ON THE BOUNDARY-VALUE PROBLEM ASSOCIATED WITH A GENERAL TWISTED TUBE WITH A UNIFORM NON-ROTATING SECTION*

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Abstract. An orthogonal curvilinear coordinate system is used to formulate the Dirichlet problem of potential theory associated with the interior of a general twisted tube with a uniform non-rotating section. Two solution schemes are presented for a class of finite and infinite tube geometries. The boundary-value problems associated with a tube of uniform circular section are discussed as an example.

Introduction. To date the linear boundary-value problems of mathematical physics associated with the interior of a tube have admitted solutions only for a limited number of tube geometries. Solutions have been obtained mainly for linear boundary-value problems which involve a torus with circular section. It appears, however, that solutions do not exist for boundary-value problems associated with tubes which include the effects of curvature and torsion and the effects of sections other than circular. Recently [1] it has been shown that an orthogonal curvilinear coordinate system can be constructed for the interior of a general twisted tube which has a uniform non-rotating section, and this coordinate system is employed in this paper to formulate the linear boundary-value problems.

The method of solution presented in what follows is based on an iterative scheme and can be used for a wide range of linear boundary-value problems. In the interests of brevity and clarity, however, we will only consider the Dirichlet problem of potential theory.

1. The coordinate system. We denote the interior and boundary of a tube in R_3 by D_3 and ∂D_3 respectively. The orientation of the tube is specified by a curve L (Fig. 1) which has a prescribed unit tangent vector $\mathbf{t}_1(\xi^1)$. The coordinate ξ^1 is the arc length along L from the origin O to the point O'. The intersection with $D_3 \cup \partial D_3$ of the plane $\xi^1 = \text{const.}$ normal to L which passes through O' is denoted by $D_2 \cup \partial D_2$.

It has been shown [1] that if the tube section remains undistorted and does not rotate about $\mathbf{t}_1$ as ξ^1 varies then an orthogonal curvilinear coordinate system can be constructed when the unit normal to ∂D_2 which lies in the plane $\xi^1 = \text{const.}$ is prescribed. These coordinates are denoted by ξ^i, $i = 1, 2, 3$ where $\xi^2 = \text{const.}$ on ∂D_2 for all values of ξ^1 and ξ^3.

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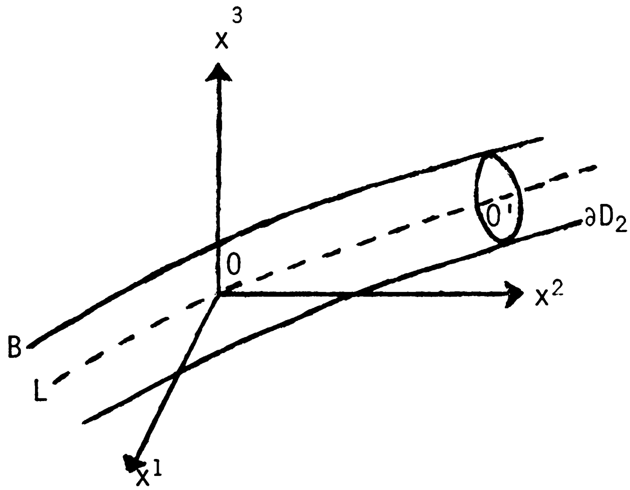

FIG. 1.

The orthogonal curvilinear coordinate system is specified by the base vectors $\mathbf{a}_i$, $i = 1, 2, 3$ where

$$\mathbf{a}_1/A_1 \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \sin\phi \\ \sin\theta \cos\phi \end{pmatrix} = \mathbf{t}_1, \tag{1.1}$$

$$\mathbf{a}_2/A_2 \equiv \begin{pmatrix} \sin\theta \sin\psi \\ \cos\phi \cos\psi - \sin\phi \cos\theta \sin\psi \\ -\sin\phi \cos\psi - \cos\phi \cos\theta \sin\psi \end{pmatrix}, \tag{1.2}$$

