frac{j_n}{kr} P_n - \mathbf{e}_\theta i\omega \sum_0^\infty G_n \left(\frac{j_n}{kr} + j_n'\right) P_n^1. \tag{37}$$

Outside the sphere we must use h_n instead of j_n in $\mathbf{B}_s$ and $\mathbf{E}_s$. The boundary conditions for H_φ and E_θ at $r = a$ yield

$$\begin{aligned} G_n^0 &= \frac{i\mu_0 k_1 Id1}{4\pi} (2n+1) j_n(k_1 r_0) \frac{k_1 j_{n1} \dfrac{h_{n1}'}{k_1 a} - \dfrac{h_{n1}}{a}\left(\dfrac{j_n}{k_1 a} + j_n'\right)}{k_0 h_{n0}\left(\dfrac{j_n}{k_1 a} + j_n'\right) - \dfrac{\mu_0}{\mu_1} k_1 j_{n1}\left(\dfrac{h_n}{k_0 a} + h_n'\right)}, \\ G_n^1 &= \frac{i\mu_1 k_1 Id1}{4\pi} (2n+1) j_n(k_1 r_0) \frac{k_0 h_{n0} \dfrac{h_{n1}'}{k_1 a} - \dfrac{\mu_0}{\mu_1} \dfrac{h_{n1}}{a}\left(\dfrac{h_n}{k_0 a} + h_n'\right)}{k_0 h_{n0}\left(\dfrac{j_n}{k_1 a} + j_n'\right) - \dfrac{\mu_0}{\mu_1} k_1 j_{n1}\left(\dfrac{h_n}{k_0 a} + h_n'\right)}. \end{aligned} \tag{38}$$

Using these coefficients and the expansions (36) and (37) the field vectors can be calculated everywhere. In applications the quantity actually measured is often the potential difference between two points 1 and 2, i.e., the line integral of $\mathbf{E}$ from 1 to 2. Such an integration can be done numerically.

The field due to a transverse current dipole can be calculated in a way analogous to the treatment of Sect. 4.

6. Discussion. The solutions of the vector wave equation derived above are given as series expansions in spherical wave functions. Otherwise they are analytical and exact. Their practical usefulness depends, of course, on the rate of convergence, which in turn depends on where the field is evaluated and on the geometry and parameters of the particular problem. The solutions have certain advantages as well as restrictions. They are quite general with respect to the frequency, and hence they should be as applicable at the very low frequencies used in geophysical measurements as at radio frequencies. The material parameters are assumed to be piecewise constant scalar quantities—a conventional approach in electromagnetic field problems. The wave equations (9) and (10) remain valid even if ε and σ (but not μ) depend on $\mathbf{r}$ continuously. However, the solution of Eq. (10) can be given as in Eq. (20) only for a constant k. The confinement to dipolar source fields is not a serious one, since higher multipoles are usually not used in practice, and the plane wave is actually easier to treat than the dipolar field.

The most severe limitation from the point of view of modelling is the requirement of spherical boundaries. Our method of solution can be used also in a problem having several spheres. However, if they are not concentric, use must be made of the transformation formulas of the scalar and vector spherical harmonics in a translation of the origin [2]. Mixing boundaries having altogether different characters, e.g., planes with spheres or cylinders, presents formidable difficulties. The mathematical tools needed to carry out the coordinate transformations in a form required to satisfy the boundary conditions are yet to be developed.

References

[1] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, Part II, McGraw-Hill, pp. 1865–6, 1898–1901 (1953)

[2] O. R. Cruzan, *Translational addition theorems for spherical vector wave functions*, Quart. Appl. Math. **20**, 33–40 (1962)

Appendix: Expansion of dipolar fields in terms of spherical harmonics and Bessel functions. Let a harmonically oscillating point dipole be located at $\mathbf{r}_0 = (r_0, \theta_0, \varphi_0)$ in a homogeneous space having the wave number k_0. The field of the dipole at $\mathbf{r} = (r, \theta, \varphi)$ can be expressed with the help of the well-known expansion

$$\frac{e^{ik_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} = 4\pi i k_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\theta, \varphi) Y_n^{m*}(\theta_0, \varphi_0) \begin{cases} j_n(k_0 r) h_n(k_0 r_0), & r < r_0 \\ j_n(k_0 r_0) h_n(k_0 r), & r > r_0 \end{cases} \tag{A1}$$

where j_n and h_n are the nth-order spherical Bessel functions of the first and third kind, respectively. Eq. (A1) represents an outgoing spherical wave emitted by the dipole. In the following we assume that the dipole lies on the polar axis, i.e., $\theta_0 = 0$. Then Eq (A1) reduces to

$$\frac{e^{ik_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} = ik_0 \sum_0^\infty (2n+1) j_n h_n P_n \tag{A2}$$

and

$$\begin{aligned}\mathbf{r}-\mathbf{r}_0 &= \mathbf{e}_x r \sin\theta \cos\varphi + \mathbf{e}_y r \sin\theta \sin\varphi + \mathbf{e}_z (r\cos\theta - r_0) \\ &= \mathbf{e}_r (r - r_0 \cos\theta) + \mathbf{e}_\theta r_0 \sin\theta.\end{aligned} \tag{A3}$$

