ON A GEOPHYSICAL INVISCID VORTEX*

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Abstract. We investigate the special case of an inviscid axisymmetric vortex rotating steadily in an atmosphere having an adiabatic lapse rate. We show that the analytic solution of the governing equations depends on the source strength of the vortex. The technique used is based largely on an unpublished report of Lilly [1].

Introduction. In this note, we consider an inviscid vortex rotating in a stratified environment. We develop the equations of linear momentum in combination with the conservation of energy, where the centerline vorticity and the temperature are assumed functions of altitude. The experimental results of Granger [2] were used for the centerline vorticity distribution. The type of geophysical inviscid vortex treated in this note is dictated by the type of distribution one uses to describe the potential temperature, and this is largely governed by the range in altitude one wishes to consider. For certain weak strength dust devils, one might be interested in elevations less than 10 km such that the lapse rate is greater than adiabatic. For tornados, the altitude might extend to 20 km such that the lapse rate is less than adiabatic. We shall treat the neutral stability case where the lapse rate is adiabatic. Furthermore, we shall follow closely the theoretical development used by Lilly [1] and use his relationship of the normalized temperature with elevation.

Development of the governing equation. Commencing with Lamb's equation for steady axisymmetric vortex flows, where the radial and axial velocity components of fluid motion are expressed in terms of the stream function $\psi(r, z)$, and the tangential velocity component is expressed in terms of the circulation $\Gamma(r, z)$,

$$\frac{\Gamma}{2\pi r}\left[\frac{\Gamma}{2\pi r^2}+\frac{1}{2\pi}\frac{\partial(\Gamma/r)}{\partial r}\right]+\frac{1}{r}\frac{\partial\psi}{\partial r}\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]= \frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left(\frac{V^2}{2}+gz\right) \quad (1)$$

$$\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{\Gamma}{2\pi r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial z}\left[\frac{\Gamma}{2\pi r^2}+\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\right)\right]=0 \quad (2)$$

$$\frac{1}{r}\frac{\partial\psi}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]+\frac{\Gamma}{2\pi r}\frac{\partial}{\partial z}\left(\frac{\Gamma}{2\pi r}\right)=\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left(\frac{V^2}{2}+gz\right) \quad (3)$$

Examining the second equation of linear momentum, one notes that this is nothing more than a statement of the conservation of circulation, such that $\Gamma=\Gamma(\psi)$. Considerable simplification results if the azimuthal vorticity component ζ_θ is incorporated into Eqs. (1)

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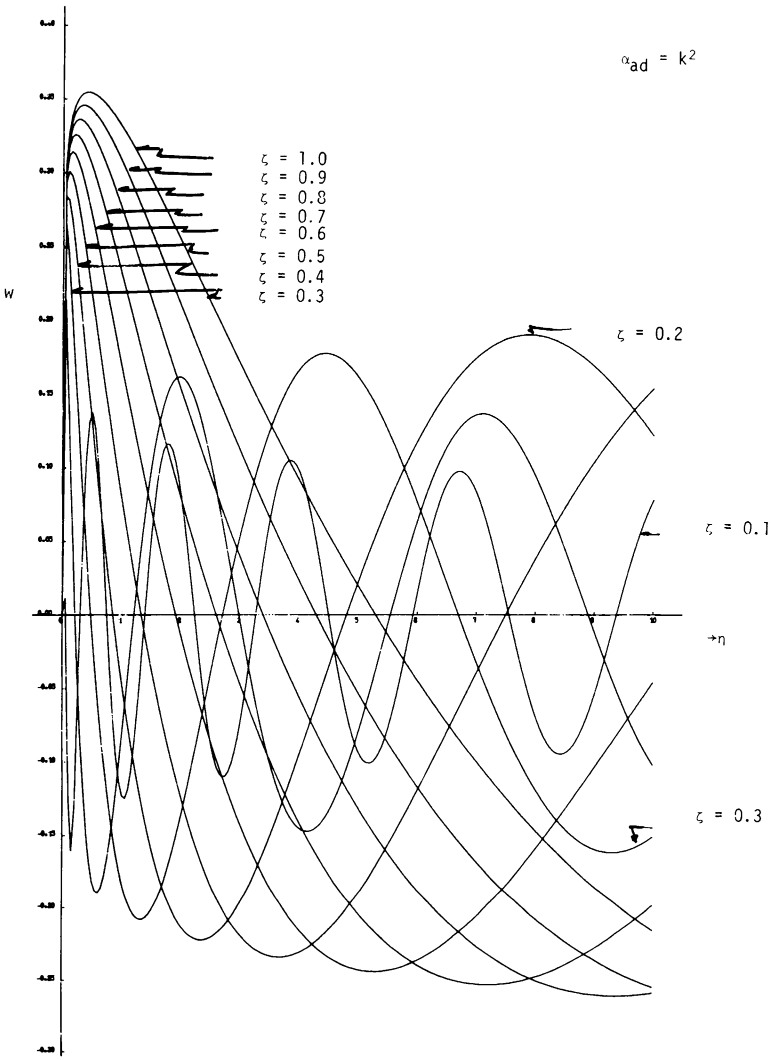