$$\mathbf{a}_3/A_3 \equiv \begin{pmatrix} -\sin\theta \cos\psi \\ \cos\phi \sin\psi + \sin\phi \cos\theta \cos\psi \\ -\sin\phi \sin\psi + \cos\phi \cos\theta \cos\psi \end{pmatrix}, \tag{1.3}$$

respectively. In Eqs. (1.1)–(1.3) A_i, $i = 1, 2, 3$ represent the magnitudes of the vectors $\mathbf{a}_i$, $i = 1, 2, 3$ and are the scaling factors for the coordinate system with $A_2 = A_3 \equiv A$. The angles θ and ϕ specify $\mathbf{t}_1$ and are prescribed twice differentiable functions of ξ^1. To preserve the orthogonality of the coordinate system A must be independent of ξ^1 and ψ must be given by

$$\psi - \psi_0 = -\int_0^{\xi^1} \frac{d\phi}{d\bar{\xi}^1} \cos\theta(\bar{\xi}^1)\, d\bar{\xi}^1, \tag{1.4}$$

where $\psi_0 \equiv \psi(\xi^1 = 0)$. In what follows it is necessary to introduce two further unit vectors given by

$$\mathbf{t}_2 = \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix}, \tag{1.5}$$

$$\mathbf{t}_3 = \begin{pmatrix} -\sin\theta \\ \sin\phi \cos\theta \\ \cos\phi \cos\theta \end{pmatrix} \tag{1.6}$$

respectively. The unit vectors $\mathbf{t}_i$, $i = 1, 2, 3$ are then mutually orthogonal.

The transformation from Cartesian coordinates x^i, $i = 1, 2, 3$ to the curvilinear coordinates ξ^i, $i = 1, 2, 3$ can be written in the form

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = T_1(\phi)^{-1} T_2(\theta)^{-1} \begin{pmatrix} 0 \\ v \\ u \end{pmatrix} + \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}, \tag{1.7}$$

where

$$T_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{1.8}$$

$$T_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{1.9}$$

and

$$\begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \int_0^{\xi^1} \mathbf{t}_1(\bar{\xi}^1)\, d\bar{\xi}^1. \tag{1.10}$$

The vector in (1.10) represents the point 0′ in D_2 and the functions $u(\xi^1, \xi^2, \xi^3)$ and $v(\xi^1, \xi^2, \xi^3)$ are given by

$$u = q\cos(\psi - \psi_0 + \mu) \tag{1.11}$$

$$v = q\sin(\psi - \psi_0 + \mu), \tag{1.12}$$

where $q = (u_0^2 + v_0^2)^{1/2}$, $q\sin\mu = v_0$, $q\cos\mu = u_0$, $u_0(\xi^2, \xi^3) = u(0, \xi^2, \xi^3)$ and $v_0(\xi^2, \xi^3) = v(0, \xi^2, \xi^3)$. The axes of the Cartesian frame $0'v$, $0'u$ coincide with the unit vectors $\mathbf{t}_2$ and $\mathbf{t}_3$.

It has been shown that $v_0 + iu_0$ and $Ae^{-i\psi}$ are analytic functions of the complex variable $\xi^2 + i\xi^3$ in $D_2 \cup \partial D_2$ for all values of ξ^1. Complex variable methods can be employed to determine u_0, v_0, A and ψ uniquely in $D_2 \cup \partial D_2$ for all values of ξ^1. Moreover, if $\kappa(\xi^1)$ is the curvature of L and $Q \equiv \max_{(\xi^2, \xi^3) \in \partial D_2} q$ with $\kappa Q < 1$ for all ξ^1, then the scaling factor A_1 is given by

$$A_1 = 1 - u\dot{\theta} - v\dot{\phi}\sin\theta, \tag{1.13}$$

where the dot notation denotes the operation $d/d\xi^1$. Further properties of the coordinate system are established in the appendix and will be used throughout the analysis.

