

## ON A LINEAR AGE-DEPENDENT POPULATION DIFFUSION MODEL\*

BY

MICHEL R. LANGLAIS

*Purdue University*

**1. Introduction.** We discuss here a model described by Gurtin [3] for age-dependent populations with diffusion in a bounded set  $\Omega$  of  $\mathbb{R}^N$ .

In this linear theory the age-space structure is studied through the population distribution  $u(t, a, x)$  where  $t$  is a time,  $a$  age ( $0 < a < A$ ) and  $x$  spatial position. The evolution of  $u$  is governed by the equation (balance law):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha = 0 \quad (1)$$

where  $\mu = \mu(t, a, x)$  is the death-modulus and  $\nabla \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha$  is the flux of population by spatial diffusion. Here  $\Delta$  is the Laplacian and  $\nabla$  is the gradient in  $\mathbb{R}^N$ . The reader is referred to Hoppensteadt [6] for equations of this form, but without diffusion.

We assume that the birth process is given by the birth law:

$$u(t, 0, x) = \int_0^A \beta(t, a, x) u(t, a, x) \, da \quad (2)$$

( $\beta$  is the birth modulus), that there is no diffusion through the boundary  $\partial\Omega$  of  $\Omega$ , that is:

$$\frac{\partial}{\partial \eta} \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha = 0 \quad (3)$$

where  $\partial/\partial\eta$  is the normal derivative, and that the initial population is known:

$$u(0, a, x) = u_0(a, x). \quad (4)$$

$A$  is the maximum life expectancy of the species.

The initial boundary value problem (1)–(4) will be referred to as problem (I), namely (subscripts indicate partial differentiation):

$$\begin{aligned} u_t + u_a + \mu u - \Delta \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha &= 0, & t > 0, 0 < a < A, x \in \Omega, \\ u(0, a, x) &= u_0(a, x), & 0 < a < A, x \in \Omega, \\ u(t, 0, x) &= \int_0^A \beta(t, a, x) u(t, a, x) \, da, & t > 0, x \in \Omega, \\ \frac{\partial}{\partial \eta} \int_0^A k(a, \alpha) u(t, \alpha, x) \, d\alpha &= 0, & t > 0, 0 < a < A, x \in \Omega. \end{aligned} \quad (I)$$

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\* Received November 23, 1981; revised version received January 19, 1982. Author's permanent address: U. E. R. de Mathématiques et Informatique, Université de Bordeaux 1, 33405 Talence Cedex France.

In the particular case when the kernel  $k$  is independent of the variable  $\alpha$ , the equation (1) and the boundary condition (3) can be expressed in a simpler form and problem (I) becomes problem (II), that is:

$$\begin{aligned}
 u_t + u_a + \mu u - k(a) \Delta \int_0^A u(t, \alpha, x) d\alpha &= 0, \quad t > 0, \quad 0 < a < A, \quad x \in \Omega, \\
 u(0, a, x) &= u_0(a, x), \quad 0 < a < A, \quad x \in \Omega, \\
 u(t, 0, x) &= \int_0^A \beta(t, a, x) u(t, a, x) da, \quad t > 0, \quad x \in \Omega, \\
 \frac{\partial}{\partial \eta} \int_0^A u(t, a, x) da &= 0, \quad t > 0, \quad x \in \partial\Omega.
 \end{aligned}
 \tag{II}$$

Some results concerning problem (I) are given in di Blasio and Lamberti [1]. The method emphasized here (initiated in Langlais [8]) is quite different. Under suitable assumptions this method provides existence and uniqueness in problem (II) (which is the first model derived in [3]), and is helpful for the nonlinear model investigated in Garroni and Langlais [2]. On the other hand, the hypotheses needed to solve problem (II) when  $\mu$  is not bounded appear again in the model with nonlinear diffusion studied by Langlais [9], and we expect them to be useful in the nonlinear diffusion model described in Gurtin and MacCamy [4, 5] (investigated for constant  $\mu$  by MacCamy [11]). In these two linear models the solution can become negative in a finite time (see [4, 9]). This paper is a first step towards nonlinear models.

**2. Notation and basic assumptions.**  $T$  and  $A$  are positive and finite real numbers;  $\Omega$  is a bounded open set of  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$ . The variables  $t, a$  and  $x$  lie respectively in  $(0, T), (0, A)$  and  $\Omega$ . The set  $(0, T) \times (0, A)$  is denoted  $\mathcal{Q}$  and  $Q$  is the product  $\mathcal{Q} \times \Omega$ .

Let  $\mu$  be a real-valued function on  $Q$  satisfying:

$$\begin{aligned}
 \mu &\text{ is continuous on } [0, T] \times [0, A) \times \bar{\Omega}; \\
 \mu(t, a, x) &\geq 0 \text{ on } Q; \\
 \nabla \mu &\text{ is bounded on } Q.
 \end{aligned}
 \tag{\mu}^1$$

$\mu$  is not and will not be assumed to be bounded near  $a = A$  (see further remarks).

We suppose that  $\beta$  is a real-valued function on  $Q$  such that:

$$\begin{aligned}
 \beta &\text{ is bounded on } Q; \\
 \text{there exists a constant } C_1 &\text{ such that} \\
 \int_0^A [\beta^2 + |\nabla \beta|^2](t, a, x) da &\leq C_1 \text{ in } (0, T) \times \Omega.
 \end{aligned}
 \tag{\beta}^1$$

The kernel  $k$  is a real measurable function defined in  $(0, A) \times (0, A)$ ; in problem (I) we shall suppose:

there exists a constant  $C_2$  such that for each  $\varphi \in L^2(0, A)$ :

$$\int_0^A \left[ \int_0^A k(a, \alpha) \varphi(\alpha) d\alpha \right]^2 da \leq C_2^2 \int_0^A \varphi^2(a) da:
 \tag{k}^1$$

for each  $\varphi \in L^2(0, A)$  we have:

$$\int_0^A \int_0^A k(a, \alpha)\varphi(\alpha)\varphi(a) da d\alpha \geq 0.$$

The first part of  $(k)^1$  means that the map:

$$\varphi \in L^2(0, A) \rightarrow \int_0^A k(a, \alpha)\varphi(\alpha) d\alpha \in L^2(0, A)$$

is continuous. The last part of  $(k)^1$  is a monotonicity-type condition used in [1].

