

## A MATHEMATICAL MODEL OF SOLAR FLARES\*

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**Abstract.** The phenomenon of solar flares is modeled assuming that the magnetic field is force-free and that its evolution is quasi-static. This model is simplified so as to be tractable and yields a semi-linear elliptic equation in a halfplane depending on a parameter  $\lambda$  which describes the time evolution. It is proved that there are (at least) two branches of solutions which have distinct asymptotic behaviors at infinity. The upper branch exists for all  $\lambda > 0$ , but the lower branch exists only on a finite interval  $[0, \lambda^c]$ . As stable solutions must have the same asymptotic behavior as the lower branch of solutions, and as this is impossible after  $\lambda^c$ , we contend that no stable solution exists after  $\lambda^c$  and that a solar flare is thus triggered.

**I. Modeling of solar flares.** A solar flare is a bright eruption at the surface of the sun; flares may appear within minutes and fade within an hour; they occur in the so-called active regions of the sun, near sunspots. They consist in the rapid conversion of energy stored in the form of extensive current systems flowing in part under the visible surface of the sun, the photosphere (5,000°K), and in part in a very tenuous and hot medium called the corona (10<sup>6</sup> °K). The flare energy is released in the corona under various forms: motion (up to 300 km s<sup>-1</sup>), energetic particles, and radiations in various parts of the electromagnetic spectrum. Solar flares can perturb the earth's magnetic field and therefore the transmission of radio waves.

In order to motivate the model that we present below, we shall give more information on active regions and flares: in an active region, the magnetic field is very intense; its

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order of magnitude is 2,000 gauss; it is roughly normal to the surface. The spots usually occur in pairs the elements of which are close together; one of the pair behaves like a north magnetic pole and the other like a south magnetic pole, and they are separated by a neutral line. Arches of magnetic field join the elements of the pair (Fig. 1). For a long time, the outstanding manifestation of a flare was the rapid and extreme brightening observed in a red line of emission which is characteristic of hydrogen  $H_{\alpha}$ . Two long and thin ribbons of  $H_{\alpha}$  emission appear in a direction parallel to the neutral line.

Satellite observations make it possible now to observe the corona against the visible disk of the sun by its X-ray emission. X-ray coronal emission is strongly increased in the flaring region and appears to consist in a system of many loops rooted in the  $H_{\alpha}$  emission ribbons, producing a long arcade which forms a bridge above the line of magnetic polarity reversal (Fig. 2). A system of electric currents is built up prior to the flare by the progressive displacement of the points at which the coronal lines of force are rooted in the photosphere; it is the catastrophic evolution of this structure which is thought to cause the flare. By "catastrophic" we mean that the whole structure of the lines of force is destroyed by a very sudden and brutal evolution, corresponding to the fact that, for some physical reason, the previous structure can no longer exist.

Loop arcades and  $H_{\alpha}$  ribbons possess a systematic elongated structure which suggests that, on a sufficiently coarse scale, the magnetic structure of the corona is locally endowed with translational symmetry in a direction tangent to the Sun's surface. This of course ignores the also systematic separation of the structure in individual loops, which may play an important rôle; but it is thought that, as far as the global equilibrium is concerned, this translation-symmetric model is safe enough.

The diameter of the active region is of the order of several thousand of km, and the diameter of the Sun is 700,000 km; this leads one to consider locally the sun as a half-space and the corona as the complementary half-space; it is appropriate to consider a

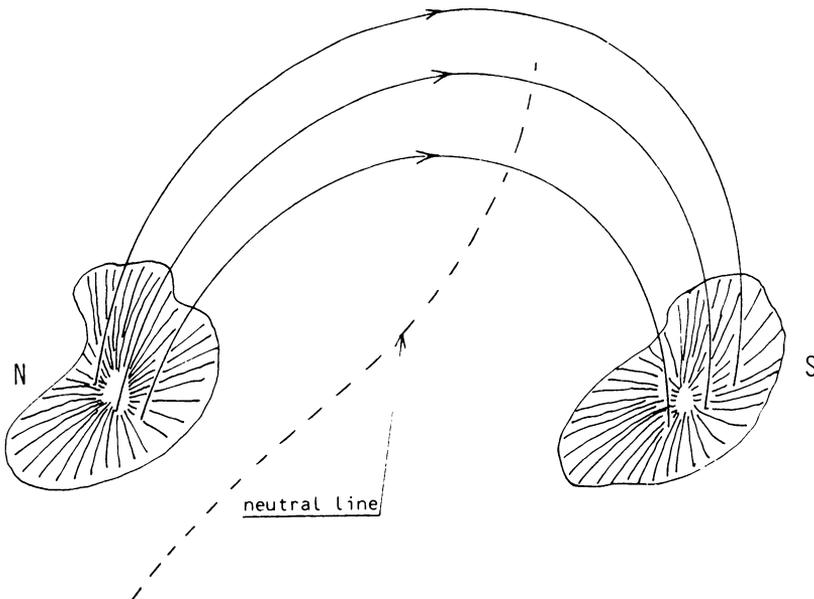


FIG. 1. Sunspots are cooler and appear darker in optical observations.