2. Formulation of the boundary-value problem. Laplace's equation in the orthogonal curvilinear coordinate system ξ^i, $i = 1, 2, 3$ has the form

$$\frac{\partial}{\partial\xi^1}\left(\frac{A^2}{A_1}\frac{\partial V}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(A_1\frac{\partial V}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(A_1\frac{\partial V}{\partial\xi^3}\right) = 0. \tag{2.1}$$

Employing the expressions for A_1 and its derivatives derived in the appendix we find, after some algebra,

$$\frac{\partial^2 V}{\partial \xi^{1^2}} + \Delta_2 V - \kappa\left[q \cos \lambda_1\left(\frac{\partial^2 V}{\partial \xi^{1^2}} + 3\Delta_2 V\right) + \cos \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^2}\right.$$
$$\left. + \sin \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^3}\right] + \dot{\kappa} q \cos \lambda_1 \frac{\partial V}{\partial \xi^1} - \tau\kappa q \sin \lambda_1 \frac{\partial V}{\partial \xi^1}$$
$$+ \kappa^2 q \cos \lambda_1\left(2 \cos \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^2} + 2 \sin \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^3} + 3q \cos \lambda_1 \Delta_2 V\right)$$
$$- \kappa^3 q^2 \cos^2 \lambda_1\left(\cos \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^2} + \sin \lambda_2 \cdot \frac{1}{A}\frac{\partial V}{\partial \xi^3} - q \cos \lambda_1 \Delta_2 V\right) = 0, \quad (2.2)$$

where

$$\Delta_2 V = \frac{1}{A^2}\left(\frac{\partial^2 V}{\partial \xi^{2^2}} + \frac{\partial^2 V}{\partial \xi^{3^2}}\right), \qquad \lambda_1 = \gamma + \psi - \psi_0 + \mu,$$

$$\lambda_2 = \gamma + \psi - \psi_0 + \nu, \qquad \tan \gamma = \frac{\phi \sin \theta}{\dot{\theta}}, \qquad \tan \nu = \frac{\partial v_0}{\partial \xi^2} \bigg/ \frac{\partial u_0}{\partial \xi^2}.$$

The curvature and torsion of L are represented by κ and τ respectively. Eq. (2.2) must be satisfied in D_3.

If B denotes the curved part of ∂D_3 then the Dirichlet condition can be imposed for the following four cases:

i) *Tube of finite length l.* In this case ∂D_3 is the union of B, $D_2 \cup \partial D_2$ ($\xi^1 = 0$) and $D_2 \cup \partial D_2$ ($\xi^1 = l$) and the boundary conditions have the form

$$V = V_0(\xi^1, \xi^3) \quad \text{on} \quad B, \qquad (2.3)$$

$$V = V_1(\xi^2, \xi^3) \quad \text{on} \quad D_2 \cup \partial D_2 \ (\xi^1 = 0), \qquad (2.4)$$

$$V = V_2(\xi^2, \xi^3) \quad \text{on} \quad D_2 \cup \partial D_2 \ (\xi^1 = l), \qquad (2.5)$$

where V_0, V_1 and V_2 are prescribed functions.

ii) *Closed tube of finite length l.* Here the boundary condition is that given by (2.3). We also require $V(\xi^1, \xi^2, \xi^3)$ to be periodic in ξ^1 with period l.

iii) *Semi-infinite tube.* If $D_2 \cup \partial D_2$ ($\xi^1 = 0$) is one bounding surface of the tube, the boundary conditions are those given by (2.3) and (2.4) and V must remain bounded everywhere in D_3.

iv) *Infinite tube.* The boundary condition in this case is that given by (2.3) with V remaining bounded everywhere in D_3. This completes the formulation of the boundary-value problem.