Figure 1. Dust Devil's Axial Velocity w (ζ.η) versus η for Various Axial Locations, ζ, for $\alpha_{ad} = k^2$

and (3). Thus

$$\frac{1}{4\pi^2 r^2}\frac{\partial(\Gamma^2/2)}{\partial\psi}\cdot\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial r}\zeta_\theta=\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left(\frac{V^2}{2}+gz\right) \tag{4}$$

comes from Eq. (1) and

$$\frac{1}{r}\frac{\partial\psi}{\partial z}\zeta_\theta+\frac{1}{4\pi^2 r^2}\frac{\partial(\Gamma^2/2)}{\partial\psi}\cdot\frac{\partial\psi}{\partial z}=\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left(\frac{V^2}{2}+gz\right) \tag{5}$$

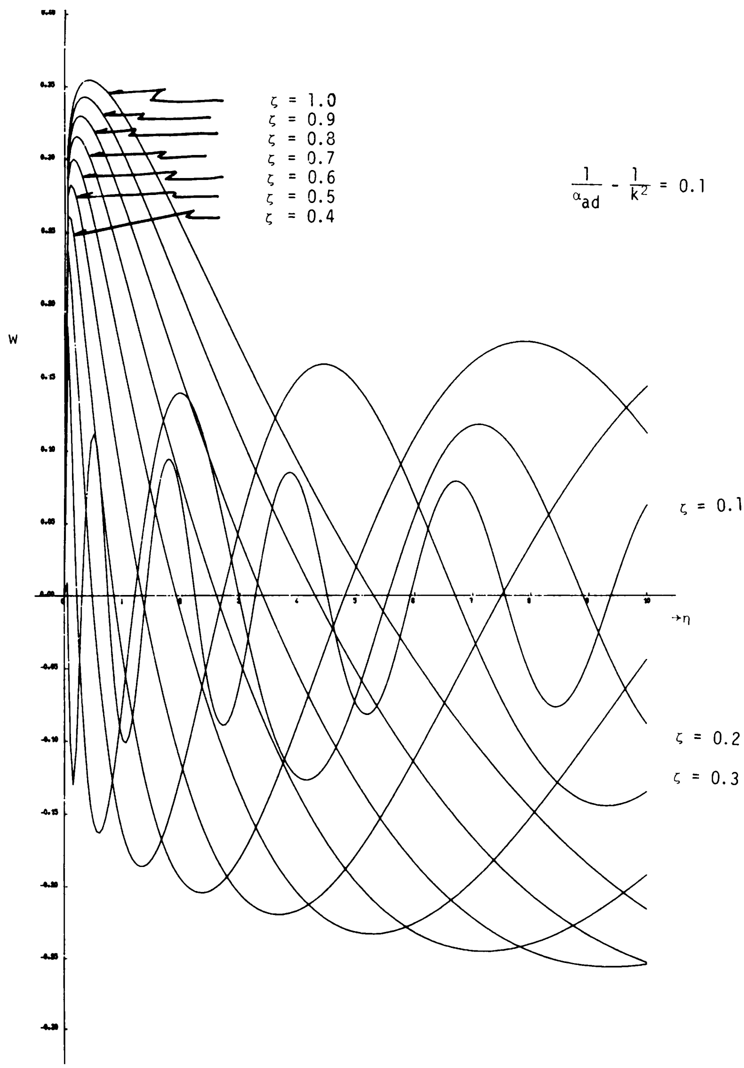

Figure 2. Dust Devil's Axial Velocity w (ζ,η) Versus η for Various Axial Locations, ζ.