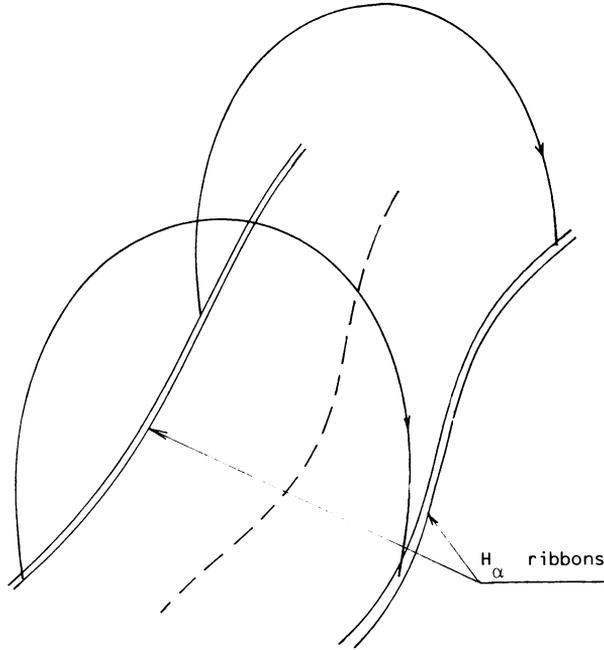


FIG. 2. By superposition of the  $H_\alpha$  and X-ray observation, loops of field appear to be rooted in the  $H_\alpha$  ribbons.

semi-infinite domain instead of a finite one, because the perturbations of the magnetic field may affect a region of the interplanetary medium which becomes rapidly larger than the sun itself.

Let us choose coordinates  $x, y, z$  as follows:  $y$  represents the radial direction;  $z$  is the direction of translational invariance; the plane  $x$ - $z$  is tangent to the surface of the sun (Fig. 3).

The magnetic field  $\mathbf{B}$  derives from the vector potential  $\mathbf{A} = (A_x, A_y, A_z)$

$$\mathbf{B} = \text{curl } \mathbf{A} = (\partial_y A_z, -\partial_x A_z, \partial_x A_y - \partial_y A_x).$$

It is convenient to set

$$\mathbf{B} = \left( \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}, B_{\parallel} \right). \quad (\text{I.1})$$

One immediately sees that along a line of force

$$\frac{dx}{\partial_y u} = -\frac{dy}{\partial_x u} = \frac{dz}{B_{\parallel}}. \quad (\text{I.2})$$

$u$  is a constant, so that curves of constant  $u$  are the projection on the  $x$ - $y$  plane, parallel to the  $z$ -axis, of actual field lines of  $\mathbf{B}$ .

The flare occurs at the end of a very slow evolution of the system of electric currents; this justifies considering a succession of equilibria, instead of a truly dynamic problem. These equilibria result from balancing several forces in the conducting gas (plasma) which constitutes the corona: Laplace forces produced by electric currents flowing in the corona ( $\mathbf{J} \times \mathbf{B}$ ), gas pressure forces, and gravity forces. Fortunately, it can be shown that gravity