3. Solution scheme. Since $\kappa Q < 1$ for all ξ^1 then, if $\varepsilon_1(< 1) \equiv \max_{\xi^1 \in L} \kappa Q$ and $\eta^1 \equiv \xi^1/Q$, we can write $\kappa Q = \varepsilon_1 f_1(\eta^1)$ where $f_1(\eta^1)$ is $O(1)$. Moreover, we will assume in what follows that $\dot{f}_1(\equiv df_1/d\eta^1)$ is $O(1)$. Also, if η^2, $\eta^3 \equiv \xi^2$, ξ^3 and $f \equiv q/Q$ then, using (A.13), we have $A = Qg$ where $g \equiv ((\partial f/\partial \eta^2)^2 + (\partial f/\partial \eta^3)^2)^{1/2}$. If τQ is $O(1)$ for all η^1 then we can write $\tau Q = f_2(\eta^1)$ where $f_2(\eta^1)$ is $O(1)$. Eq. (2.2) can now be written in the form

$$V_{,11} + \nabla^2 V = \varepsilon_1[f_1 fc_1(V_{,11} + 3\nabla^2 V) + f_1 c_2 V_{,2} + f_1 s_2 V_{,3} - f_1 \dot{f}_1 fc_1 V_{,1} + f_1 f_2 fs_1 V_{,1}]$$
$$- \varepsilon_1^2 f_1^2 fc_1(2c_2 V_{,2} + 2s_2 V_{,3} + 3fc_1 \nabla^2 V) + \varepsilon_1^3 f_1^3 f^2 c_1^2(c_2 V_{,2} + s_2 V_{,3} - fc_1 \nabla^2 V), \quad (3.1)$$

where

$$(\ )_{,1} \equiv \partial(\ )/\partial\eta^1, \qquad (\ )_{,j} \equiv \frac{1}{g}\frac{\partial(\ )}{\partial\eta^j}, \qquad j = 2, 3, \qquad \nabla^2 V \equiv \frac{1}{g^2}\left(\frac{\partial^2 V}{\partial\eta^{22}} + \frac{\partial^2 V}{\partial\eta^{32}}\right)$$

and c_i, $s_i \equiv \cos\lambda_i$, $\sin\lambda_i$, $i = 1, 2$. We will seek a solution of (3.1) in the form

$$V = \sum_{n=0}^{\infty} \varepsilon_1^n V^{(n)}. \tag{3.2}$$

The system of equations for $V^{(n)}$, $n \geq 0$, is

$$V^{(0)}_{,11} + \nabla^2 V^{(0)} = 0, \tag{3.3}$$

together with

$$V^{(n)}_{,11} + \nabla^2 V^{(n)} = U^{(n-1)}, \ n \geq 1, \tag{3.4}$$

where $U^{(n-1)}$ is given by

$$\begin{aligned} U^{(n-1)} = {} & f_1 f c_1 (V^{(n-1)}_{,11} + 3\nabla^2 V^{(n-1)}) + f_1 c_2 V^{(n-1)}_{,2} + f_1 s_2 V^{(n-1)}_{,3} \\ & - f_1 \dot{f}_1 f c_1 V^{(n-1)}_{,1} + f_1 f_2 f s_1 V^{(n-1)}_{,1} - f_1^2 f c_1 (2c_2 V^{(n-2)}_{,2} + 2s_2 V^{(n-2)}_{,3} \\ & + 3 f c_1 \nabla^2 V^{(n-2)}) + f_1^3 f^2 c_1^2 (c_2 V^{(n-3)}_{,2} + s_2 V^{(n-3)}_{,3} - f c_1 \nabla^2 V^{(n-3)}), \ n \geq 1, \end{aligned} \tag{3.5}$$

with $V^{(m)}$, and all derivatives of $V^{(m)}$, identically zero when $m < 0$. Eq. (3.3) can be considered as Laplace's equation for a *cylindrical* coordinate system (η^1, η^2, η^3) with scaling factors 1, g, g and is solved subject to the given Dirichlet boundary condition. Eqs. (3.4) are Poisson type and are solved subject to homogeneous boundary conditions.

We introduce a Green's function $G(\eta^1, \eta^2, \eta^3; \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3)$ which satisfies

$$G_{,11} + \nabla^2 G = \frac{\delta(\eta^1 - \bar{\eta}^1)\,\delta(\eta^2 - \bar{\eta}^2)\,\delta(\eta^3 - \bar{\eta}^3)}{g^2} \tag{3.6}$$

The boundary condition imposed on G is that it vanish on the bounding surfaces of the cylinder with section D_2 and generators $\eta^2 =$ const. In cases (iii) and (iv) above G and $\partial G/\partial\eta^1$ must approach zero uniformly as η^1 becomes infinite.