In the particular case where the kernel  $k$  does not depend on  $\alpha$  we shall simply need:

$$k \in L^2(0, A); \quad k(a) \geq 0 \quad \text{in } (0, A). \tag{k}^2$$

Now if  $U$  is any open set of either  $\mathbb{R}$  or  $\mathbb{R}^2$  and if  $H$  is any of the Sobolev spaces of order 1 or 2  $H^1(\Omega)$  and  $H^2(\Omega)$ , then  $L^2(U, H)$  is the Hilbert space of measurable and square integral function  $v: U \rightarrow H$  (see Lions and Magenes [10]).

**3. Statement of results.** The first result concerns problem (I), that is the initial boundary value problem (1)–(4).

**THEOREM 1.** Let  $(\mu, \beta, k)$  satisfy  $(\mu)^1, (\beta)^1, (k)^1$  and

$$\Delta\mu \in L^\infty(Q), \quad \partial\mu/\partial\eta = 0 \text{ in } \mathcal{O} \times \partial\Omega; \tag{\mu}^2$$

$$\int_0^A |\Delta\beta|^2(t, a, x) da \leq C_1 \text{ in } (0, T) \times \Omega, \quad \frac{\partial\beta}{\partial\eta} = 0 \text{ in } \mathcal{O} \times \partial\Omega. \tag{\beta}^2$$

Then given any  $u_0$  in  $L^2(0, A; H^2(\Omega))$  such that  $\partial u_0/\partial\eta = 0$  in  $(0, A) \times \partial\Omega$  there exists a unique  $u$  belonging to  $L^2(\mathcal{O}; H^2(\Omega))$ , a solution to problem (I) and verifying  $\partial u/\partial\eta = 0$  in  $\mathcal{O} \times \partial\Omega$ .

It is worth while to notice that the initial boundary conditions (2–4) make sense provided the solution belongs to  $L^2(\mathcal{O}; H^2(\Omega))$ .

As the solution  $u$  lies in  $L^2(\mathcal{O}; H^2(\Omega))$ , the properties of  $k$  ensure that  $\int_0^A k(a, \alpha)u(t, \alpha, x) d\alpha$  lies in  $L^2(\mathcal{O}; H^2(\Omega))$ ; hence the Neumann boundary condition (3) makes sense and is fulfilled because  $u$  satisfies the homogeneous boundary Neumann condition. Moreover, from Eq. (1) we deduce that  $u_t + u_a + \mu \cdot u$  belongs to  $L^2(Q)$ . Now take any  $A_0, 0 < A_0 < A$ , and set  $\mathcal{O}_0 = (0, T) \times (0, A_0), Q_0 = \mathcal{O}_0 \times \Omega$ ; then  $\mu$  is bounded on  $Q_0$  and  $u_t + u_a$  lies in  $L^2(Q_0)$ . Thus bearing in mind that  $u$  is in  $L^2(Q_0)$ , initial conditions (2) and (4) make sense.

When the data  $u_0$  is not smooth enough or when the conditions  $(\mu)^2$  and  $(\beta)^2$  are not fulfilled we can prove the existence and uniqueness of a suitable weak solution. Hence Theorem 1 can be considered as a regularity theorem. But it is more interesting to view it as a basic result from which we can derive particular cases.

Let us assume first that:

$$k(a, \alpha) = h(a)h(\alpha), \quad h \in L^2(0, A).$$

This assumption is stronger than  $(k)^1$ . The Neumann boundary condition (3) becomes:

$$\frac{\partial}{\partial\eta} \int_0^A h(\alpha)u(t, \alpha, x) d\alpha = 0 \text{ in } (0, T) \times \partial\Omega. \tag{5}$$

Condition  $(k)^3$  allows us to remove  $(\mu)^2$  and  $(\beta)^2$  in Theorem 1.

**THEOREM 2.** Suppose that  $(\mu)^1, (\beta)^1, (k)^3$  are satisfied; then given any  $u_0$  in  $L^2(0, A; H^1(\Omega))$  there exists a unique  $u$  in  $L^2(\mathcal{O}; H^1(\Omega))$ , a solution to problem (I) and such that:

$$\int_0^A h(\alpha)u(t, \alpha, x) \, d\alpha \in L^2(0, T; H^2(\Omega)). \tag{6}$$

We investigate now the particular case when the kernel  $k$  is independent of  $\alpha$ , that is, problem (II). Assume first that  $k$  lies in  $C^1([0, A])$  and that:

$$\text{there exists a real constant } k_0 \text{ such that } k(a) \geq k_0 > 0 \text{ in } (0, A) \tag{k}^4$$

If we let  $u(t, a, x) = k(a)v(t, a, x)$  in  $\mathcal{Q}$ , then, at least formally,  $v$  is a solution of the equations:

$$\begin{aligned} k(v_t + v_a) + (\mu k + k_a)v - k \Delta \int_0^A k(\alpha)v(t, \alpha, x) \, d\alpha &= 0 \text{ in } \mathcal{Q}, \\ v(0, a, x) &= [k(a)]^{-1}u_0(a, x) \text{ in } (0, A) \times \Omega \\ v(t, 0, x) &= [k(0)]^{-1} \int_0^A \beta(t, a, x)k(a)v(t, a, x) \, da \text{ on } (0, T) \times \Omega, \\ \frac{\partial}{\partial \eta} \int_0^A k(a)v(t, a, x) \, da &= 0 \text{ on } \mathcal{O} \times \partial\Omega. \end{aligned} \tag{III}$$

This boundary-value problem with  $(k)^4$  and the problem (I) with  $(k)^3$  have the same qualitative properties. We can prove that when  $(\mu)^1, (\beta)^1, (k)^2$  and  $(k)^4$  are satisfied for any  $u_0$  given in  $L^2(0, A; H^1(\Omega))$  there exists a unique  $v$  verifying (6)—see Theorem 2—a solution to problem (III). From  $v$  we get  $u$ , a solution to problem (II).

