PRECESSIONAL ADVANCES OF A LOCK-FOWLER MISSILE*

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Abstract. It is proved that for apses-to-apses motions which are purely direct (retrograde), a Lock-Fowler missile satisfies the Halphen inequality. The result is independent of all launching conditions.

1. Introduction. In a previous paper, subsequently referred to as (A), Rath and Namboodiri [1] obtained conditions under which a Lock-Fowler missile would have the same apsidal limits as its simulator (the dynamically equivalent Lagrange gyro). In the present paper we have considered the precessional advances of a Lock-Fowler missile, as far as possible without putting restrictions on its launching conditions (as was done in (A)).

Our main observation in this paper (Sec. 3) is that the missile has the same lower apsidal limit as its simulator for all types of direct and retrograde motions of the missile, independent of all launching conditions. Whenever restrictions on launching conditions of the missile have been imposed to show that it might have similar apsidal limits as its simulator, they are different from those considered in (A) and are much less stringent (Theorems 3 and 4). So far as the asymptotic motions of the missile are concerned, they are qualitatively the same as its simulator for purely direct (retrograde) motions and there is an obvious qualitative difference in case of the combined (grapevine) motion (Sec. 5). Symbols and notation of this paper are the same as those given in (A).

2. Basic equations and the apsidal angle. The nutational and precessional motion of a Lock-Fowler missile may be characterized by the following equations:

$$(dz/dt)^2 = 4\beta H(z), \tag{2.1}$$

$$d\phi/dt = \Omega(\lambda - z)/2z(1 - z), \tag{2.2}$$

where

$$z = (1/2)(1 + \cos\delta), \tag{2.3}$$

$$\lambda = (F + \Omega)/2\Omega, \tag{2.4}$$

$$\Omega = AN/B, \tag{2.5}$$

$$4\beta H(z) = z(1 - z)[E_1 - \alpha(2z - 1) - \beta(2z - 1)^2] - \Omega^2(\lambda - z)^2$$

$$= 4\beta\left[\prod_{i=1}^{4}(z - z_i)\right], \qquad E_1 = E + \alpha + \beta, \tag{2.6}$$

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ϕ and δ are the angles of precession and nutation respectively, E and F are constants of energy and angular momentum (see (A), (2.1) and (2.2)), A and B are the axial and transverse moments of inertia and N is the axial spin of the missile. The parameters α, β and s are given by

$$\alpha = (\Omega^2/2s)(1 - 4qs), \qquad \beta = q\Omega^2,$$
$$s = A^2N^2/4\beta\mu(0), \tag{2.7}$$

where $\mu(0)$ is a certain aerodynamic parameter with reference to linear angular motion of the missile.

It may be noted that for real motions z_i ($i = 1, \ldots, 4$), the zeros of the polynomial $H(z)$ are real and satisfy the inequalities

$$z_1 \leq 0 \leq z_2 < z < z_3 \leq 1 \leq z_4. \tag{2.8}$$

By defining certain symmetric functions of the roots as

$$L = [-(z_1 z_2 z_3 z_4)]^{1/2}, \tag{2.9}$$
$$L' = [(1 - z_1)(1 - z_2)(1 - z_3)(z_4 - 1)]^{1/2}, \tag{2.10}$$

it can be shown that

$$|\lambda| L' = L|\lambda - 1|. \tag{2.11}$$

(See (A), (2.22).) From (2.1) and (2.2) we have

$$\frac{d\phi}{dz} = \frac{\Omega(\lambda - z)}{2z(1 - z)[H(z)]^{1/2}} = g(z), \tag{2.12}$$

say, which shows that the precessional advance during the passage of the body axis of the missile from z_2 to z_3 is given by

$$\Phi = \int_{z_2}^{z_3} g(z)\, dz. \tag{2.13}$$

The factor $(\lambda - z)/z(1 - z)$ in $g(z)$ may be decomposed into partial fractions and using (2.11) we have for $\Omega > 0$

$$\Phi = \operatorname{sign}(\lambda)\Phi_1 + \operatorname{sign}(\lambda - 1)\Phi_2, \tag{2.14}$$

where

$$\Phi_i = \int_{z_2}^{z_3} g_i(z)\, dz \ (i = 1, 2), \tag{2.15}$$
$$g_1(z) = L/2z[H(z)]^{1/2}, \tag{2.16}$$
$$g_2(z) = L'/2(1 - z)[H(z)]^{1/2}. \tag{2.17}$$