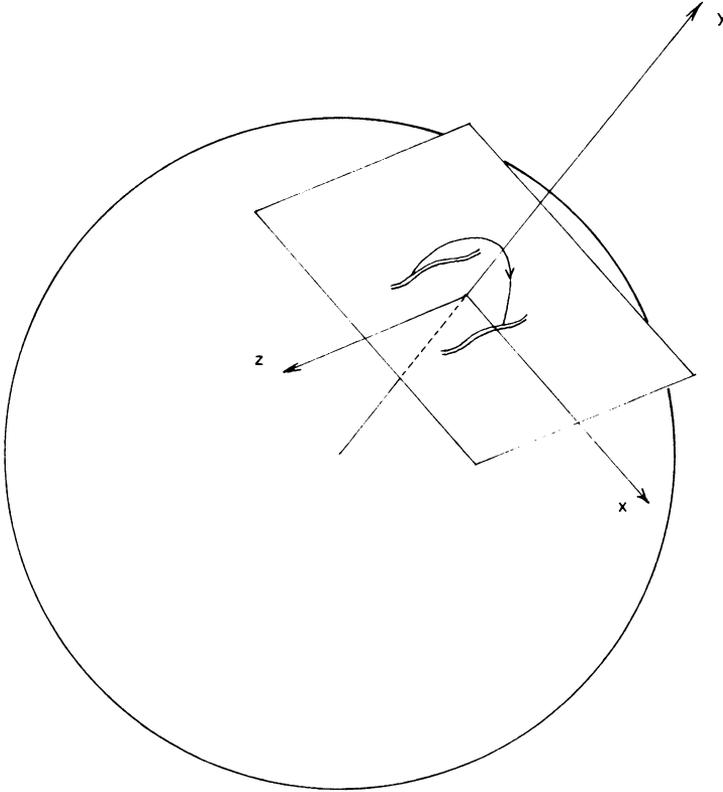


FIG. 3. The system of coordinates;  $z$  is along the direction of local translational invariance, and  $y$  is radial.

and pressure forces can be only a tiny fraction of any well-developed Laplace force. This is because the corona is very tenuous while the magnetic field is large. To a very good approximation, the magnetic structure and current flow must be free of Laplace forces:

$$\mathbf{J} \times \mathbf{B} = 0. \quad (\text{I.3})$$

This approximation, termed a force-free approximation, ceases to be valid near the visible surface of the sun.

The electric current density  $\mathbf{J}$  is, from Ampère's equation,  $\mathbf{J} = \text{curl } \mathbf{B}/\mu_0$ , so

$$J_x = \frac{1}{\mu_0} \frac{\partial B_{\parallel}}{\partial y}; \quad J_y = -\frac{1}{\mu_0} \frac{\partial B_{\parallel}}{\partial x}; \quad J_z = -\frac{1}{\mu_0} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (\text{I.4})$$

The vector equation (I.3), with (I.4), gives three equations, one of which expresses that  $B_{\parallel}$  and  $u$  are not independent;

$$\frac{\partial B_{\parallel}}{\partial x} \frac{\partial u}{\partial y} = \frac{\partial B_{\parallel}}{\partial y} \frac{\partial u}{\partial x}, \quad (\text{I.5})$$

so that locally  $B_{\parallel} = B_{\parallel}(u)$ , and the other two equations are both equivalent to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial u} \left( \frac{B_{\parallel}^2(u)}{2} \right) = 0. \quad (\text{I.6})$$

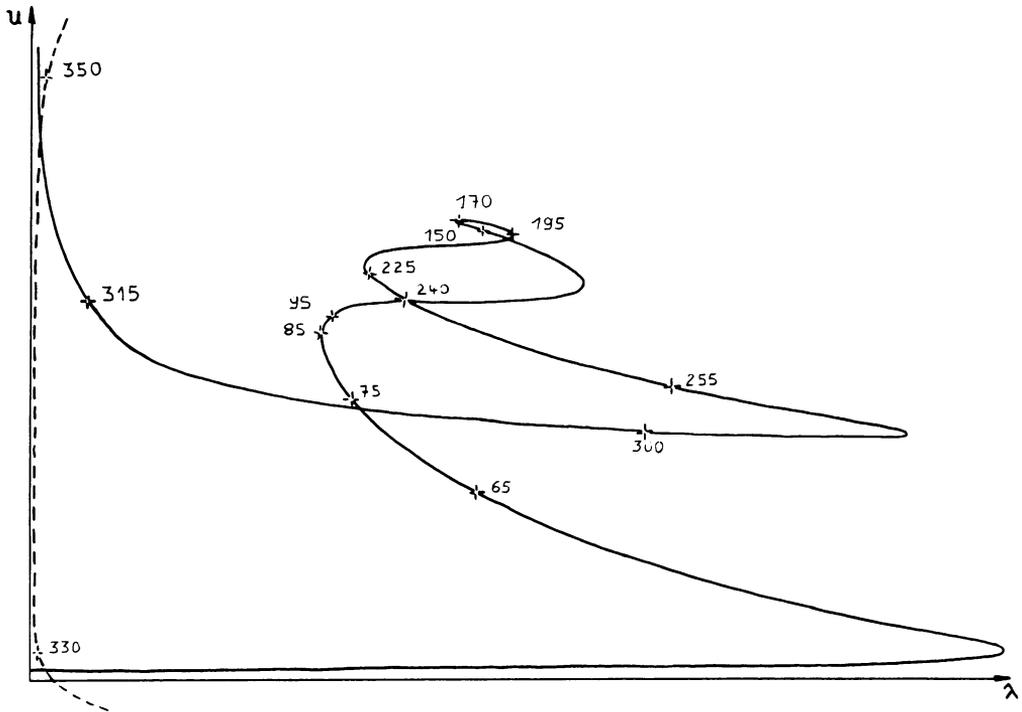


FIG. 4. Following the curve of solutions with  $\lambda$  in abscissa and  $u(x_0, y_0)$  in ordinate, where  $(x_0, y_0)$  is some definite point; the dotted line is the last part of the curve, with a scale four times smaller in  $u$ . The numbers indicated at particular points are the number of iterations.