*Remark 1.* Up to now we did not suppose  $\mu$  to be bounded at  $a = A$  (see  $(\mu)^1$ ). Actually if  $\mu$  is rapidly increasing at  $a = A$  then any of the solutions whose existence has been previously established vanishes at  $a = A$ . More precisely, the conditions  $u(t, A, x) = 0$  in  $(0, T) \times \Omega$  and the two conditions:

$$\begin{aligned} 0 < t < A, \quad x \in \Omega \lim_{a \rightarrow A} \int_0^t \mu(\tau, a - t + \tau, x) \, dt &= +\infty \\ A < t < T, \quad x \in \Omega \lim_{a \rightarrow A} \int \mu(t - a + \alpha, \alpha, x) \, d\alpha &= +\infty \end{aligned} \tag{\mu}^3$$

are equivalent (see Langhaar [7], [2] and [8]: this property is independent of the diffusion term in (1)).

Unfortunately, we have not been able to solve Problem (II) when  $(\mu)^3$  is fulfilled (except when  $(k)^4$  is true) without additional hypotheses on  $u, k$  and  $u_0$ .

**THEOREM 3.** Let  $(\mu)^1, (\beta)^1, (k)^2$  be satisfied. Let  $u_0$  be in  $L^2((0, A) \times \Omega)$  and verify  $\int_0^A u_0(a, x) \, da \in H^1(\Omega)$ . Assume either:

there exist two constants  $\lambda_1, \lambda_2$  such that

$$\mu^2 - \mu_t - \mu_a \geq \lambda_1\mu + \lambda_2 \text{ on } \mathcal{Q}; \tag{i}$$

$$\mu(0, a, x)u_0 \in L^2((0, A) \times \Omega); \tag{ii} \tag{\mu}^4$$

there exists a constant  $M$  such that

$$\int_0^A \mu^2(t, a, x)k^2(a) da \leq M \text{ on } (0, T) \times \Omega; \quad (\text{iii})$$

or:

$$\text{there exists a constant } m \text{ such that } \int_0^A \mu(t, a, x) da \leq m \text{ in } (0, T) \times \Omega. \quad (\mu)^5$$

Then there exists a unique  $u$  in  $L^2(Q)$  satisfying (6)— $\mu u \in L^2(Q)$  when  $(\mu)^4$  is true—a solution to problem (II).

*Remark 2.* When  $(\mu)^3$  is satisfied and when  $k$  is a constant then  $(\mu)^4(\text{iii})$  is not fulfilled; nevertheless, to conclude we merely apply Theorem 2. More generally,  $(k)^4$ ,  $(\mu)^3$ , and  $(\mu)^4(\text{iii})$ , are not consistent, but we have previously dealt with  $(k)^4$ .

The conditions  $(\mu)^3$  and  $(\mu)^4(\text{i})$  are realized when:

$$\begin{aligned} \mu(a) &= q(A - a)^{-p}, \quad q > 0 \text{ and } p > 1 \text{ or } q \geq 1 \text{ and } p = 1, \\ \mu(a) &= q \exp + 1/A - a, \quad q > 0. \end{aligned}$$

**4. Proofs.** We first discuss a preliminary result from which Theorem 1 is proved, for bounded  $\mu$ , using a fixed-point method. Then we turn to the general case. Theorem 3 is proved along the same lines, but we shall point out the differences in the first and last steps. The proof of Theorem 2 is omitted.

Now let  $u$  be a solution to problem (I) or problem (II) and set:

$$u(t, a, x) = e^{\lambda t}v(t, a, x) \text{ in } Q, \lambda \text{ constant};$$

then  $v$  is the solution to problem (I) or problem (II) with  $\mu$  replaced by  $\mu + \lambda$ . This change of unknown function will be done throughout this section, for suitable positive values of  $\lambda$ , and  $v$  will be simply denoted  $u$ .

4-1. *Preliminary results.* We investigate the following initial-boundary value problem:

$$\begin{aligned} u_t + u_a + \lambda u - \Delta \int_0^A k(\alpha, a)u(t, \alpha, x) d\alpha &= f \text{ in } Q, \\ u(0, a, x) &= u_0(a, x) \text{ in } (0, A) \times \Omega, \\ u(t, 0, x) &= b(t, x) \text{ in } (0, T) \times \Omega, \\ \frac{\partial}{\partial \eta} \int_0^A k(a, \alpha)u(t, \alpha, x) da &= 0 \text{ in } \mathcal{O} \times \partial\Omega. \end{aligned} \quad (\text{IV})$$

Here  $\lambda$  is a positive constant. In order to get the existence of a solution we introduce the eigenfunctions of the Neumann problem in  $\Omega$ :

$$\begin{aligned} -\Delta w_j &= v_j w_j \text{ in } \Omega, \quad \partial w_j / \partial \eta = 0 \text{ in } \partial\Omega, \\ \int_{\Omega} w_j^2 &= 1, \quad \int_{\Omega} w_j \cdot w_i dx = 0 \quad j \neq i. \end{aligned}$$

Expressing the data on the form:

$$\begin{aligned}
 f(t, a, x) &= \sum_j f_j(t, a)w_j(x), \\
 b(t, x) &= \sum_j b_j(t)w_j(x), \quad u_0(a, x) = \sum_j u_{0,j}(a)w_j(x),
 \end{aligned}
 \tag{7}$$

it is appealing to seek as a formal solution to problem (III) the series

$$u(t, a, x) = \sum_j u_j(t, a)w_j(x).$$

It turns out that for suitable data  $(f, b, u_0)$  this gives a solution.