3. Precessional advances of the missile in the case of purely direct and retrograde motion. It is interesting to note that for purely direct (retrograde) motions the missile has the same apsidal lower limits as that of an equivalent common top. This result has been established in (A) by imposing certain conditions on the roots of the fundamental quartic $H(z)$. From the present analysis one can see that such restrictions are not necessary.

To establish the result we state

PROPOSITION I.

$$\Phi_1 = \pi/2 + P_1, \tag{3.1}$$

$$\Phi_2 = -\pi/2 + P_2, \tag{3.2}$$

where P_1 $(i = 1, 2)$ are two positive integrals.

To prove the proposition, we shall apply the method of complex integration. Let us regard Rz as a complex variable; this makes $g_i(z)$ $(i = 1, 2)$ and $[H(z)]^{1/2}$ double-valued functions of z, but they are uniquely defined on the Riemann sheets bounded by the cuts c_j $(j = 1, 2, 3)$ and the circle c_0 with its center at the origin (see Fig. 1). If we write

$$z - z_k = r_k \exp(i\theta_k),\ 0 \le \theta_k < 2\pi \qquad (k = 1, \ldots, 4), \tag{3.3}$$

the signs of $[H(z)]^{1/2}$ on the cuts have been fixed and are shown in Fig. 1. Now integrating $g_i(z)$ around the contours c_0—c_1—c_2—c_3 and applying Cauchy's residue theorem we have

$$\int_{c0} g_i(z)\, dz - \int_{c1} - \int_{c2} - \int_{c3} = 2\pi i R, \tag{3.4}$$

where R represents the sum of residues of the integrand concerned.

As usual, $\int_{c0} g_i(z)\, dz = 0$ when the radius of the circle c_0 becomes infinitely large and the contributions due to the remaining integrals are

$$\int_{c1} = \int_{-\infty}^{z1} g_1(z)\, dx, \tag{3.5}$$

$$\int_{c2} = \int_{z2}^{z3} g_1(z)\, dz = -2\Phi_1, \tag{3.6}$$

$$\int_{c3} = \int_{z4}^{\infty} g_1(z)\, dz. \tag{3.7}$$

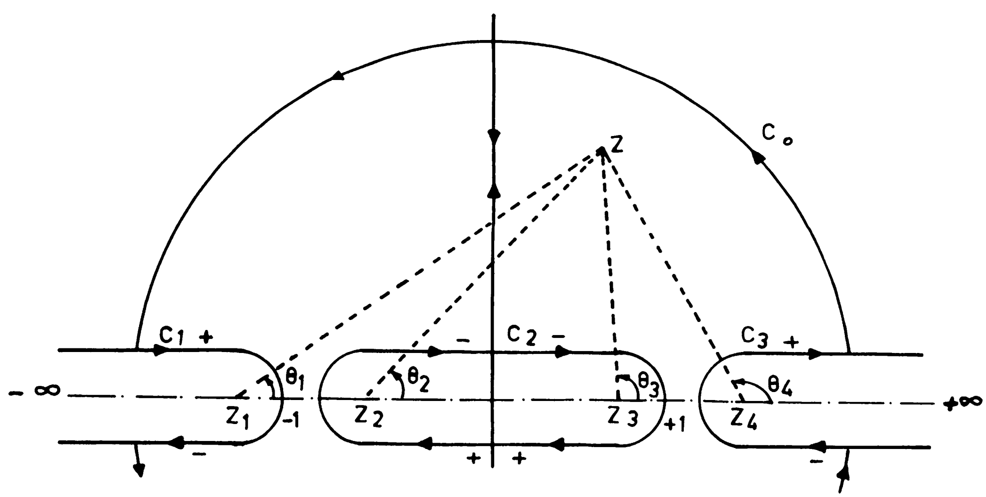

FIG. 1. Riemann sheets for $g(z)$ and $g_i(z)$.

