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ANALYSIS OF A FREE BOUNDARY PROBLEM IN PARTIAL LUBRICATION*

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Abstract. Hydrodynamic lubrication is concerned with a particular form of creeping flow between surfaces in relative motion where cavitation takes place. The determination of the free boundary of the cavitation area is then of fundamental importance for the computation of the characteristics of the mechanisms. Different conditions at the free boundary have been introduced. We study two of them and compare corresponding solutions with respect to film extent and pressure repartition.

Introduction. The resolution of the so-called Reynolds elliptic equation using a variational inequation modeling is well known [1, 2, 3, 4], both from mathematical and numerical aspects. Unfortunately, since the cavitation area is restricted to rest in the divergent portion of the bearing, the solution obtained in this way may be unrealistic and does not always respect the mass flow conservation law in the cavitation area, especially when the supply line is not located at the maximum gap.

Numerous models have been introduced in order to explain the various aspects of the cavitation phenomena [5, 6, 7, 8]. If tensile strength and inertial effects are neglected, [5] gives a good basis for further developments by relating, in a one-dimensional cavitated convex slider bearing, the mass flow, the supply pressure and various breakdown conditions, the supply position being located at infinity. If the supply position and the supply pressure are given, as usually in a journal bearing, the differential problem studied in [5] becomes a two-point boundary value problem where not only the breakdown position but also the beginning of the oil film are unknown. It is common practice that a regular condition on the gradient of the pressure is taken at the film rupture whereas an eventual discontinuity is allowed for this gradient at film reformation (as in the articles by Elrod [6 p. 37, Floberg [6 p. 31, 7 p. 138 and 9, 10]).

In the present paper, we study the mathematical aspects of this modeling (problem (P)) for an infinitely long journal bearing with zero supply and cavitation pressure whatever the eccentricity ε and the supply position ϕ. It will be noted that we must recall the variational inequation modeling (problem (PV)), especially the study of the film extent, before giving an existence and uniqueness theorem for the problem (P).

It is of interest to notice that the solution of problem (P) is always less than that of problem (PV). The coincidence of both solutions is possible only under precise operating

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conditions; i.e., the input mass flow at the supply line must be the greatest compatible with the external boundary conditions.

1. Physical problem and preliminary results. The lubricating region $\Omega =]0, 2\pi[$ of the bearing can be divided into two distinct zones. In the first zone $\Omega+$, where the fluid film is complete, the usual Reynolds equation (1) applies and the pressure $p(x)$ is positive. In the second zone Ω_0, which is cavitated, only an unknown fraction $\theta(x)$ of the film gap $h(x)$ is assumed to be occupied by the fluid and the pressure is assumed to be zero (see Fig. 1).

The unknown boundary (σ) between $\Omega+$ and Ω_0 is the free boundary. Applying the continuity equation to the dimensionless mass flow [6], $\theta h - h^3\,(dp/dx)$, we obtain the following problem (P) with (θ, p, σ) as unknowns:

Problem (P):

on $\Omega+$: $$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) = \frac{dh}{dx}, \qquad \theta = 1, \qquad p > 0; \tag{1}$$

on Ω_0: $$\frac{d}{dx}(\theta h) = 0, \qquad 0 < \theta < 1, \qquad p = 0; \tag{2}$$

on (σ): $$h^3\frac{dp}{dx} = (1-\theta)h; \tag{3}$$

on the supply line: $$p(0) = p(2\pi) = 0. \tag{4}$$

For a journal bearing, $h(x)$ is defined by (5) where ε and ϕ are geometrical data:

$$h(x) = 1 - \varepsilon\cos(x-\phi); \qquad 0 < \varepsilon < 1, \quad 0 \le \phi < 2\pi, \quad \phi \ne \pi. \tag{5}$$

Remarks. Due to the lack of side leakage, the input and output mass flow must be equal, but the supply line may be a discontinuity line for θ if the film starts at $x = 0$.