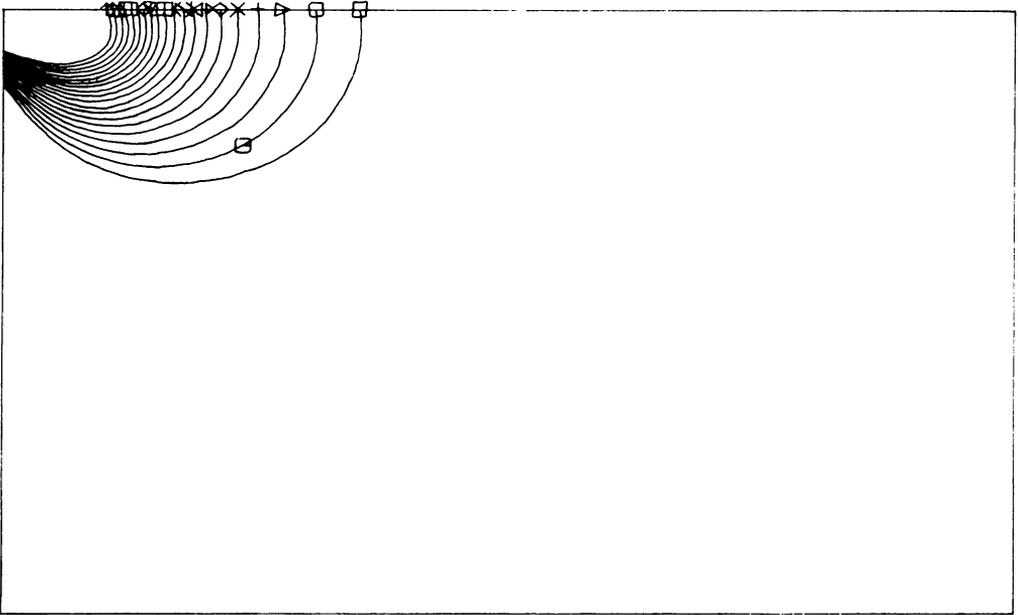
$B_{\parallel}$  is determined by the distance  $l(u)$  between the projections on the  $z$ -axis of the root points of the line of force  $u$ . This distance is one of the quantities which observation can give more easily, and it is the quantity which is most likely to be a monotonic function of time.  $B_{\parallel}$  and  $l$  are related by the equation of the line of force  $dz/B_{\parallel} = dx/\partial_y u$  which can be integrated along a line of force  $u(x, y) = \bar{u}$  to give

$$l(\bar{u}) = B(\bar{u}) \int_{x-(\bar{u})}^{x+(\bar{u})} \frac{dx}{\partial_y u}. \tag{I.7}$$

The derivative under the integral sign has to be evaluated at the points  $(x, y)$  such that  $u(x, y) = \bar{u}$ . This works for points  $(x, y)$  which can be connected by a field line to the boundary.

Other considerations must be used to specify the value of  $B_{\parallel}(x)$  on those lines which are not connected to the boundary. Up to now, no precise statement concerning this choice is available. We shall come back later to this question.

The system of equations (I.6) and (I.7) should in principal be solved for a set of functions  $l(u, t)$  describing progressive shearing of the root points of the field lines. This can be considered a very difficult problem; in order to present a tractable model, the assumption that the function  $B_{\parallel}(u)$  is known has been proposed. The physical significance of this assumption can be understood by noting that the vertical current  $J_z$ , emitted at the

FIG. 5.  $N = 65$ .

boundary  $y = 0$  in the domain of interest is, at a point  $(x, 0)$ ,

$$J_y = -\frac{1}{\mu_0} \frac{\partial B_{\parallel}}{\partial u} \frac{\partial u}{\partial x}.$$

