SOLUTION OF STRUCTURAL OPTIMIZATION PROBLEMS BY PIECEWISE LINEARIZATION*

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Summary. Structural optimization problems tend to be nonlinear and often also non-convex. In this paper it is proposed to reduce a general class of such problems to linear programming problems through piecewise linearization. They can then be solved by the highly effective linear programming computer codes currently available.

1. Introduction. Structural optimization problems tend to be nonlinear and often also non-convex. However, as stated e.g. by Morris in [1], they can frequently be cast in a form where both the objective function and the constraints are fractions of polynomials in the design variables. Such problems can readily be piecewise linearized, and a global optimum can be found through mixed integer programming. If the problem is convex, the linearization results in a linear programming problem without integer variables.

Linear programming codes are now commercially available which can solve problems with a large number of variables within a reasonably short time. Thus piecewise linearization represents a solution technique which should be seriously considered for a large class of structural optimization problems.

2. Transformation to geometric programming. Following [1], the problem considered will be of the following form:

$$\text{minimize} \quad f(x)/g(x)$$

subject to

$$g_j(x)/l_j(x) \leq k_j, \qquad j = 1, \ldots, J,$$

$$x > 0, \quad g(x) > 0, \quad l_j(x) > 0, \qquad j = 1, \ldots, J,$$

where $f(x)$, $h(x)$, $g_j(x)$ and $l_j(x)$ are polynomials in the design variables which are components of the vector x.

This problem is equivalent to the following problem:

$$\text{minimize} \quad f(x)/y$$

subject to

$$h(x)/y \geq 1,$$

$$g_j(x)/(k_j z_j) \leq 1, \qquad j = 1, \ldots, J,$$

$$l_j(x)/z_j \geq 1, \qquad j = 1, .., J,$$

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$$x > 0, \quad y > 0, \quad z_j > 0, \qquad j = 1, \ldots, J.$$

This is a special case of the general geometric programming problem:

$$\begin{aligned}&\text{minimize} \quad y_0(x)\\&\text{subject to}\\&x = (x_1, \ldots, x_Q) > 0,\\&y_m(x) \le 1, \qquad m = 1, \ldots, M.\end{aligned}$$

Here

$$y_m(x) = \sum_{t \in P_m} c_{mt} \prod_{q=1}^{Q} x_q^{a_{mtq}} - \sum_{t \in N_m} c_{mt} \prod_{q=1}^{Q} x_q^{a_{mtq}}.$$

The coefficients c_{mt} are positive, and for each m the sets P_m and N_m of indices are disjoint.

The transformation $x_q = e^{z_q}$ gives the following form of the problem above:

$$\begin{aligned}&\text{minimize} \quad \sum_{t \in P_0} e^{w_{0t}} - \sum_{t \in N_0} e^{w_{0t}}\\&\text{subject to}\\&\sum_{t \in P_m} e^{w_{mt}} - \sum_{t \in N_m} e^{w_{mt}} \le 1,\\&w_{mt} = \sum_q a_{mtq} z_q + \ln c_{mt}.\end{aligned}$$

3. Piecewise linearization. The functions $e^{w_{mt}}$ will now be piecewise linearized using conventional techniques (see e.g. Salkin [2]).

The variable w_{mt} is supposed to vary between the limits W_{mt1} and W_{mtI}, and the piecewise linear approximation is to be exact for $w_{mt} = W_{mti}$, $i = 1, \ldots, I$, where $W_{mt1} < W_{mt2} < \cdots < W_{mtI}$.

The function $e^{w_{mt}}$ is replaced by $\sum_{i=1}^{I} \alpha_{mti} e^{W_{mti}}$ where the new variables $\alpha_{mti} \ge 0$ satisfy $\sum_{i=1}^{I} \alpha_{mti} = 1$. w_{mt} is replaced by $\sum_{i=1}^{I} \alpha_{mti} W_{mti}$. These substitutions are sufficient for all $t \in P_m$. This means that for design problems where the sets N_m are empty the substitutions lead to a linear programming problem which can readily be solved with available codes.

For $t \in N_m$ the following additional constraints are required in order to ensure valid linearizations:

$$\begin{aligned}&\alpha_{mt1} \le \delta_{mt1},\\&\alpha_{mti} \le \delta_{mt,\,i-1} + \delta_{mti}, \qquad i = 2, \ldots, I-1,\\&\alpha_{mtI} \le \delta_{mt,\,I-1}\\&\sum_{i=1}^{I-1} \delta_{mti} = 1, \qquad \delta_{mti} = 0 \quad \text{or} \quad 1.\end{aligned}$$

These additional constraints make the problem a mixed integer programming problem. Even if such problems are harder to solve than problems without integer variables, they can still in many cases be successfully solved using the highly sophisticated codes for mixed integer programming which are commercially available.

4. Final remarks. Although the piecewise linearization approach has been presented here in the context of structural optimization where its use is especially relevant, its potential area of application is much wider. In [3] Duffin and Peterson have demonstrated that any nonlinear algebraic program, i.e. a problem of the form:

$$\text{minimize} \quad y(x)$$

$$\text{subject to}$$

$$h_k(x) \geq 0,$$

$$x \geq 0,$$

where y and h_k are real-valued functions formed by addition, subtraction or exponentiation to real powers, can be transformed to a geometric programming problem, and is thus amenable to piecewise linearization and hence in many cases solvable by linear programming codes.

References

[1] A. J. Morris, *Generalization of dual structural optimization problems in terms of fractional programming*, Quart. Appl. Math. **36**, 115–119 (1978)

[2] H. M. Salkin, *Integer programming*, Addison-Wesley, 1975, pp. 5–6

[3] R. J. Duffin and E. L. Peterson *Geometric programming with signomials*, J. Optimization Theor. Appl. **11**, 3–35 (1973)