Similarly, for $g_2(z)$ we have

$$\int_{c_1} = \int_{-\infty}^{z_1} g_2(z)\,dz, \tag{3.8}$$

$$\int_{c_2} = \int_{z_2}^{z_3} g_2(z)\,dz = -2\Phi_2, \tag{3.9}$$

$$\int_{c_3} = -\int_{z_4}^{\infty} g_2(z)\,dz. \tag{3.10}$$

The residues of $g_1(z)$ at $z = 0$ and $g_2(z)$ at $z = 1$ are $1/2i$ and $-1/2i$ respectively. Now, using integrals (3.5)—(3.10) in (3.4), we have (3.1) and (3.2) where

$$P_1 = \frac{1}{2}\int_{z_4}^{\infty} g_1(z)\,dz - \frac{1}{2}\int_{-\infty}^{z_1} g_1(z)\,dz > 0, \tag{3.11}$$

and

$$P_2 = \frac{1}{2}\int_{-\infty}^{z_1} g_2(z)\,dz - \frac{1}{2}\int_{z_4}^{\infty} g_2(z)\,dz > 0. \tag{3.12}$$

Thus, Proposition I is proved.

Based on the above results we shall prove the following:

THEOREM 1.

$$\Phi < -\pi/2 \qquad \text{when } \lambda \le 0, \tag{3.13}$$

$$\Phi > \pi/2 \qquad \text{when } \lambda \ge 1. \tag{3.14}$$

To prove the above theorem, we shall first consider $\lambda < 0$. Now from (2.14) we have

$$\begin{aligned}\Phi &= -(\Phi_1 + \Phi_2)\\ &= -\pi/2 - P_1 - \Phi_2 \qquad \text{(due to (3.1))}\\ &< -\pi/2 \qquad (\text{since } \Phi_2, P_1 > 0).\end{aligned} \tag{3.15}$$

However, when $\lambda = 0$, the precessional advance defined by the integral $\int_{z_2}^{z_3} g(z)\,dz$ degenerates to

$$\Phi = -\int_{z_2}^{z_3} \frac{\Omega\,dz}{2(1-z)[H(z)]^{1/2}}. \tag{3.16}$$

Evaluating the integral (3.16) around the contour (given in Fig. 1) we get

$$\Phi = -\pi/2 - \frac{1}{2}\int_{z_4}^{\infty} + \frac{1}{2}\int_{-\infty}^{z_1} \frac{\Omega\,dz}{(1-z)[H(z)]^{1/2}}, \tag{3.17}$$

where we note that $\text{Res}(1) = (\Omega/2)[H(1)]^{1/2} = 1/2i$.

As the last two terms in the r.h.s. of (3.17) are positive, it now follows that

$$\Phi < -\pi/2 \quad \text{when} \quad \lambda = 0. \tag{3.18}$$

Thus (3.15) and (3.18) together establish the first part of Theorem 1, i.e. (3.13).

Coming to the second part, we note that when $\lambda = 1$

$$\begin{aligned}\Phi &= \Phi_1 \\ &= \pi/2 + P_1, \qquad \text{(Proposition 1)} \\ &> \pi/2 \qquad (\text{since } P_1 > 0),\end{aligned}$$

and when $\lambda > 1$

$$\begin{aligned}\Phi &= \Phi_1 + \Phi_2 \qquad \text{(due to (2.14))} \\ &= \pi/2 + P_1 + \Phi_2 \\ &> \pi/2 \qquad (\text{as } P_1, \Phi_2 > 0).\end{aligned}$$

Hence Theorem 1 is established.

In case of purely direct and retrograde motions the following two theorems may also be proved.

THEOREM 2. $\Phi > \pi$ whenever $1 < \lambda \leq z_4$.

THEOREM 3. $\Phi < -\pi$ when $z_1 \leq \lambda < 0$.

The result given in Theorem 2 is true in the case of the equivalent common top [2], whereas Theorem 3 is peculiar to the missile itself. We shall proceed to prove Theorem 3.