With the aid of Green's theorem, Eqs. (3.3)–(3.6), and the boundary conditions we obtain for case (i)

$$\begin{aligned} & V^{(0)}(\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3) \\ & = \iint_{\eta^2=\text{const}} \frac{\partial G}{\partial\eta^2} V_0 \, d\eta^1 \, d\eta^3 + \iint_{Q\eta^1=l} \frac{\partial G}{\partial\eta^1} V_2 g^2 \, d\eta^2 \, d\eta^3 - \iint_{\eta^1=0} \frac{\partial G}{\partial\eta^1} V_1 g^2 \, d\eta^2 \, d\eta^3, \end{aligned} \tag{3.7}$$

and

$$V^{(n)}(\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3) = \iiint G U^{(n-1)} g^2 \, d\eta^1 \, d\eta^2 \, d\eta^3, \ n \geq 1, \tag{3.8}$$

where the volume integral is evaluated throughout the interior of the cylinder. For cases (ii), (iii) and (iv) the solutions for $V^{(n)}$, $n \geq 1$, are again given by (3.8). The solution for $V^{(0)}$ in case (ii) has the form

$$V^{(0)}(\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3) = \iint\limits_{\eta^2 = \text{const.}} \frac{\partial G}{\partial \eta^2} V_0 \, d\eta^1 \, d\eta^3, \tag{3.9}$$

since V and G are periodic in η^1 with period l/Q. In case (iii) the solution for $V^{(0)}$ is given by

$$V^{(0)}(\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3) = \iint\limits_{\eta^2 = \text{const.}} \frac{\partial G}{\partial \eta^2} V_0 \, d\eta^1 \, d\eta^3 - \iint\limits_{\eta^1 = 0} \frac{\partial G}{\partial \eta^1} V_1 g^2 \, d\eta^2 \, d\eta^3, \tag{3.10}$$

since the integral over the surface $Q\eta^1 = l$ in (3.7) approaches zero as $l \to \infty$. Similarly, in case (iv), the solution for $V^{(0)}$ will be given by (3.9).

When $\tau Q < 1$ for all η^1 we can write $\tau Q = \varepsilon_2 f_2(\eta^1)$ where $\varepsilon_2(<1) \equiv \max_{\xi^1 \in L} \tau Q$ and $f_2(\eta^1)$ is again $O(1)$. Eq. (2.2) in this case has the form

$$\begin{aligned} V_{,11} + \nabla^2 V = {} & \varepsilon_1 (f_1 f c_1 (V_{,11} + 3\nabla^2 V) + f_1 c_2 V_{,2} + f_1 s_2 V_{,3} \\ & - f_1 \dot{f}_1 f c_1 V_{,1}) + \varepsilon_1 \varepsilon_2 f_1 f_2 f s_1 V_{,1} - \varepsilon_1^2 f_1^2 f c_1 (2c_2 V_{,2} + 2s_2 V_{,3} \\ & + 3 f c_1 \nabla^2 V) + \varepsilon_1^3 f_1^3 f^2 c_1^2 (c_2 V_{,2} + s_2 V_{,3} - f c_1 \nabla^2 V). \end{aligned} \tag{3.11}$$

Using the summation notation of the tensor calculus, we can write the solution of (3.11) in the form

$$V = \sum_{n=0}^{\infty} \varepsilon_1^r \varepsilon_2^s V_{rs}^{(n)}, \qquad r + s = n. \tag{3.12}$$