Noting that

$$-\frac{1}{rr_0}\frac{\partial}{\partial \cos\theta}\frac{e^{ik_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} = e^{ik_0|\mathbf{r}-\mathbf{r}_0|}\left(\frac{ik_0}{|\mathbf{r}-\mathbf{r}_0|^2} - \frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}\right), \tag{A4}$$

differentiating term by term, and using the properties of the Legendre functions

$$\sin^2\theta \frac{dP_n^m}{d\cos\theta} = (n+1)\cos\theta\, P_n^m - (n-m+1)P_{n+1}^m, \tag{A5}$$

$$(2n+1)\cos\theta\, P_n^m = (n-m+1)P_{n+1}^m + (n+m)P_{n-1}^m, \tag{A6}$$

$$\begin{aligned}(2n+1)\sin\theta P_n^m &= P_{n+1}^{m+1} - P_{n-1}^{m+1} \\ &= (n+m)(n+m-1)P_{n-1}^{m-1} - (n-m+1)(n-m+2)P_{n+1}^{m-1},\end{aligned} \tag{A7}$$

we obtain

$$e^{ik_0|\mathbf{r}-\mathbf{r}_0|}\left(\frac{ik_0}{|\mathbf{r}-\mathbf{r}_0|^2} - \frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}\right) = -\frac{ik_0}{rr_0 \sin\theta}\sum_1^\infty (2n+1) j_n h_n P_n^1. \tag{A8}$$

Let the moment of a magnetic dipole be $\mathbf{m}_d = (m_d, \theta_d, 0)$, since we can choose $\varphi_d = 0$ without loss of generality. Then

$$\begin{aligned}(\mathbf{r}-\mathbf{r}_0)\times\mathbf{m}_d = &-\mathbf{e}_r m_d r_0 \sin\theta_d \sin\theta \sin\varphi + \mathbf{e}_\theta m_d \sin\theta_d (r - r_0\cos\theta)\sin\varphi \\ &+ \mathbf{e}_\varphi m_d \sin\theta_d (r\cos\theta - r_0)\cos\varphi - \mathbf{e}_\varphi m_d r \cos\theta_d \sin\theta.\end{aligned} \tag{A9}$$

The expansion of the potential of a magnetic dipole is given by the product of Eq. (A8) and Eq. (A9). That of an electric dipole can be obtained with the help of Eq. (A2).

The expansions of the dipolar fields can be written compactly in terms of the vector functions $\mathbf{L}_n^m$, $\mathbf{M}_n^m$ and $\mathbf{N}_n^m$ (see Eq. (18)). For $m = 0$ and $m = 1$ these functions are

$$\begin{aligned}\mathbf{L}_n^0 &= \mathbf{e}_r j_n' P_n - \mathbf{e}_\theta \frac{j_n}{kr} P_n^1, \\ \mathbf{M}_n^0 &= \mathbf{e}_\varphi j_n P_n^1, \\ \mathbf{N}_n^0 &= \mathbf{e}_r n(n+1)\frac{j_n}{kr} P_n - \mathbf{e}_\theta \left(\frac{j_n}{kr} + j_n'\right) P_n^1,\end{aligned} \tag{A10}$$

$$\mathbf{L}_n^1 = \mathbf{e}_r j_n' P_n^1 e^{i\varphi} + \frac{1}{(2n+1)\sin\theta} \frac{j_n}{kr}$$
$$\cdot \{\mathbf{e}_\theta\{n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1\} + \mathbf{e}_\varphi i(2n+1)P_n^1\}e^{i\varphi},$$

$$\mathbf{M}_n^1 = \frac{j_n}{(2n+1)\sin\theta}\{\mathbf{e}_\theta i(2n+1)P_n^1 - \mathbf{e}_\varphi\{n^2P_{n+1}^1 - (n+1)^2P_{n-1}^1\}\}e^{i\varphi}, \tag{A11}$$

$$\mathbf{N}_n^1 = \mathbf{e}_r n(n+1)\frac{j_n}{kr} P_n^1 e^{i\varphi} + \frac{1}{(2n+1)\sin\theta}\left(\frac{j_n}{kr} + j_n'\right)$$
$$\cdot \{\mathbf{e}_\theta\{n^2 P_{n+1}^1 - (n+1)^2 P_{n-1}^1\} + \mathbf{e}_\varphi i(2n+1)P_n^1\}e^{i\varphi}.$$