Integrating $g(z)$ over the contour (Fig. 1), we have straightaway

$$-2\Phi + \int_{-\infty}^{z_1} g(z)\, dz - \int_{z_4}^{\infty} g(z)\, dz = 2\pi i R, \tag{3.19}$$

where

$$R = R(1) + R(0), \qquad R(1) = -\Omega(\lambda - 1)/2[H(1)]^{1/2}. \tag{3.20}$$

But $[H(1)]^{1/2} = \pm i\Omega|\lambda - 1|$ (see (2.6)). Since $\arg[H(1)]^{1/2} = \pi/2$ in the present case we choose the positive square root. Since $\lambda - 1 < 0$ by stipulation, we have

$$R(1) = -\Omega(\lambda - 1)/2i|\lambda - 1| = 1/2i. \tag{3.21}$$

Similarly,

$$R(0) = \frac{\Omega\lambda}{-2i\Omega|\lambda|} = 1/2i, \tag{3.22}$$

since $\arg[H(0)]^{1/2} = 3\pi/2$. Hence $R = 1/i$ and finally from (3.19) we have

$$\Phi = -\pi - Q, \tag{3.23}$$

where

$$Q = -\int_{-\infty}^{z_1} + \int_{z_4}^{\infty} g(z)\, dz > 0. \tag{3.24}$$

for $z_1 \leq \lambda < 0$. Hence $\Phi < -\pi$, which proves Theorem 3. Theorem 2 can be proved similarly.

4. Precessional advances in the case of grapevine motion. In the case of grapevine motion there exists qualitative differences in the two systems [3]. In particular, when $0 < \lambda < 1$, Φ has been observed to be negative [see (A), Sec. 8], and hence no precise lower bound for Φ can be claimed. However, the following result appears to be interesting.

To be precise, we shall prove the following:

THEOREM 4.

$$\Phi > \pi \quad \text{whenever } \tfrac{1}{2} < \lambda \le 1, \qquad z_4 \ge \lambda/(2\lambda - 1).$$

To prove the theorem we have from (2.14)

$$\begin{aligned}\Phi &= \Phi_1 - \Phi_2 \\ &= \pi + P_1 - P_2 \qquad \text{(due to Proposition 1).}\end{aligned} \tag{4.1}$$

Now an elementary transformation shows that

$$\begin{aligned}P_1 - P_2 = (L/2)&\int_{z_4}^{\infty} [(1 - R)z - 1][z(z - 1)\{H(z)\}^{1/2}]^{-1}\, dz \\ &+ (L/2)\int_{|z_1|}^{\infty} [(1 - R)z + 1][z(z + 1)\{H(-z)\}^{1/2}]^{-1}\, dz,\end{aligned}$$

where $R = L'/L = |(1 - \lambda)/\lambda|$ (due to (2.11)).

A sufficient condition for $P_1 - P_2 > 0$ is now

$$0 \le 1 - |(1 - \lambda)/\lambda| \le z_4^{-1},$$

which is same as

$$z_4 \ge \lambda/(2\lambda -), \quad \text{for} \quad 1/2 < \lambda \le 1. \tag{4.2}$$

Hence Theorem 4 follows from (4.1).

It may be observed that $\lambda/(2\lambda - 1)$ has the minimum value 1 and tends to ∞ as $\lambda \to 1/2$. Thus as λ approaches 1/2, z_4 must be indefinitely large, whereas when $\lambda \to 1$, z_4 may possibly tend to 1, which is its minimum admissible value. In such a case there is a possibility of asymptotic motion where $\Phi \to \infty$. One may expect that when $1/2 < \lambda \le 1$, $\Phi(\lambda) \to \infty$ as $\lambda \to 1 - 0$. But, as we shall see in the next section, this assertion cannot be established.