comes from Eq. (3). Eqs. (4) and (5) are combined to yield

$$\frac{1}{4\pi^2 r^2}\, d\left(\frac{\Gamma^2}{2}\right) + \frac{\zeta_\theta}{r}\, d\psi = \frac{dp}{\rho} + d\left(\frac{V^2}{2} + gz\right), \tag{6}$$

which was originally obtained by Lilly [1].

Defining $\Phi(r, z)$ as the potential atmospheric temperature, the conservation of energy can be expressed as

$$\frac{\partial\psi}{\partial z}\frac{\partial\Phi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\Phi}{\partial z} = 0, \tag{7}$$

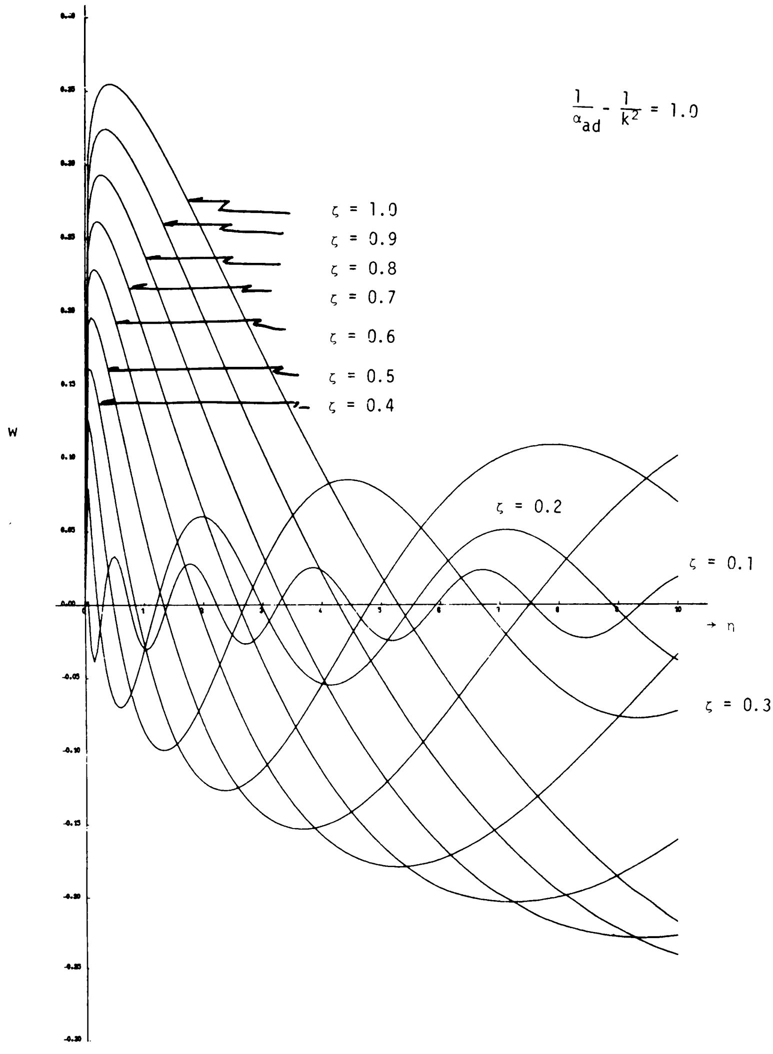

Figure 3. Dust Devil's Axial Velocity w (ζ,η) Versus η for Various Axial Locations, ζ

which states the potential temperature, like the circulation, is exclusively dependent upon the stream function.