**THEOREM 4.** Let  $k$  satisfy  $(k)^1$  and let  $(f, b, u_0)$  be such that:

$$\begin{aligned}
 f &\in L^2(\mathcal{O}; H^2(\Omega)), & \partial f / \partial \eta &= 0 \text{ in } \mathcal{O} \times \partial \Omega, \\
 b &\in L^2(0, T; H^2(\Omega)), & \partial b / \partial \eta &= 0 \text{ in } (0, T) \times \partial \Omega, \\
 u_0 &\in L^2(0, T; H^2(\Omega)), & \partial u_0 / \partial \eta &= 0 \text{ in } (0, A) \times \partial \Omega.
 \end{aligned}
 \tag{8}$$

Then there exists a unique solution  $u$  in  $L^2(\mathcal{O}; H^2(\Omega))$  to problem (IV). Moreover, this solution satisfies the boundary condition:

$$\partial u / \partial \eta = 0 \text{ on } \mathcal{O} \times \partial \Omega$$

and the a priori estimates:

$$\begin{aligned}
 \lambda \int_{\mathcal{O}} [u^2 + |\nabla u|^2] dt da dx &\leq \frac{1}{2} \int_{(0, A) \times \Omega} [u_0^2 + |\nabla u_0|^2] da dx \\
 &+ \frac{1}{2} \int_{(0, T) \times \Omega} [b^2 + |\nabla b|^2] dt dx + \int_{\mathcal{O}} [f \cdot u + \nabla f \cdot \nabla u] dt da dx,
 \end{aligned}
 \tag{9}$$

$$\begin{aligned}
 \lambda \int_{\mathcal{O}} |\Delta u|^2 dt da dx &\leq \frac{1}{2} \int_{(0, A) \times \Omega} |\Delta u_0|^2 da dx \\
 &+ \frac{1}{2} \int_{(0, T) \times \Omega} |\Delta b|^2 da dx + \int_{\mathcal{O}} \Delta f \cdot \Delta u dt da dx.
 \end{aligned}
 \tag{10}$$

*Proof.* The existence of a solution is obtained by the method of separation of variables outlined above. The solution satisfies the estimates (9) and (10).

Now let  $u^1$  and  $u^2$  be solutions of (IV). The difference  $u = u^1 - u^2$  is solution of the same problem (IV) with  $f = 0, b = 0$  and  $u_0 = 0$ . Multiplying equation (IV)<sup>1</sup> by  $u$  and integrating over  $Q$  yields:

$$\int_{\mathcal{O}} (u_t + u_a)u dt da dx + \lambda \int_{\mathcal{O}} u^2 dt da dx - \int_{\mathcal{O}} \Delta \int_0^A k(a, \alpha)u(t, a, x) d\alpha \cdot u dt da dx = 0 \tag{11}$$

Integrating by parts the last term and using  $(k)^1$  we have:

$$\begin{aligned}
 \int_{\mathcal{O}} \nabla \int_0^A k(a, \alpha)u(t, \alpha, x) d\alpha \cdot \nabla u(t, a, x) dt da dx \\
 = \int_{(0, T) \times \Omega} \left[ \int_0^A \int_0^A k(a, \alpha)\nabla u(t, \alpha, x) \cdot \nabla u(t, a, x) d\alpha da \right] dt dx \geq 0
 \end{aligned}$$

$u$  belonging to  $L^2(\mathcal{O}; H^2(\Omega))$ . From equation (IV)<sup>1</sup> we get  $u_t + u_a \in L^2(Q)$ . We can integrate by parts the first term in (11); this gives:

$$\int_Q (u_t + u_a)u \, dt \, da \, dx = \frac{1}{2} \left[ \int_{(0, A) \times \Omega} u^2(T, a, x) \, da \, dx + \int_{(0, T) \times \Omega} u^2(t, A, x) \, dt \, dx \right] \geq 0,$$

because  $u$  vanishes on  $t = 0$  and  $a = 0$ . It is easy now to prove that  $u = 0$ ; from (11) we derive  $\int_Q u^2(t, a, x) \, dt \, da \, dx \leq 0$ .

**4.2 Proof of Theorem 1 for bounded  $\mu$ .** We assume that  $\mu$  is bounded. Let  $\lambda$  be a positive constant large enough compared with  $C_i$  (see the hypotheses on  $\beta$ ), with the norms of  $\mu$  and  $\Delta\mu$  in  $L^\infty(Q)$  and with the norms of  $\nabla\mu$  in  $[L^\infty(Q)]^N$ . We deal with  $v = e^{-\lambda t}u$  still denoted  $u$ .

We want to prove Theorem 1 using a fixed-point method and (IV), (9), (10). Let  $E$  be the Hilbert space:

$$F = \{u \in L^2(\mathcal{O}; H^2(\Omega)), \partial u / \partial \eta = 0 \text{ in } \mathcal{O} \times \partial\Omega\}.$$

Assuming that the hypotheses of theorem 1 ( $\mu$  bounded) are satisfied, for any  $w$  in  $E$  there exists a unique  $u = Sw$  in  $E$ , a solution to

$$u_t + u_a + \lambda u - \Delta \int_0^A k(a, \alpha)u(t, \alpha, x) \, d\alpha = -\mu w \text{ in } Q,$$

$$u(0, a, x) = u_0(a, x) \text{ in } (0, A) \times \Omega,$$

$$u(t, 0, x) = \int_0^A \beta(t, a, x)w(t, a, x) \, da \text{ in } (0, T) \times \Omega,$$

$$\frac{\partial}{\partial \eta} \int_0^A k(a, \alpha)u(t, \alpha, x) \, d\alpha = 0 \text{ in } \mathcal{O} \times \partial\Omega.$$

Thus we define a map  $S: E \rightarrow E$ . Its fixed points are the solutions to problem (I) in  $E$ . From (9) and (10) we deduce that  $S$  is continuous.