Let us note that $\xi = \theta(0-) = \theta(2\pi-)$; then we have

$$\xi h(0) = \theta(0+)h(0) - h^3(0)\frac{dp}{dx}(0+). \tag{6}$$

(3) implies that at the end $(\sigma+)$ of a non-cavitated area (i.e. $dp/dx \le 0$) we have

$$\theta = 1, \qquad dp/dx = 0 \tag{7}$$

whereas at the beginning $(\sigma-)$ of $\Omega+$, θ and dp/dx have a jump.

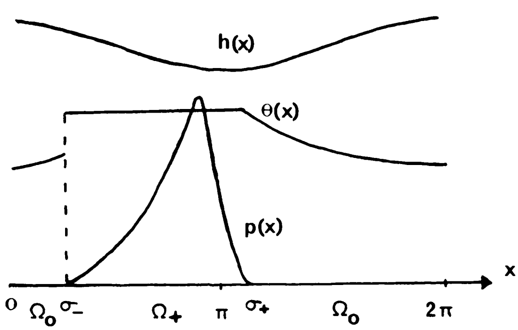

FIG. 1. Typical $h(x)$, $p(x)$, $\theta(x)$ aspects.

Then the following results hold for the shape of the cavitated area and the existence of a lower bound for the input mass flow.

THEOREM 1. If $\Omega+$ is not empty, $\Omega+$ is a connected set and $dh/dx > 0$ on $\sigma+$.

Proof. Let $[b_1, a_2]$ be a cavitated area between two non-cavitated zones Ω_{1+} and Ω_{2+}; we have, by integrating (1) in Ω_{1+} and Ω_{2+}:

$$h^3(x)\frac{dp}{dx} = h(x) - h_i^*, \qquad i = 1, 2 \tag{8}$$

where h_i^* is the film thickness at the point where $p(x)$ is maximum in each Ω_{i+}. From (2) and (7) we have: $h_1^* = h_2^* = h(b_1) = \theta(a_{2-})h(a_2)$, and this is impossible for the given film gap (5).

COROLLARY 1. For all ξ with $\xi h(0) \le h_{\min}$ ($h_{\min} = 1 - \varepsilon$), problem (P) has the unique solution $p = 0$, $\theta(x) = \xi h(0)/h(x)$.

Proof. This is obvious from (2), (8) because existence of a point $\sigma+$ would imply:

$$\xi h(0) = \theta(\sigma+)h(\sigma+) = h(\sigma+) > h_{\min}.$$

2. The variational inequality modeling (Problem (PV)). We recall here the mathematical formulation [1] of problem (PV) and study the shape of the non-cavitated area Ω_{R+}.

Let $H_0^1(\Omega)$ be the Sobolev space of square-integrable functions with square-integrable derivatives, which are zero for $x = 0$ and $x = 2\pi$. Let K be the closed convex set defined by: $K = \{\phi \in H_0^1(\Omega), \phi \ge 0 \text{ a.e. in } \Omega\}$.

It is well known [1, 2, 3] that there is a unique function $p_R \in K$ which is a solution of the variational inequality:

$$\int_\Omega h^3 \frac{dp_R}{dx}\frac{d(\phi - p_R)}{dx}\,dx \ge \int_\Omega \frac{dh}{dx}(\phi - p_R)\,dx \qquad \forall\phi, \quad \phi \in K.$$

p_R is continuously differentiable and satisfies:

$$\text{on} \quad \Omega_{R0} = \{x \in \Omega / p_R(x) = 0\}, \qquad p_R = 0, \ \frac{dh}{dx} > 0, \tag{9}$$

$$\text{on} \quad \Omega_{R+} = \{x \in \Omega / p_R(x) > 0\}, \qquad \frac{d}{dx}\left(h^3\frac{dp_R}{dx}\right) = dh/dx,$$

$$\text{on} \quad \sigma_R \text{ (the Reynolds free boundary surface):} \qquad p_R = \frac{dp_R}{dx} = 0. \tag{10}$$

Moreover, we have the following result:

THEOREM 2. Each connected set of Ω_{R+} has at most one free boundary, so Ω_{R+} begins at $x = 0$ or ends at $x = 2\pi$.