Since the adiabatic process is limiting in regards to vertical stability, gas particles that move vertically at constant entropy experience a change of temperature according to the familiar rate expression for an ideal gas

$$d(\ln \Phi) = C_p\, d(\ln T) - R\, d(\ln p) \tag{8}$$

Substituting Eq. (8) into Eq. (6) results in

$$\frac{1}{4\pi^2 r^2}\frac{d(\Gamma^2/2)}{d\psi} + \frac{\zeta_\theta}{r} + \frac{C_p T}{\Phi}\frac{d\Phi}{d\psi} = \frac{d}{d\psi}\left(\frac{V^2}{2} + gz + C_p T\right) \tag{9}$$

Before applying the thermodynamic relationship given by Eq. (8) to Lilly's momentum equation (6), we first should obtain a reasonable approximation for the temperature variation with elevation. The atmosphere is conditionally unstable above a relatively shallow moist layer with a weak inversion separating the moisture from the region of instability. A particular application would be a tornado that grows with a mesoscale squall line. For this example, the instability level increases, inversion is eliminated, and the lapse rate $\alpha > \alpha_{ad.}$. Thus,

$$T\, d(\ln \Phi) = -|\alpha_{ad.} - \alpha|\, dz. \tag{10}$$

For an isentropic atmosphere,

$$\frac{T}{\Phi} = \frac{-gz}{C_p \Phi_0}, \tag{11}$$

such that upon integrating Eq. (10)

$$\Phi(\psi) = \Phi_{\max} \frac{z_\infty(\psi)}{H}, \tag{12}$$

where $\Phi_{\max}$ is the atmospheric potential temperature at the altitude z equal H, and z_∞ is the altitude of the streamline ψ at an infinite radial extent. The analysis is adapted to mid-latitude atmospheres for spring through fall where $\alpha_{ad.} - \alpha = 3°$ K/km.

Attention is next focused on the particular class of geophysical vortices experimentally studied by Granger [2] having axial variation of centerline vorticity governed by

$$\psi(r, z) = -\frac{kz\Gamma(r, z)}{2\pi}, \tag{13}$$

where the constant k is based on the source strength of the vortex and is a parameter to be varied in the analysis.

Substituting the assumptions of temperature and vorticity into the momentum equation (9), one obtains the governing differential equation of a class of inviscid vortex motions.

$$\frac{(1-F)F^{2/3}}{k^2\eta\xi^2} - F^{2/3}\left[\frac{4H^2}{r_0^2}\frac{\partial^2 F}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial^2 F}{\partial \xi^2}\right] - B(F - \xi) = 0 \tag{14}$$

where

$$F = 1 - \psi/\psi_\infty, \tag{15}$$

$$\eta = (r/r_0)^2, \tag{16}$$

$$\xi = z/H, \tag{17}$$

$$B = -\frac{g\Phi_{\max} H^3 r_0^2}{3\Phi_0 \psi_\infty^2}, \tag{18}$$

and ψ_∞ is the stream function at z_∞ equals zero such that at z_∞ equals H, the stream function is zero. We seek solutions of Eq. (14) subject to boundary conditions.

At the vortex centerline r equal zero, the radial and circumferential velocity components