Let  $w^1$  (resp.  $w^2$ ) be in  $E$  and set  $u^1 = Sw^1$  (resp.  $u^2 = Sw^2$ ). The function  $u = u^1 - u^2$  is a solution to problem (IV) with:

$$f = -\mu w \text{ in } Q, \quad w = w^1 - w^2 \text{ in } Q.$$

$$b(t, x) = \int_0^A \beta(t, a, x)w(t, a, x) \, da \text{ in } (0, T) \times \Omega, \tag{12}$$

$$u_0(a, x) = 0 \text{ in } (0, A) \times \Omega.$$

Before employing the estimates (9) (10), we need to bound the right-hand side of (9), (10) when  $f, b$  and  $u_0$  are given by (12).

First, the properties of  $\beta$ —that is,  $(\beta)^1, (\beta)^2$ —give:

$$\int_{(0, T) \times \Omega} [b^2 + |\nabla b|^2] \, dt \, dx \leq 3C_1 \int_Q [w^2 + |\nabla w|^2] \, dt \, da \, dx, \tag{13}$$

$$\int_{(0, T) \times \Omega} |\Delta b|^2 \, dt \, dx \leq 4C_1 \int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2] \, dt \, da \, dx.$$

Here we used the Holder inequality. The same calculation for  $\mu$  leads to:

$$\int_Q [f \cdot u + \nabla f \cdot \nabla u] \, dt \, da \, dx \leq C_3 \left[ \int_Q (u^2 + |\nabla u|^2) \, dt \, da \, dx \right]^{1/2} \cdot \left[ \int_Q (w^2 + |\nabla w|^2) \, dt \, da \, dx \right]^{1/2}, \tag{14}$$

$$\int_Q \Delta f \cdot \Delta u \, dt \, da \, dx \leq C_4 \left[ \int_Q |\Delta u|^2 \, dt \, da \, dx \right]^{1/2} \left[ \int_Q (w^2 + |\nabla w|^2 + |\Delta w|^2) \, dt \, da \, dx \right]^{1/2}$$

where:

$$\begin{aligned} C_3 &= 2|\mu|_{L^\infty(Q)} + |\nabla \mu|_{L^\infty(Q)}, \\ C_4 &= |\mu|_{L^\infty(Q)} + |\Delta \mu|_{L^\infty(Q)} + 2|\nabla \mu|_{L^\infty(Q)}. \end{aligned} \tag{15}$$

If we substitute (13) and (14) into the estimate (9), using the Schwarz inequality we obtain

$$\left( \lambda - \frac{C_3}{2} \right) \int_Q [u^2 + |\nabla u|^2] \, dt \, da \, dx \leq \frac{3C_1 + C_3}{2} \int_Q [w^2 + |\nabla w|^2] \, dt \, da \, dx. \tag{16}$$

If we use (13), (14), (10) and the Schwarz inequality we have:

$$\left( \lambda - \frac{C_4}{2} \right) \int_Q |\Delta u|^2 \, dt \, da \, dx \leq \frac{4C_1 + C_4}{2} \int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2] \, dt \, da \, dx. \tag{17}$$

Remembering that  $\lambda$  was chosen very large compared with  $(C_1, C_3, C_4)$ , we easily derive from (16), (17) that there exists a constant  $K = K(\lambda, C_1, C_3, C_4)$  that satisfies  $0 < K < 1$  and such that:

$$\int_Q [u^2 + |\nabla u|^2 + |\Delta u|^2] \, dt \, da \, dx \leq K \int_Q [w^2 + |\nabla w|^2 + |\Delta w|^2] \, dt \, da \, dx.$$

But  $w = w^1 - w^2$  and  $u = u^1 - u^2 = Sw^1 - Sw^2$ . Therefore we proved that  $S$  has a unique fixed point. Hence for bounded  $\mu$  the problem (1-4) has a unique solution belonging to the space  $E$ .

**4.3 Proof of Theorem 1: Existence.** In the general case  $\mu$  is not bounded near  $a = A$ . Let  $\lambda$  be a positive constant very large compared with  $C_1$  and the norms of  $\nabla \mu$  and  $\Delta \mu$  in  $L^\infty(Q)$ . Once more we deal with  $e^{-\lambda t}u$ , still denoted  $u$ . There exists a sequence  $(\mu_n)_n$  such that:

$$\begin{aligned} \mu_n &\in L^\infty(Q), \\ |\nabla \mu_n|_{L^\infty(Q)} + |\Delta \mu_n|_{L^\infty(Q)} &\leq C_5 \text{ independent of } n, \\ \mu_n(t, a, x) &= \mu(t, a, x) \text{ in } (0, T) \times (0, A - 1/n) \times \Omega. \end{aligned} \tag{18}$$

For each  $n$  there exists a unique  $u_n$  belonging to  $E$ , a solution of the problem (I) with  $\mu$  changed into  $\mu_n + \lambda$ . But  $u_n$  is the solution to problem (IV) with:

$$f = -\mu_n u_n \text{ in } Q, \quad b(t, x) = \int_0^A \beta(t, a, x) u_n(t, a, x) \, da \text{ in } (0, T) \times \Omega.$$