Now  $u$  is known on the boundary because it is prescribed by the solar activity, and thus  $J_y$  will also be known provided  $\partial B_{\parallel}/\partial u$  or, equivalently,  $\frac{1}{2}(\partial/\partial u)(B_{\parallel}^2(u)) = F(u)$  is known. Instead of problem (I.6)–(I.7) with a set of functions  $l(u, t)$ , we shall consider problem (I.6) with a set of functions  $F(u, t)$  which we choose for simplicity as

$$F(u, t) = \lambda(t)f(u), \quad \lambda > 0.$$

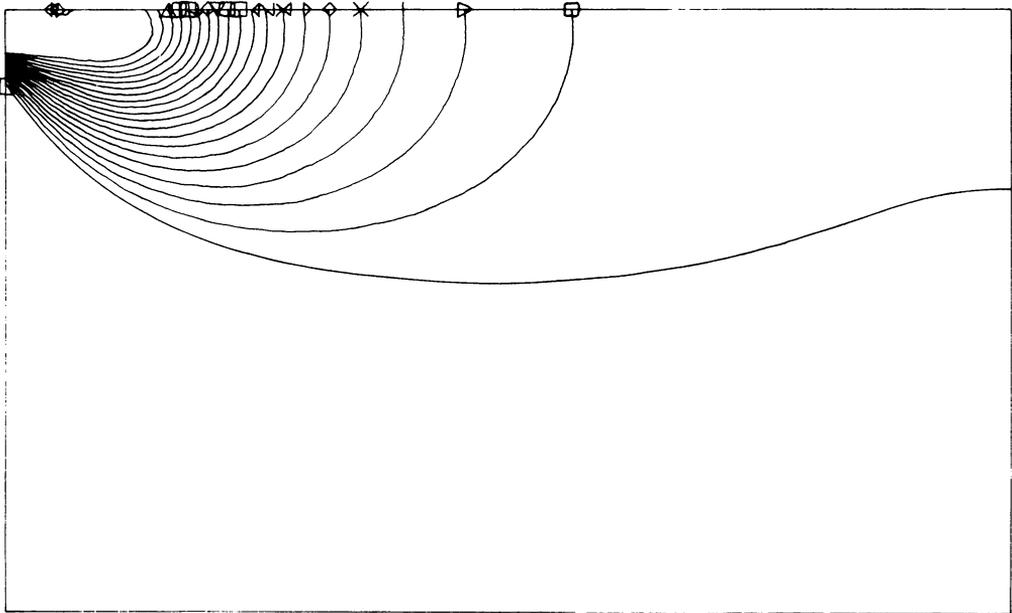
The evolution of the structure is modeled, requiring that  $\lambda$  be an increasing function of time. We recall that, up to a multiplicative factor,  $F(\bar{u})$  is the current density flowing on the line  $u = \bar{u}$  in the  $z$  direction. It also can be related to the other meaningful quantity  $J_y(x, 0)$  which represents the current density supplied by the interior of the sun to the external corona near point  $x$ . Actually,  $\partial B_{\parallel}/\partial u$  may be obtained from  $F(u)$ , provided that  $B_{\parallel}(u(x, 0))$  is known. Here we assume it to be zero.

By knowing the boundary conditions  $u(x, 0)$ , which do not change during the deformation,  $J_y$  can be calculated and is equal to

$$J_y(x, 0) = \sqrt{\lambda} j(x),$$

where  $j(x)$  is a known function of  $x$ .

With this in mind, we can see that some conditions must be imposed on  $f(u)$ . First, one has to exclude the possibility that  $J_y(x, 0)$  becomes infinite. This is obtained by im-

FIG. 6.  $N = 75$ .

posing regularity conditions on the function  $f(u)$ , and by demanding that

- (1)  $u(x, 0)$  be bounded and have continuous derivatives;
- (2)  $\lim_{|x| \rightarrow \infty} u(x, 0) = 0$ .

Notice that only derivatives of  $u$  are physically meaningful.

Moreover, the choice of  $f$  must be such that the current driven on the line of force  $u$  is consistent with physics. Physics can prescribe currents driven by the interior of the sun into the corona. It is not presently able to prescribe the current in field lines which do not connect to the boundary. A reasonable choice is to assume zero current in these regions. We then add the requirement:  $f(u)$  vanishes for values of  $u$  referring to field lines which do not connect to the boundary. In particular, if  $[a, b]$  is the interval of values of  $u(x, 0)$ ,  $x \in \mathbb{R}$ , we demand that

- (3)  $f(u) = 0$  if  $u \notin (a, b)$ .