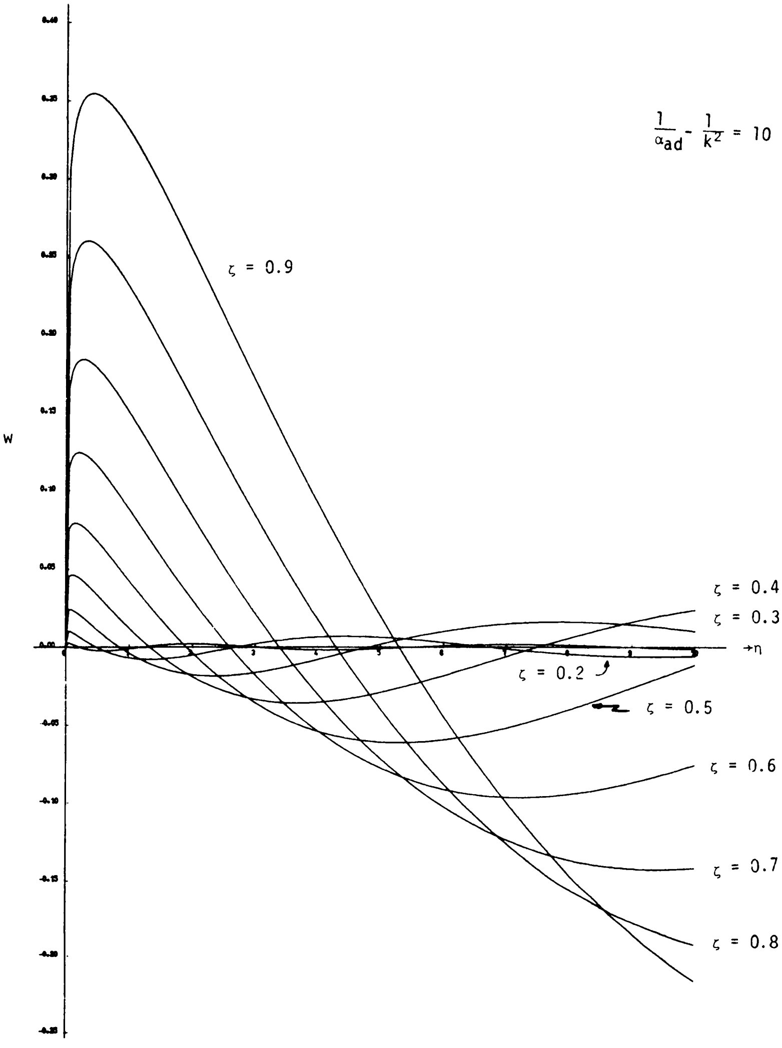

Figure 4. Dust Devil's Axial Velocity w (ζ,η) Versus η for Various Axial Locations, ζ.

vanish and the axial velocity w is a maximum, resulting in

$$\frac{\partial F(0, \xi)}{\partial \xi} = 0, \tag{19}$$

$$F(0, \xi) = 1.0, \tag{20}$$

$$\frac{\partial F(0, \xi)}{\partial \eta} = 0. \tag{21}$$

At an infinite radial extent, all three velocity components vanish, such that

$$\frac{\partial F(\infty, \xi)}{\partial \xi} = \frac{\partial F(\infty, \xi)}{\partial \eta} = F(\infty, \xi) = 0. \tag{22}$$