Again we use the estimates (9), (10) and we need some preliminary calculations. It is obvious that the inequalities (13) are valid with  $u_n$  instead of  $w$ . Now we have:

$$\int_Q f \cdot u_n + \nabla f \cdot \nabla u_n \, dt \, da \, dx = - \int_Q \mu_n [u_n^2 + |\nabla u_n|^2] \, dt \, da \, dx - \int_Q u_n \cdot \nabla \mu_n \cdot \nabla u_n \, dt \, da \, dx.$$

hence from (18) and Schwarz and Holder inequalities we obtain:

$$\begin{aligned} \int_Q [f \cdot u_n + \nabla f \cdot \nabla u_n] \, dt \, da \, dx \\ \leq - \int_Q \mu_n [u_n^2 + |\nabla u_n|^2] \, dt \, da \, dx + \frac{C_5}{2} \int_Q [u_n^2 + |\nabla u_n|^2] \, dt \, da \, dx. \end{aligned}$$

Along the same lines we transform  $\int_Q \Delta f \cdot \Delta u_n \, dt \, da \, dx$  into:

$$- \int_Q \mu_n \cdot |\Delta u_n|^2 \, dt \, da \, dx - \int_Q [2\nabla \mu_n \cdot \Delta u_n \nabla u_n + \Delta \mu_n \cdot u_n \Delta u_n] \, dt \, da \, dx;$$

therefore:

$$\begin{aligned} \int_Q \Delta f \cdot \Delta u_n \, dt \, da \, dx \\ \leq - \int_Q \mu_n |\Delta u_n|^2 \, dt \, da \, dx + C_6 \int_Q [u_n^2 + |\nabla u_n|^2 + |\Delta u_n|^2] \, dt \, da \, dx, \end{aligned}$$

where  $C_6$  is a constant depending only on  $C_5$ .

Substituting these results in (9) and (10) yields, respectively:

$$\int_Q \left[ \lambda + \mu_n - \frac{3C_1}{2} - \frac{C_5}{2} \right] \left( [u_n^2 + |\nabla u_n|^2] \, dt \, da \, dx \leq \frac{1}{2} \int_{(0, A) \times \Omega} [u_0^2 + |\nabla u_0|^2] \, da \, dx. \quad (19) \right.$$

$$\begin{aligned} \int_Q [\lambda + \mu_n] |\Delta u_n|^2 \, dt \, da \, dx - C_6 \int_Q [|u_n|^2 + |\nabla u_n|^2 + |\Delta u_n|^2] \, dt \, da \, dx \\ \leq \frac{1}{2} \int_{(0, A) \times \Omega} |\Delta u_0| \, da \, dx. \quad (20) \end{aligned}$$

But  $\lambda$  has been chosen large. So from (19) and (20) we can deduce that the sequence  $(u_n)$  is bounded in the  $L^2(\mathcal{O}; H^2(\Omega))$  norm. The clue to getting this estimate is that  $\mu$  is non-negative. This allows us to remove the hypothesis “ $\mu$  bounded.”

Hence there exists  $u$  in  $L^2(\mathcal{O}; H^2(\Omega))$  and a sub-sequence  $(u_{n_p})_p$  (which we simply denote  $(u_p)$ ) such that:

$$u_p \xrightarrow{p \rightarrow \infty} u \text{ weakly in } L^2(\mathcal{O}; H^2(\Omega)).$$

Each  $u_p$  satisfies the homogeneous Neumann boundary condition; it follows that  $u$  satisfies the same homogeneous Neumann boundary condition.

Now  $(\Delta u_n)_n$  is bounded in  $L^2(Q)$ ; from the equation

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p = \Delta \int_0^A k(a, \alpha) u_p(t, \alpha, x) \, d\alpha \text{ in } Q, \quad (21)$$

we conclude that the sequence  $(\partial u_p / \partial t + \partial u_p / \partial a + \mu_p u_p)$  is bounded in the  $L^2(Q)$ -norm. For a subsequence still denoted with the  $p$ -indices we have:

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p \xrightarrow{p \rightarrow \infty} h \text{ weakly in } L^2(Q).$$

But the choice of the sequence  $(\mu_n)$ —see (18)<sup>3</sup>—ensures that:

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} + \mu_p u_p \xrightarrow{p \rightarrow \infty} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u \text{ in } \mathcal{D}'(Q),$$

that is, in the distributional sense in  $Q$ . Thus  $h = u_t + u_a + \mu u$  and, letting  $p \rightarrow +\infty$  in (21), we obtain that  $u$  satisfies Eq. (1).

We may notice that weakly in  $L^2((0, T) \times \Omega)$  we have:

$$u_p(t, 0, x) = \int_0^A \beta(t, a, x) u_p(t, a, x) \, da \xrightarrow{p \rightarrow \infty} \int_0^A \beta(t, a, x) u(t, a, x) \, da. \quad (22)$$

Let  $A_0$  be such that  $0 < A_0 < A$  and define  $\mathcal{O}_0 = (0, T) \times (0, A_0)$ ; for  $n \geq n(A_0)$   $\mu_n(t, a, x) = \mu(t, a, x)$  in  $\mathcal{O}_0$ . Hence we have  $\mu \in L^\infty(\mathcal{O}_0)$  and from Eq. (21) we get  $(\partial u_p / \partial t + \partial u_p / \partial a)_p$  bounded in  $L^2(\mathcal{O}_0)$ . Thus:

$$u_p \xrightarrow{p \rightarrow \infty} u \text{ weakly in } L^2(\mathcal{O}_0),$$

$$\frac{\partial u_p}{\partial t} + \frac{\partial u_p}{\partial a} \xrightarrow{p \rightarrow \infty} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} \text{ weakly in } L^2(\mathcal{O}_0),$$

and by continuity it follows that:

$$u_p(0, a, x) \xrightarrow{p \rightarrow \infty} u(0, a, x) \text{ weakly in } L^2((0, A_0) \times \Omega),$$

$$u_p(t, 0, x) \xrightarrow{p \rightarrow \infty} u(t, 0, x) \text{ weakly in } L^2(0, T) \times \Omega.$$

From (22) we deduce that  $u$  satisfies the initial condition (2). For each  $u_p$  verifying (4) on  $(0, A) \times \Omega$  we have:

$$u(0, a, x) = u_0(a, x) \text{ in } (0, A_0) \times \Omega;$$

this equality being true for any  $A_0$  lying in  $(0, A)$  is true in the open set  $(0, A) \times \Omega$ . So the initial condition (4) is fulfilled.