Also, for the sake of simplicity and for no other reason, we assume  $f(u)$  to be strictly positive on  $(a, b)$ . This is the case for  $J_z$  flowing always in the same direction and corresponds to the simplest situation conceivable which represents a simple current bridge:

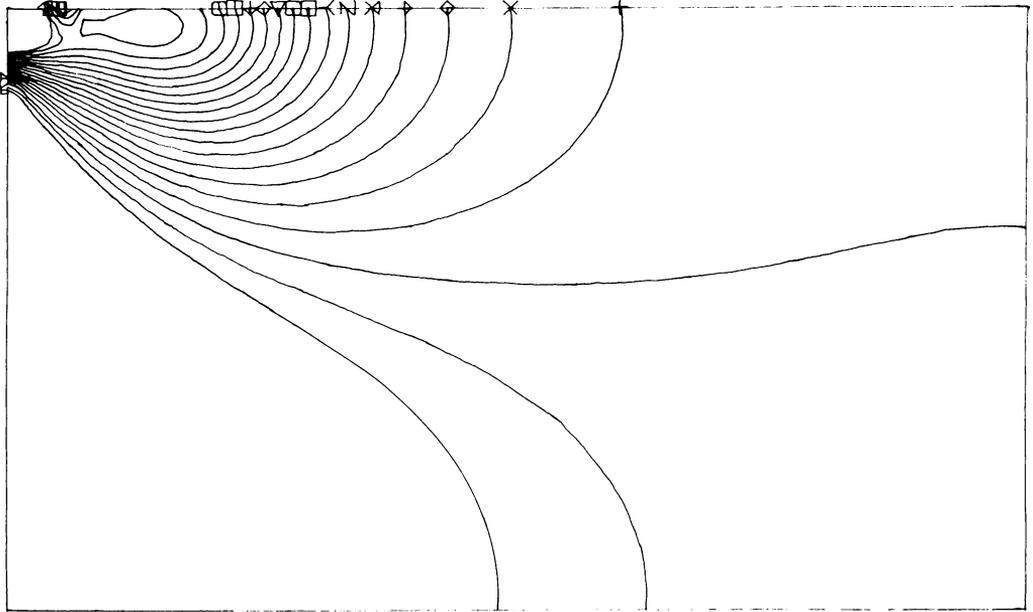
- (4)  $f(u) \geq 0$ .

Physically different effects can be expected when this condition is not fulfilled.

The main questions that we wish to answer are the following:

- (1) Do there exist solutions of this problem?
- (2) How many solutions exist?
- (3) What is the shape of their level lines (since this shape is an essential indication of the stability of the solutions)?

It is the purpose of the mathematical part of this paper to give some answers to the

FIG. 7.  $N = 85$ .

above questions; these answers will be discussed in the conclusion together with the information given by the numerical study of the problem.

## II. Mathematical results.

*II.1. Introduction.* In accordance with Sec. I, we consider the following boundary-value problem depending on the nonnegative parameter  $\lambda$ :

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega = \mathbb{R} \times (0, +\infty), \quad u(x, 0) = g(x), \quad \forall x \in \mathbb{R}. \quad (\text{II.1})$$

We seek twice continuously differentiable solutions and we demand that they be bounded and non-negative.

We make the following assumptions on  $f$  and  $g$ :

The function  $g$  and its first two derivatives  $g'$  and  $g''$  are bounded and continuous on  $\mathbb{R}$ ;  $g''$  is Hölder-continuous:

$$|g''(x) - g''(y)| \leq l|x - y|^\nu, \quad \forall x, y \in \mathbb{R}, \quad (\text{II.2})$$

The function  $f$  is Lipschitz-continuous from  $\mathbb{R}^+$  into itself with Lipschitz constant  $k$ :

$$|f(\alpha) - f(\beta)| \leq k|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}^+. \quad (\text{II.3})$$

They also satisfy the following assumptions:

There exists a positive number  $\alpha_0$  such that

$$\text{if } \alpha \in (0, \alpha_0), f(\alpha) > 0; \quad \text{if } \alpha \geq \alpha_0, f(\alpha) = 0, \quad (\text{II.4})$$