Comparing the vector potential of a radial magnetic dipole (see Eq. (23)) and Eq. (A10) we see immediately that

$$\mathbf{A}_P = \frac{i\mu_0 k_0 m_d}{4\pi r_0}\sum_1^\infty (2n+1)h_n \mathbf{M}_n^0. \tag{A12}$$

To obtain the expansion of the potential of a transverse dipole, we replace $\cos\varphi$ and $i\sin\varphi$ in Eq. (27) with $e^{i\varphi}$. To get the r-component right, the expansion must obviously be

$$\mathbf{A}_P = \sum_1^\infty M_n \mathbf{M}_n^1 + C_0 r_0 k_0 \sum_1^\infty \frac{2n+1}{n(n+1)} h_n \mathbf{N}_n^1, \tag{A13}$$

since $\nabla \cdot \mathbf{A}_P = 0$ and the $\mathbf{L}$-functions are consequently not needed. To match the θ-components we must have

$$\sum_1^\infty iM_n j_n P_n^1 + C_0 r_0 k_0 \sum_1^\infty h_n\left(\frac{j_n}{kr} + j_n'\right)\left(\frac{n}{n+1}P_{n+1}^1 - \frac{n+1}{n}P_{n-1}^1\right)$$
$$\equiv C_0 \sum_1^\infty h_n \frac{j_n}{r}\{r_0 n P_{n+1}^1 + r_0(n+1)P_{n-1}^1 - (2n+1)rP_n^1\}. \tag{A14}$$

Using the properties of the Bessel functions

$$(2n+1)j_n' = nj_{n-1} - (n+1)j_{n+1}, \qquad (2n+1)\frac{j_n}{kr} = j_{n-1} + j_{n+1} \tag{A15}$$

and collecting the coefficients of $j_n P_n^1$, we obtain after some algebra

$$M_n = \frac{iC_0(2n+1)}{n(n+1)}\{(n+1)h_n - k_0 r_0 h_{n+1}\}. \tag{A16}$$

This coefficient also gives the correct φ-component. Thus the expansion of the potential of a transverse magnetic dipole is

$$\mathbf{A}_P = \sum_1^\infty \frac{C_0(2n+1)}{n(n+1)}\{i\{(n+1)h_n - k_0 r_0 h_{n+1}\}\mathbf{M}_n^1 + k_0 r_0 h_n \mathbf{N}_n^1\}. \tag{A17}$$

The potential of a current dipole can also be expanded. For example, the potential of a radial dipole, Eq. (32), can be written as

$$\mathbf{A}_P = \mathbf{e}_r C_1 \sum_0^\infty j_n h_n\{(n+1)P_{n+1} + nP_{n-1}\} - \mathbf{e}_\theta C_1 \sum_0^\infty j_n h_n(P_{n+1}^1 - P_{n-1}^1). \tag{A18}$$

Using Eq. (A15) it is easy to show that

$$
\begin{aligned}
(n+1)\mathbf{L}_{n+1}^{0} + \mathbf{N}_{n+1}^{0} &= \mathbf{e}_r(n+1)j_n P_{n+1} - \mathbf{e}_\theta j_n P_{n+1}^{1} \\
-\, n\mathbf{L}_{n-1}^{0} + \mathbf{N}_{n-1}^{0} &= \mathbf{e}_r n j_n P_{n-1} + \mathbf{e}_\theta j_n P_{n-1}^{1};
\end{aligned}
\tag{A19}
$$

hence

$$
\begin{aligned}
\mathbf{A}_P &= C_1 \sum_{0}^{\infty} \{(n+1)\mathbf{L}_{n+1}^{0} + \mathbf{N}_{n+1}^{0}\} - C_1 \sum_{1}^{\infty} (n\mathbf{L}_{n-1}^{0} - \mathbf{N}_{n-1}^{0}) \\
&= C_1 \sum_{0}^{\infty} (n\mathbf{L}_{n}^{0} + \mathbf{N}_{n}^{0}) - C_1 \sum_{0}^{\infty} \{(n+1)\mathbf{L}_{n}^{0} - \mathbf{N}_{n}^{0}\} \\
&= C_1 \sum_{0}^{\infty} (2\mathbf{N}_{n}^{0} - \mathbf{L}_{n}^{0}).
\end{aligned}
\tag{A20}
$$

These expansions in terms of the vector functions are particularly useful when coordinate transformations are needed in order to satisfy the boundary conditions.