Proof. Integrating (1) on $\Omega_{R+} =]a, b[$, where Eq. (10) holds, if p_R is maximum for $x = x^*$, we have

$$\frac{dp_R}{dx}(a) = \frac{dp_R}{dx}(b) = \frac{dp_R}{dx}(x^*) = 0$$

and then $h(a) = h(b) = h(x^*)$, which is impossible from (5).

So p_R has one of the three patterns shown in Fig. 2. The usual one-hump pressure

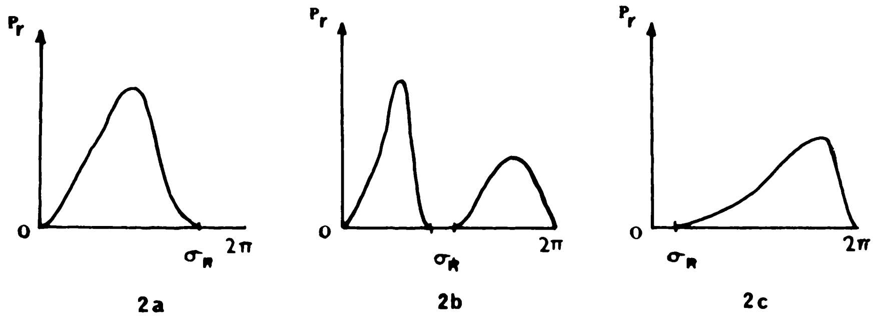

FIG. 2. Typical $p_R(x)$ profiles.

profile starting at the supply line (Fig. 2a), a two-hump profile (Fig. 2b) or a one-hump profile ending at the supply line (Fig. 2c). Numerical computations demonstrate (Fig. 4ⓥ) that for a fixed eccentricity, all these three configurations are actually obtained as the supply angle ϕ varies. For each of these three shapes, we study now whether or not p_R may also be solution of problem (P).

COROLLARY 2. If p_R has the (Fig. 2a) profile, then $(p_R, \theta_R, \sigma_R)$, with $\theta_R(x)$ defined on Ω_{R0} by $h(\sigma_R)/h(x)$, is the solution of problem P corresponding to the value ξ_R of the inlet flow given by:

$$\xi_R h(0) = h(0) - h^3(0) \frac{dp_R}{dx}(0).$$

COROLLARY 3. If p_R has the (Fig. 2b) profile, the result of Corollary 2 still holds with p_R limited to its first hump only; p_R itself is not a solution of the initial problem due to Theorem 1.

COROLLARY 4. If p_R has the (Fig. 2c) profile, it is not a solution of the problem (P); otherwise we would have from (2), (7):

$$\forall x \in \Omega_{R0} \qquad \theta(x)h(x) = \theta(\sigma_R)h(\sigma_R) = h(\sigma_R)$$

and then (9) implies $\theta(x) > \theta(\sigma_R) = 1$.

3. Existence and uniqueness theorem for problem (P). We prove that problem (P) has a family of solutions related to the input mass flow ξ and increasing with respect to ξ. Moreover, all solutions of problem (P) are less than p_R and the domain of validity of ξ is directly related to the graph of p_R.

THEOREM 3. Let us suppose that p_R in Sec. 2 has a (Fig. 2a) or (Fig. 2b) shape. Then for each value of ξ, $\xi \in \,]\,(1-\varepsilon)/h(0), \xi_R]$, there exists a unique solution (p, θ, σ) of problem (P) and p is a monotone function of ξ.