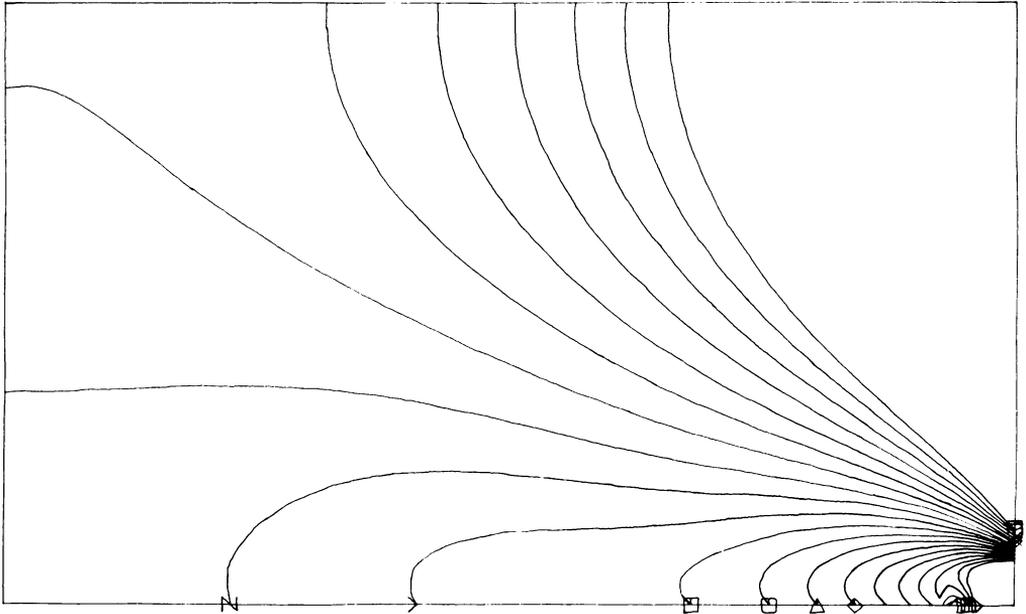


FIG. 8.  $N = 95$

There exist a positive constant  $C$  and a number  $s > 3$  such that  $f(\alpha) \leq C\alpha^s$  in a neighborhood of zero. (II.5)

There exist numbers  $L$  and  $n(s)$  such that  $L > 0$ ,  $n(s)$  lies in  $(2/(s - 1), 1)$ , and  $g$  satisfies

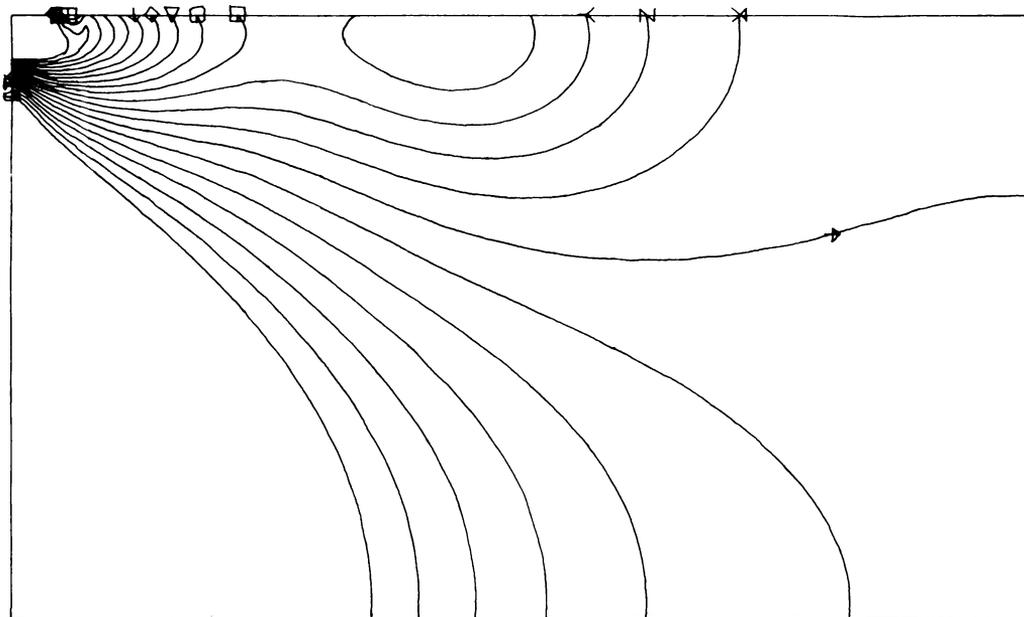
$$0 \leq g(x) \leq L(1 + |x|)^{-2n(s)}, \quad \forall x \in \mathbb{R}. \quad \text{(II.6)}$$

Of the above hypotheses, (II.2) and (II.3) are mild regularity assumptions; (II.4) is the translation of hypotheses (3) and (4) of the model, with the simplification that  $f$  is assumed positive between  $a$  and  $b$  and that  $a$  is normalized to be zero. We do not assume that  $g$  takes its values in the interval  $[0, \alpha_0]$ , because the mathematical treatment ignores this condition.

The only hypotheses that cannot be justified on physical grounds are (II.5), (II.6). They are necessary to obtain the lower branch of solutions, using the method described below, and to give the precise asymptotic behavior of the upper branch of solutions.

We shall prove that (II.1) possesses at least two branches of solutions  $u_\lambda$  and  $\bar{u}_\lambda$ , which can be discriminated by their asymptotic behavior in the half-plane. Moreover, there exists a critical value of the parameter  $\lambda$ , say  $\lambda^c$ , such that for  $\lambda > \lambda^c$ , any solution has the same asymptotic behavior as  $\bar{u}_\lambda$ , and this behavior is incompatible with all the level lines connecting one point of the boundary to another.