5. Asymptotic motions: continuity and existence of conjugate motion. When we say $\Phi(\lambda) = \Phi(z_2) - \Phi(z_3) = \int_{z_2}^{z_3} g(z)\, dz$ (advances in precession), Φ is not defined as such when $\lambda = 0$ or 1, except by the integral $\int_{z_2}^{z_3} g(z)\, dz$, which becomes divergent at these points where the motion is asymptotic. For continuity of asymptotic motions at these points we must have

$$\begin{aligned}&\Phi^+(1) = \Phi^-(1) = \Phi(1) = \infty, \\ &\Phi^+(0) = \Phi^-(0) = \Phi(0) = -\infty,\end{aligned} \tag{5.1}$$

where

$$\Phi^+(1) = \underset{\lambda = 1 + 0}{Lt} \Phi(\lambda), \qquad \Phi^-(1) = \underset{\lambda = 1 - 0}{Lt} \phi(\lambda),$$

and $\Phi^+(0)$, $\Phi^-(0)$ are similarly defined. To test the continuity of asymptotic motion across $\lambda = 1$ (for definiteness), we give a positive variation to λ so that

$$\Phi(\lambda) = \Phi(1 + \overline{\lambda - 1}) = \Phi(1) + (\lambda - 1)\Phi'(1) + \cdots, \tag{5.2}$$

where $\Phi'(1) = \int (dz/2z[H(z)]^{1/2}) > 0$ and $\lambda - 1 > 0$. This implies that $\Phi^+(1) > \Phi(1)$ and therefore $\Phi^+(1) = \infty$. Similarly, it can be shown that $\Phi^-(0) < \Phi(0)$ and therefore $\Phi^-(0) = -\infty$.

The left-hand continuity across $\lambda = 1$ and similarly the right-hand continuity across $\lambda = 0$ for asymptotic motions cannot be as easily established; as we see from (5.2), $\Phi^-(1)$ and similarly $\Phi^-(0)$ assume the indeterminate form $\infty - \infty$. However, if the system admits of conjugate motion (across $\lambda = 1$) as in case of the common top, continuity may be established, [2]. But the present system does not admit of any such motion. This may be seen from the following.

Proposition 2. If the three parameters F, Ω and E represent the motion of the missile with the fundamental quartic $H(z)$, then the parameters $F_1 = \Omega$, $\Omega_1 = F$, $E_{1c} = E + (\Omega^2 - F^2)$ determine the conjugate motion (across $\lambda = 1$) if and only if $q = 0$.

To prove the result, choose general initial conditions $z = z_0$, $z' = z'_0$, which determine F, Ω and E (dependent on F and Ω) and $H(z_0)$. Thus we have from (2.1) and (2.6)

$$E = \frac{z_0'^2 + \Omega^2(\lambda - z_0)^2}{z_0(1 - z_0)} + 2(z_0 - 1)[\alpha + 2\beta z_0]. \tag{5.3}$$

Now we choose another set of constants $F_1 = \Omega, \Omega_1 = F$ (depending on the same initial conditions) and claim that there exists an E_{1c} (depending on F and Ω) such that F_1, Ω_1 and E_{1c} generate the same quartic $H(z_0)$. From this, we must have

$$E_{1c} = \frac{z_0'^2 + \Omega_1(\lambda_1 - z_0)^2}{z_0(1 - z_0)} + 2(z_0 - 1)[\alpha_1 + 2\beta_1 z_0], \tag{5.4}$$

where

$$\lambda_1 = (F + \Omega)/(2F), \qquad \alpha_1 = \Omega/2s(1 - 4qsF^2/\Omega^2),$$
$$\beta_1 = qF^2, \qquad s_1 = s(F^2/\Omega^2). \tag{5.5}$$

It is assumed that

$$\mu(0) = B\Omega^2/4s = BF^2/4s_1. \tag{5.6}$$

Now subtracting (5.3) from (5.4) and using (5.5), we have after some simplification

$$E_{1c} = E + (F^2 - \Omega^2)[4q(z_0 - 1)^2 - 1]. \tag{5.7}$$

Thus the proof of Proposition 2 is obvious from (5.6).

In view of the above analysis and the example given in Sec. 8 of (A) which claims that Φ need not be positive when $0 < \lambda \leq 1$, it is futile to look for left-hand continuity of asymptotic motion across $\lambda = 1$ for the present system.

References

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