Proof. Let b be the unique point of Ω defined by:

$$h(b) = \xi h(0) \quad \text{and} \quad h'(b) > 0. \tag{11}$$

From Corollaries 2, 3 and (9) we have:

$$b < \sigma_R. \tag{12}$$

We will denote by p the solution of the retrograde Cauchy problem:

$$h^3(x)\frac{dp}{dx} = h(x) - h(b), \qquad p(b) = 0. \tag{13}$$

The sign of $h'(b)$ implies that $p(b-) > 0$. Let $w = p - p_R$; we have from (9)–(12)

$$\forall x \in]0, b] \qquad h^3\frac{dw}{dx}(x) = h(\sigma_R) - h(b) > 0. \tag{14}$$

$w(b) = -p_R(b) < 0$ from Corollaries 2, 3, so $w(b) < 0$ and $p(0) < 0$. Then there exists a positive abscissa a satisfying

$$p(x) > 0 \quad \text{on} \quad]a, b[,$$

$$p(x) < 0 \quad \text{on} \quad]-\infty, a[.$$

Uniqueness of point a is given by (12) and (5) and prevents dp/dx from being null more than twice.

Taking $\theta(x) = \xi h(0)/h(x)$ on $[0, a[\cup [b, 2\pi]$, we have $0 < \theta(x) < 1$, from (11) and (5). Moreover, (13) may be written:

$$h^3\frac{dp}{dx}(a+) = h(a) - h(b) = h(a)\left[1 - \frac{h(b)}{h(a)}\right] = h(a)[1 - \theta(a-)].$$

So (3) holds with $a = \sigma-$ and $b = \sigma+$, Uniqueness is given by (11) and (13). The monotonicity of p with respect to ξ is shown by considering that b is a monotone function of ξ and using the same argument as in (14).

THEOREM 4. If p_R has a (Fig. 2c) shape, then for each value of ξ, $\xi \in]\,(1-\varepsilon)/h(0), 1]$, there is a unique solution (p, θ, σ); $p(x)$ is monotone with respect to ξ.

Proof. The proof is the same as in Theorem 3, the sign of $w = p - p_R$ being studied on the set $[\sigma_R, b]$ which is not empty.

Remarks. In Theorem 3, p_R (or the first hump of p_R) is the limiting case corresponding to $\xi = \xi_R$, whereas in Theorem 4, p_R is a strictly upper solution of the problem P.

The monotonicity of p as a function of ξ implies that for $\xi > \xi$ max, with ξ max $= \xi_R$ (Theorem 3) or ξ max $= 1$ (Theorem 4), problem (P) has no solution satisfying (4).

4. Numerical results and concluding remarks. Fig. 3 describes the set of values of ξ

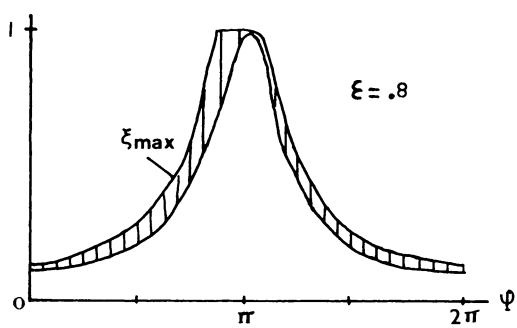

FIG. 3. Domain of validity for the parameter ξ as a function of supply angle ϕ.