The system of equations for $V_{rs}^{(n)}$, $n \geq 0$, is

$$V_{00,11}^{(0)} + \nabla^2 V_{00}^{(0)} = 0 \tag{3.13}$$

and

$$V_{rs,11}^{(n)} + \nabla^2 V_{rs}^{(n)} = U_{rs}^{(n-1)}, \qquad r + s = n, \qquad n \geq 1, \tag{3.14}$$

where

$$\begin{aligned} U_{rs}^{(n-1)} = {} & f_1 f c_1 (V_{r-1s,11}^{(n-1)} + 3\nabla^2 V_{r-1s}^{(n-1)}) + f_1 c_2 V_{r-1s,2}^{(n-1)} \\ & + f_1 s_2 V_{r-1s,3}^{(n-1)} - f_1 \dot{f}_1 f c_1 V_{r-1s,1}^{(n-1)} + f_1 f_2 f s_1 V_{r-1s-1,1}^{(n-2)} \\ & - f_1^2 f c_1 (2c_2 V_{r-2s,2}^{(n-2)} + 2s_2 V_{r-2s,3}^{(n-2)} + 3 f c_1 \nabla^2 V_{r-2s}^{(n-2)}) \\ & + f_1^3 f^2 c_1^2 (c_2 V_{r-3s,2}^{(n-3)} + s_2 V_{r-3s,3}^{(n-3)} - f c_1 \nabla^2 V_{r-3s}^{(n-3)}), \end{aligned} \tag{3.15}$$

with $V_{ij}^{(m)}$, $i + j = m$, and all derivatives of $V_{ij}^{(m)}$ identically zero when either $i < 0$ or $j < 0$.

The solution of (3.13)–(3.15) subject to the Dirichlet boundary condition is formally the same as the case when τQ is $O(1)$. For brevity we can omit the details.

4. The general twisted tube with uniform circular section. For a tube with a uniform circular section the functions u and v are given by [1]

$$u(\eta^1, \eta^2, \eta^3) = a e^{\eta^2} \sin(\eta^3 - (\psi - \psi_0)) \tag{4.1}$$

and

$$v(\eta^1, \eta^2, \eta^3) = a e^{\eta^2} \cos(\eta^3 - (\psi - \psi_0)), \tag{4.2}$$

where a is the section radius, $\psi - \psi_0$ is given by (1.4) and η^3 is the angle between $O'P$

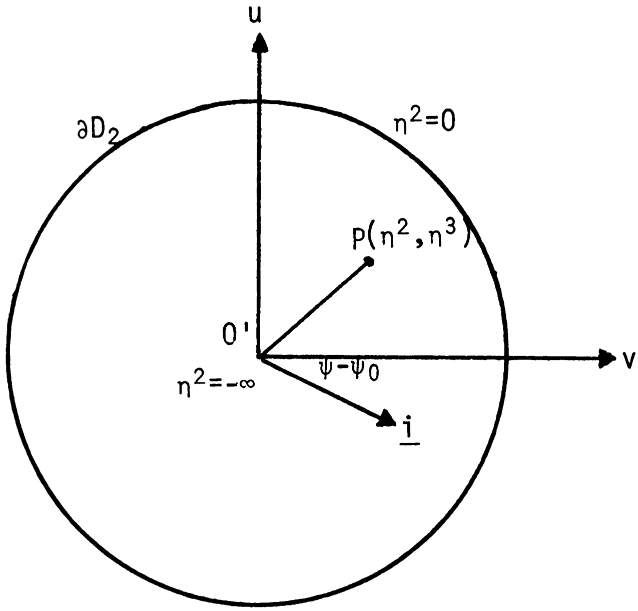

FIG. 2.

and **i** (see Fig. 2). The vector **i** is defined by

$$\mathbf{i} = \cos(\psi - \psi_0)\mathbf{t}_2 - \sin(\psi - \psi_0)\mathbf{t}_3, \tag{4.3}$$

and does not rotate about $\mathbf{t}_1$ as η^1 varies. Moreover, $\eta^2 = 0$ on B and $\eta^2 = -\infty$ on L. Also $Q = a$, $q = A = ae^{\eta^2}$ and $\mu = \nu = \pi/2 - \eta^3$ so that $\lambda_1 = \lambda_2 = \gamma + \psi - \psi_0 + \pi/2 - \eta^3 \equiv \lambda$, $c_1 = c_2 \equiv c$, and $s_1 = s_2 \equiv s$.