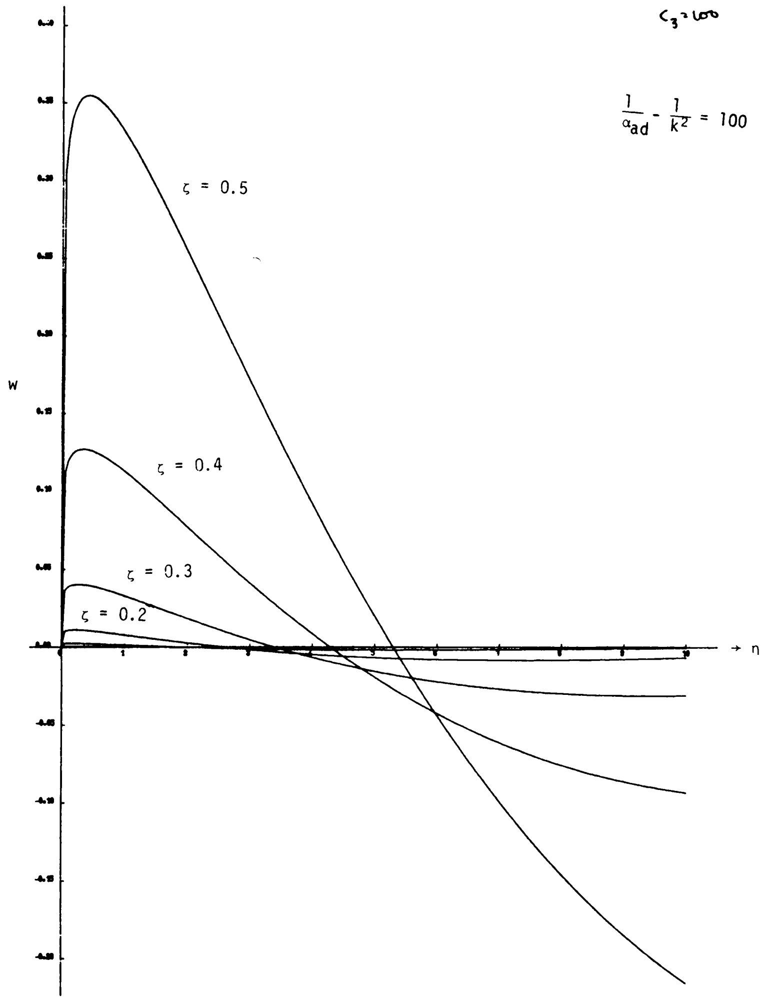

Figure 5. Dust Devil's Axial Velocity w (ζ,η) Versus η for Various Axial Locations, ζ.

Along the ground the axial velocity must be zero so that

$$\frac{\partial F(\eta, 0)}{\partial \eta} = 0. \tag{23}$$

There are several methods available to solve Eq. (14). The example below shows one such technique.

Example. Consider the case of an inviscid vortex rotating steadily in a neutrally stable atmosphere. The similarity transform

$$\psi(\xi, \eta) = A\eta^a \xi^b g(\varepsilon), \tag{24}$$

is substituted into Eqs. (14) and (15), where the similarity variable ε is defined as

$$\varepsilon = \eta^c \xi^d \tag{25}$$

with the result

$$4[a(a-1)\eta^{-1}\xi^2 g + (a^2 + 2ac - a)\,\eta^{-1}\xi^2\varepsilon g' + c^2\eta^{-1}\xi^2\varepsilon^2 g'']$$
$$+ \alpha_{ad}[b(b-1)g + (d^2 + 2bd - d)\varepsilon g' + d^2\varepsilon^2 g''] + \left(\frac{\alpha_{ad}}{c}\right)^2 g = 0. \quad (26)$$

Values of a, b, c, d were sought which would collapse Eq. (26) to ordinary form. Values

$$a = 0.618034 \quad (27)$$

$$b = 0.5 + [0.25 + (1/\alpha_{ad} - 1/k^2)]^{1/2} \quad (28)$$

$$c = 1.0 \quad (29)$$

$$d = -2.0 \quad (30)$$

were obtained using the boundary conditions given by Eqs. (19)–(23). The analytic solution of Eq. (26) resulted in

$$\frac{\psi}{\psi_\infty}(r, z) = (r/r_0)^{1.236068}(z/H)^b J_1\left(\frac{rH}{r_0 z}\right) \quad (31)$$

for the dimensionless stream function.

The above solution is similar in form to the result obtained by M. G. Hall [3] for a class of geophysical vortex flows that might be found in hurricanes, leading-edge vortices and bath-tub vortices. Using the stream function of Eq. (31), the dimensionless axial velocity w was evaluated. The results are presented in Figs. 1–5 for a range of values of atmospheric and swirl conditions, $(1/\alpha_{ad.} - 1/k^2)$. The figures show that there can be significantly large axial velocities in the vortex for $(1/\alpha_{ad.} - 1/k^2) > 10$ and at the higher elevations. This is not surprising, as many investigators (e.g. Granger [2]) have shown this to be the case for both viscous and inviscid vortex flows. Though the present vortex is inviscid, the slopes of the curves in Figs. 1–5 indicate stretching of vorticity. Part is due to a thermal buoyant jet, but the main reason is due to the imposition of the source (or sink) by the condition given by Eq. (13). One further notes in the figures a continuous shortening of vortex tubes parallel to the axis. A physical explanation might be that air moves into the jet, stretching the tubes as they are being drawn across the outer region of the jet. It could possibly be that this convection amplification of vorticity is the key in both the formation and persistence of certain geophysical vortices, among which might be the tornado and the dust devil.

References

[1] D. K. Lilly, *Tornado dynamics*, NCAR Manuscript 69–117 (1969)
[2] R. A. Granger, *A steady axisymmetric vortex flow*, Geophysical Fluid Dynamics 3, 45–88, (1972)
[3] M. G. Hall, Progress in aeronautical sciences 7 1966)