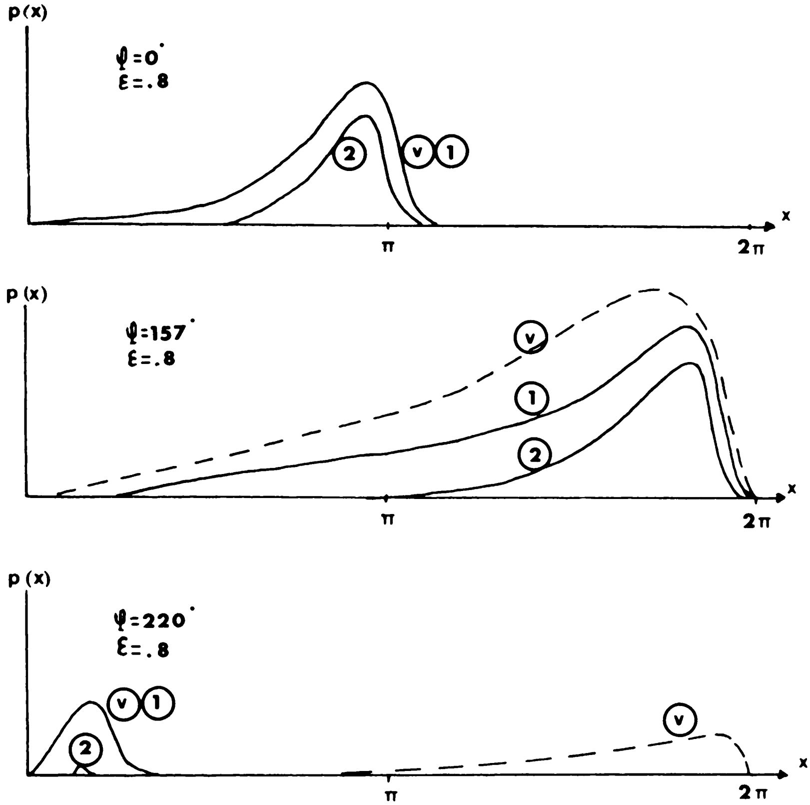

FIG. 4. Pressure profiles for the Reynolds auxiliary problem Ⓥ ; the initial problem for $\xi = \xi$ max ① , and $\xi = \xi$ max/2 ② .

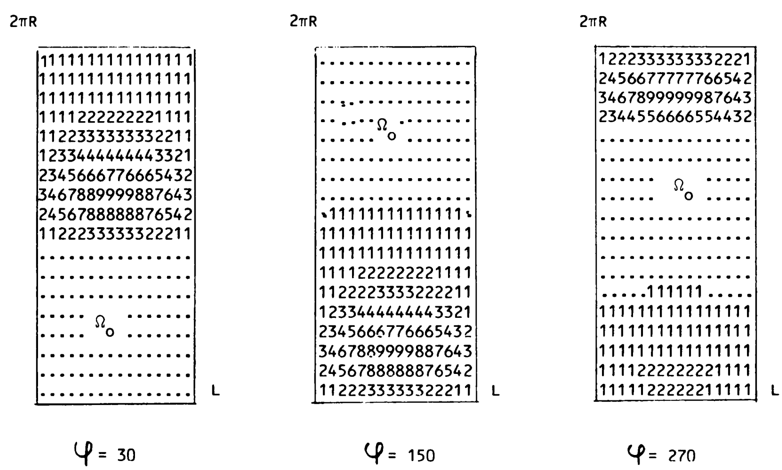

FIG. 5. Normalized pressure repartition for a 2D bearing using (PV) modelling.

for which problem P has a nontrivial solution with respect to ϕ, for a given eccentricity ε. The upper bound ξ max, if it is different from one, is numerically calculated from the p_R solution using an iterative procedure [11]. Various pressure profiles appear in Fig. 4. Fig. 5 illustrates that the typical configurations of the non-cavitated area described by the present one-dimensional study are fair approximations for not-so-long bearings, even with a length L equal to its radius R. So we expect that a further comparison of both modellings in the two-dimensional case would give same results, using the same ideas:

a) the film extent is governed not only by boundary conditions on the pressure but also by the input mass flow on the supply line,

b) the input mass flow is bounded by values predicted from the solution p_R,

c) the violation of mass flow conservation law in the (PV) modelling introduces unrealistic non-cavitated areas; this difficulty disappears in the (P) modelling.

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