For convenience we introduce η, r, ξ where $\eta = \eta^1$, $r = e^{\eta^2}$ and $\xi = \xi^3$. Then, using standard techniques [2], the Green's functions for the four boundary-value problems can be written in the form

$$\text{(i):}\quad G = \frac{2a}{\pi l}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi a\eta}{l}\right)\sin\left(\frac{m\pi a\bar{\eta}}{l}\right)\cos n(\xi - \bar{\xi})$$

$$\times\left(\frac{K_n\left(\frac{m\pi a}{l}\right)}{I_n\left(\frac{m\pi a}{l}\right)}I_n\left(\frac{m\pi ar}{l}\right)I_n\left(\frac{m\pi a\bar{r}}{l}\right) - H_n\left(\frac{m\pi ar}{l},\frac{m\pi a\bar{r}}{l}\right)\right),$$

$$0 < \eta,\bar{\eta} < l/a, \qquad 0 < r,\bar{r} < 1, \qquad 0 < \xi,\bar{\xi} < 2\pi, \tag{4.4}$$

$$\text{(ii):}\quad G = \frac{2a}{\pi l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\cos\left(\frac{2m\pi a}{l}(\eta - \bar{\eta})\right)\cos n(\xi - \bar{\xi})$$

$$\times\left(\frac{K_n\left(\frac{2m\pi a}{l}\right)}{I_n\left(\frac{2m\pi a}{l}\right)}I_n\left(\frac{2m\pi ar}{l}\right)I_n\left(\frac{2m\pi a\bar{r}}{l}\right) - H_n\left(\frac{2m\pi ar}{l},\frac{2m\pi a\bar{r}}{l}\right)\right),$$

$$0 < \eta,\bar{\eta} < l/a, \qquad 0 < r,\bar{r} < 1, \qquad 0 < \xi,\bar{\xi} < 2\pi, \tag{4.5}$$

$$\text{(iii)}: \quad G = \frac{2}{\pi^2}\sum_{n=0}^{\infty}\cos n(\xi - \bar{\xi})\int_0^{\infty}\sin\alpha\eta\,\sin\alpha\bar{\eta}\left(\frac{K_n(\alpha)}{I_n(\alpha)}I_n(\alpha r)I_n(\alpha\bar{r}) - H_n(\alpha r, \alpha\bar{r})\right)d\alpha,$$

$$0 < \eta,\bar{\eta} < \infty, \qquad 0 < r,\bar{r} < 1, \qquad 0 < \xi,\bar{\xi} < 2\pi, \tag{4.6}$$

$$\text{(iv)}: \quad G = \frac{1}{\pi^2}\sum_{n=0}^{\infty}\cos n(\xi - \bar{\xi})\int_0^{\infty}\cos\alpha(\eta - 5)\left(\frac{K_n(\alpha)}{I_n(\alpha)}I_n(\alpha r)I_n(\alpha\bar{r}) - H_n(\alpha r, \alpha\bar{r})\right)d\alpha,$$

$$-\infty < \eta,\bar{\eta} < \infty, \qquad 0 < r,\bar{r} < 1, \qquad 0 < \xi,\bar{\xi} < 2\pi, \tag{4.7}$$

where

$$\begin{aligned} H_n(\alpha_1, \alpha_2) &= I_n(\alpha_1)K_n(\alpha_2), \qquad 0 < \alpha_1 < \alpha_2, \\ &= I_n(\alpha_2)K_n(\alpha_1), \qquad 0 < \alpha_2 < \alpha_1, \end{aligned} \tag{4.8}$$

and I_n, K_n are the modified Bessel functions of the first and second kind respectively.

The methods of Sec. 3 can now be employed to obtain the solutions of the boundary-value problems once the tube and boundary data are given.