We can also derive that  $\mu^{1/2}u$  belongs to  $L^2(\mathcal{O}; H^2(\Omega))$ .

**4.4 Proof of Theorem 1: Uniqueness.** Let  $\lambda$  be large enough with respect to  $C_1$ ; we perform the change of unknown function  $u(t, a, x) = e^{\lambda t} v(t, a, x)$  and we deal with  $v$  which we simply denote  $u$ .

Let  $u$  be in  $L^2(\mathcal{O}; H^2(\Omega))$ , a solution to problem (I), and satisfy the homogeneous Neumann condition on  $\mathcal{O} \times \partial\Omega$ , and let  $u(0, a, x) = 0$  in  $(0, A) \times \Omega$ .

For any  $A_0$  such that  $0 < A_0 < A$  we let  $Q_0 = (0, T) \times (0, A_0) \times \Omega$ . Multiplying Eq. (1) by  $u$  and integrating over  $Q_0$  yields:

$$\int_{Q_0} (u_t + u_a + \mu u + \lambda u)u \, dt \, da \, dx - \int_{Q_0} \left[ \Delta \int_0^{A_0} k(a, \alpha)u(t, \alpha, x) \, d\alpha \right] u \, dt \, da \, dx = 0. \tag{23}$$

We already know that  $u_t + u_a$  lie in  $L^2(Q_0)$  because  $\mu \in L^\infty(Q_0)$ . So we can integrate by parts the first term in the left-hand side of (23); we obtain, using (2) and  $u(0, a, x) = 0$ :

$$\begin{aligned} \frac{1}{2} \int_{(0, A_0) \times \Omega} u^2(T, a, x) \, da \, dx + \frac{1}{2} \int_{(0, A_0) \times \Omega} u^2(t, A_0, x) \, dt \, dx \\ + \int_{Q_0} (\lambda + \mu)u^2 \, dt \, da \, dx - \frac{1}{2} \int_{(0, T) \times \Omega} \left[ \int_0^{A_0} \beta \cdot u \, da \right]^2 \, dt \, dx. \end{aligned}$$

Moreover, we can integrate by parts the second integral in the left-hand side of (23). These two calculations lead to the estimate:

$$\begin{aligned} \lambda \int_{Q_0} u^2(t, a, x) \, dt \, da \, dx \\ + \int_{(0, T) \times \Omega} \left[ \nabla \int_0^{A_0} k(a, \alpha)u(t, \alpha, x) \, d\alpha \cdot \nabla \int_0^{A_0} u(t, a, x) \, da \right] \, dt \, dx \\ \leq \frac{1}{2} \int_{(0, T) \times \Omega} \left[ \int_0^{A_0} \beta u \, da \right]^2 \, dt \, dx. \end{aligned}$$

So if we let  $A_0 \rightarrow A$ , using the properties of the kernel  $k$  we get

$$\lambda \int_Q u^2 \, dt \, da \, dx \leq \frac{1}{2} \int_{(0, T) \times \Omega} \left[ \int_0^A \beta u \, da \right]^2 \, dt \, dx.$$

From the condition  $(\beta)^1$  we derive:

$$\left( \lambda - \frac{C_1}{2} \right) \int_Q u^2 \, dt \, da \, dx \leq 0 \quad \text{and} \quad u = 0 \text{ in } Q.$$

4.5 *Sketch of the proof of Theorem 3.* We must modify the first step, because (9), (10) are not necessarily fulfilled when we change  $(k)^1$  into  $(k)^2$ .

When  $(k)^2$  is satisfied the analogue to problem (IV) is:

$$\begin{aligned} u_t + u_a + \lambda u - k(a) \Delta \int_0^A u(t, a, x) \, da &= f \text{ in } Q, \\ u(0, a, x) &= u_0(a, x) \text{ in } (0, A) \times \Omega, \\ u(t, 0, x) &= b(t, x) \text{ in } (0, T) \times \Omega, \\ \frac{\partial}{\partial \eta} \int_0^A u(t, a, x) \, da &= 0 \text{ in } \mathcal{O} \times \partial\Omega. \end{aligned} \tag{V}$$

Let  $Q_T = (0, T) \times \Omega$ . Assuming:

$$\begin{aligned} f \in L^2(Q), \quad u_0 \in L^2((0, A) \times \Omega), \\ \int_0^A u_0(a, x) da \in H^1(\Omega), \quad b \in L^2((0, A) \times \Omega), \end{aligned} \tag{24}$$

then any  $u$  in  $L^2(Q)$  verifying (6), that is:

$$\int_0^A u(t, a, x) da = P(t, x) \in L^2(0, T; H^2(\Omega)), \tag{6}$$

and a solution to problem (V) is such that:

$$\begin{aligned} P_t - K \Delta P + \lambda P &= \int_0^A f(t, a, x) da + b(t, x) - u(t, A, x) \text{ in } Q_T, \\ P(0, x) &= \int_0^A u_0(a, x) da = p_0(x) \text{ in } \Omega, \\ \partial P / \partial \eta &= 0 \text{ in } (0, T) \times \partial \Omega, \end{aligned} \tag{VI}$$

where we have used  $k = \int_0^A k(a) da$ . This is obtained by integrating Eq. (V)<sup>1</sup> with respect to the variable  $a$ . The hypotheses on  $(f, b, u_0)$  and  $(u, P)$  are consistent. Now  $u$  is the solution to:

$$\begin{aligned} u(0, a, x) &= u_0(a, x) \text{ in } (0, A) \times \Omega, \\ u(t, 0, x) &= b(t, x) \text{ in } (0, T) \times \Omega, \\ u_t + u_a + \lambda u &= k(a) \Delta P + f \text{ in } Q. \end{aligned} \tag{VII}$$

If  $u$  belongs to  $L^2(Q)$  and satisfies (6) and if  $(f, b, u_0)$  satisfies (24), a priori bounds can be derived from Eq. (VI) and (VII).