References

[1] J. C. Murray, *Curvilinear coordinate systems associated with a class of boundary-value problems*, J. Inst. Math. Applics. **25**, 397–411 (1980)

[2] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, McGraw-Hill, 1953

Appendix. If **N** denotes the principal normal and **B** the binormal to L, the Serret-Frenet equations have the form

$$\dot{\mathbf{t}}_1 = \kappa\mathbf{N}, \qquad \dot{\mathbf{B}} = -\kappa\mathbf{N}, \qquad \dot{\mathbf{N}} = -\kappa\mathbf{t}_1 + \tau\mathbf{B}, \tag{A.1–A.3}$$

where κ and τ denote the curvature and torsion, respectively, of L. Moreover, from [1], we have

$$\dot{\mathbf{t}}_1 = \dot{\phi}\sin\theta\mathbf{t}_2 + \dot{\theta}\mathbf{t}_3, \tag{A.4}$$

$$\dot{\mathbf{t}}_2 = -\dot{\phi}\cos\theta\mathbf{t}_3 - \dot{\phi}\sin\theta\mathbf{t}_1, \tag{A.5}$$

$$\dot{\mathbf{t}}_3 = -\dot{\theta}\mathbf{t}_1 + \dot{\phi}\cos\theta\mathbf{t}_2, \tag{A.6}$$

so that

$$\kappa\mathbf{N} = \dot{\phi}\sin\theta\mathbf{t}_2 + \dot{\theta}\mathbf{t}_3, \tag{A.7}$$

and, since $\mathbf{B} = \mathbf{t}_1 \times \mathbf{N}$,

$$\kappa\mathbf{B} = -\dot{\theta}\mathbf{t}_2 + \dot{\phi}\sin\theta\mathbf{t}_3. \tag{A.8}$$

Therefore $\kappa^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2$ and the angle γ can be defined by

$$\dot{\theta} = \kappa\cos\gamma, \qquad \dot{\phi}\sin\theta = \kappa\sin\gamma. \tag{A.9}$$

After differentiating (A.8) we find, using (A.2), (A.5)–(A.8), and (A.9),

$$\tau = -\dot{\gamma} - \dot{\phi}\cos\theta. \tag{A.10}$$

The scaling factor A_1 may be written [1]

$$A_1 = 1 - \kappa q \cos \lambda_1, \tag{A.11}$$

where $\lambda_1 \equiv \psi - \psi_0 + \mu - \gamma$. Therefore, using (1.4) and (A.10), we obtain

$$\partial A_1/\partial \xi^1 = -\dot{\kappa} q \cos \lambda_1 + \tau \kappa q \sin \lambda_1. \tag{A.12}$$

The scaling factor A is given by [1]

$$A^2 = \left(\frac{\partial u_0}{\partial \xi^2}\right)^2 + \left(\frac{\partial v_0}{\partial \xi^2}\right)^2 = \left(\frac{\partial q}{\partial \xi^2}\right)^2 + \left(\frac{\partial q}{\partial \xi^3}\right)^2, \tag{A.13}$$

so that the angle ν can be defined by

$$\partial v_0/\partial \xi^2 = A \sin \nu, \qquad \partial u_0/\partial \xi^2 = A \cos \nu. \tag{A.14}$$

With the aid of the equations $q \sin \mu = v_0$ and $q \cos \mu = u_0$ together with the Cauchy-Riemann equations for u_0 and v_0 we can show that

$$q(\partial \lambda_1/\partial \xi^2) = q(\partial \mu/\partial \xi^2) = A \sin(\nu - \mu), \tag{A.15}$$

$$q(\partial \lambda_1/\partial \xi^3) = q(\partial \mu/\partial \xi^3) = -A \cos(\nu - \mu), \tag{A.16}$$

$$\partial q/\partial \xi^2 = A \cos(\nu - \mu), \tag{A.17}$$

$$\partial q/\partial \xi^3 = A \sin(\nu - \mu). \tag{A.18}$$

Finally, Eqs. (A.15)–(A.18) enable us to prove that

$$\partial A_1/\partial \xi^2 = -\kappa A \cos \lambda_2 \tag{A.19}$$

and

$$\partial A_1/\partial \xi^3 = -\kappa A \sin \lambda_2, \tag{A.20}$$

where $\lambda_2 \equiv \psi - \psi_0 + \nu - \gamma$.