For the parabolic equation (VI) we have:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla P(T, x)|^2 dx + \int_{Q_T} [\lambda |\nabla P|^2 + K |\Delta P|^2] dt dx \\ = \frac{1}{2} \int_{\Omega} |\nabla p_0|^2 dx + \int_{Q_T} \left[ \int_0^A f(t, a, x) da + b - u(t, A, x) \right] [-\Delta P] dt dx. \end{aligned} \tag{25}$$

As for the first-order equation (VII), integrating by parts yields:

$$\begin{aligned} \frac{1}{2} \int_Q u^2(t, A, x) dt dx + \lambda \int_Q u^2 dt da dx \leq \int_Q [f - k(a) \Delta P] u dt da dx \\ + \frac{1}{2} \int_{Q_T} b^2(t, x) dt dx + \frac{1}{2} \int_{(0, A) \times \Omega} u_0^2 da dx. \end{aligned} \tag{26}$$

Employing the method of separation of variables and (25), (26) we prove for problem (V) the analogue to Theorem 4. Via a fixed-point method we get Theorem 3 for bounded  $\mu$ . It suffices to use a suitable linear combination of (25) and (26).

Now we want to turn to the general case, namely, the case when  $\mu$  is not bounded.  $\mu$  can be approximated by a sequence  $(\mu_n)$  satisfying (18); for each integer there is a unique  $u_n$  in  $L^2(Q)$  verifying (6) and solution to problem (II); that is,  $u_n$  and  $P_n(t, x) = \int_0^A u(t, a, x) da$  are solutions to (VI) and (VII) with:

$$f = -\mu_n u_n \text{ in } Q, \quad b = \int_0^A \beta u_n da \text{ in } (0, T) \times \Omega. \tag{27}$$

From (26) we can get an estimate on  $(\mu_n^{1/2} \cdot u_n)_n$  in  $L^2(Q)$  but not on the term  $-\int_0^A \mu_n \cdot u_n da$  that appears on the right-hand side of (25) when  $f$  is given by (27). However, when the condition  $(\mu)^5$  is satisfied we have:

$$\left| \int_0^A \mu_n \cdot u_n da \right| \leq \int_0^A \mu_n^{1/2} \mu_n^{1/2} |u_n| da \leq m^{1/2} \cdot \left( \int_0^A \mu_n u_n^2 da \right)^{1/2}.$$

This is enough to obtain that  $(u_n)$  and  $(\mu_n^{1/2} \cdot u_n)$  (resp.  $P_n$ ) are bounded in  $L^2(Q)$  (resp.  $L^2(0, T; H^2(\Omega))$ ) and to prove Theorem 3 by letting  $n \rightarrow \infty$ .

When  $(\mu)^5$  is not fulfilled we shall derive from  $(\mu)^4$  that  $(\mu_n u_n)$  is bounded in the  $L^2(Q)$ -norm. If we multiply Eq. (VII) that  $u_n$  satisfies by  $\mu_n^2 u_n$  and if we integrate over  $Q$  we obtain:

$$\begin{aligned} & \int_Q \left[ \mu_n^2 + \lambda \mu_n - \frac{\partial \mu_n}{\partial t} - \frac{\partial \mu_n}{\partial a} \right] \mu_n u_n^2 dt da dx \\ & \leq \frac{1}{2} \int_{(0, A) \times \Omega} \left[ \mu_n^2(0, a, x) u_0^2(a, x) da dx + \frac{1}{2} \int_{Q_T} \mu_n^2(t, 0, x) \left[ \int_0^A \beta u_n da \right]^2 dt dx \right. \\ & \qquad \qquad \qquad \left. - \int_Q [\mu_n k \Delta P_n] \cdot [\mu_n u_n] dt da dx. \right. \end{aligned}$$

We can choose the sequence  $(\mu_n)$  so that each  $\mu_n$  satisfies  $(\mu)^4$  uniformly with respect to  $n$ . Thus:

$$\begin{aligned} & (\lambda + \lambda_1) \int_Q \mu_n^2 u_n^2 dt da dx + \lambda_2 \int_Q \mu_n u_n^2 dt da dx \\ & \leq C_7 + C_8 \int_Q u_n^2 dt da dx + M^{1/2} \cdot \left[ \int_{Q_T} |\Delta P_n|^2 dt dx \right]^{1/2} \left[ \int_Q \mu_n^2 u_n^2 dt da dx \right]^{1/2} \tag{28} \end{aligned}$$

where  $C_7$  and  $C_8$  are independent of  $n$ . From (28), (25) and (26) are satisfied by  $u_n$  and  $P_n$  when  $(f, b)$  is given by (27) and the inequality

$$\left| \int_0^A \mu_n u_n da \right| \leq A^{1/2} \left( \int_0^A \mu_n^2 u_n^2 da \right)^{1/2}.$$

We derive that if  $\lambda$  has been chosen large enough:

$$\int_Q [u_n^2 + \mu_n^2 u_n^2] dt da dx + \int_{Q_T} |\Delta P_n|^2 dt dx \leq C_9, \text{ independent of } n.$$

So letting  $n \rightarrow \infty$  we conclude.

**Remark 3.** An alternative method is to approximate the kernel  $k$  in such a way that the condition  $(k)^4$  is satisfied (see Sec. 3).

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