A number of authors have worked on semilinear elliptic problems in unbounded domains. Let us cite R. Chiapinelli, T. Küpper, and C. A. Stuart who studied differential problems on the half-line in Hilbert spaces (see [5, 15, 16, 17, 23, 24]). K. Kirchgässner and J. Scheurle [14] studied a problem in  $\mathbb{R} \times \Omega$ , where  $\Omega$  is bounded. A. Bahri, H. Berestycki, M. J. Esteban and P. L. Lions have studied problems in  $\mathbb{R}^n$  by variational and topological techniques (see [2, 3, 4, 8, 18], and the authors quoted there).

FIG. 9.  $N = 150$ .

Some of the results that we present here are true only in the particular case of problem (II.1) with hypotheses (II.2)–(II.6). Some others will be generalized in a subsequent work. It seems that this work is the first occurrence of bifurcation in a Frechet space, as we obtain it in Sec. II.7. Our techniques consist of a systematic use of the maximum principle together with sharp estimates and, in particular, estimates on the asymptotic behavior of the solution.

*II.2. Summary of functional results.* The main two ingredients needed in this work are a simple form of the Phragmén-Lindelöf Principle and some Agmon-Douglis-Nirenberg estimates.

*II.2a. Phragmén-Lindelöf Principle.* A consequence of the corollary of Theorem 19 of Protter and Weinberger ([21], p. 99) can be stated as follows:

**PROPOSITION 1.** Let  $\Omega$  be an open connected set in  $\mathbb{R}^n$  and let  $L$  be an operator of the form

$$Lu = - \sum_{i,j} a_{ij} u_{x_i} u_{x_j} + \sum_i b_i u_{x_i} + cu,$$

where  $a_{ij}$  is uniformly elliptic, i.e., there exists an  $\alpha > 0$  such that

$$\sum_{i,j} a_{ij} \xi_i \xi_j \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^n$$

and where  $c \geq 0$ . Assume that the coefficients  $a_{ij}$ ,  $b_i$ ,  $c$  are bounded and that there exists a

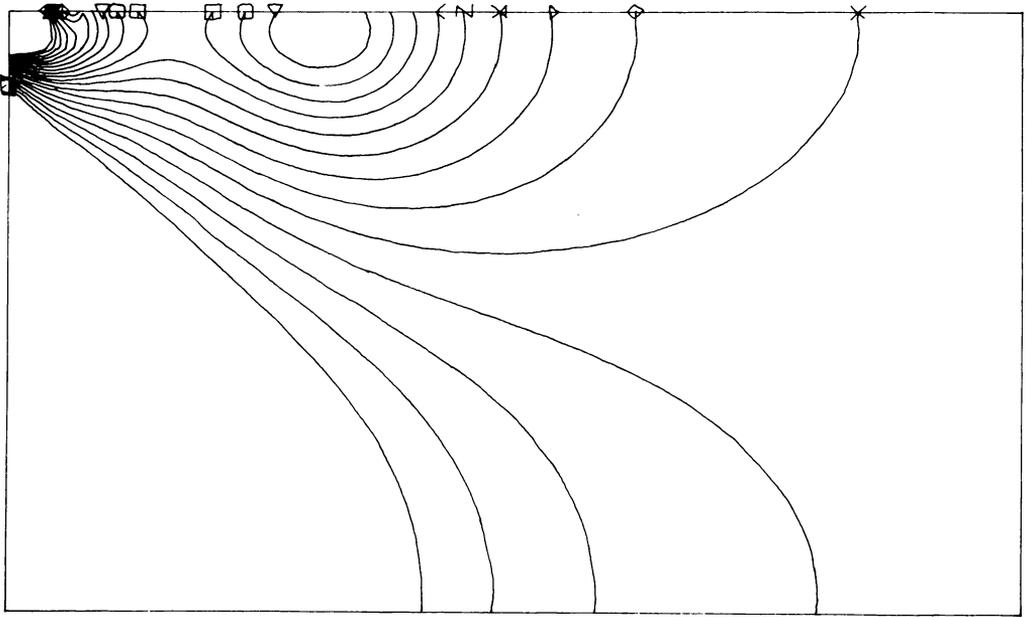


FIG. 10.  $N = 170$ .

function  $w$  of class  $C^2$  such that

$$w > 0 \quad \text{on } \Omega, \tag{II.7}$$

$$Lw \geq 0 \quad \text{on } \Omega, \tag{II.8}$$

$$\lim_{|x| \rightarrow \infty} w(x) = \infty \quad \text{if } \Omega \text{ is unbounded.} \tag{II.9}$$

Let  $u$  be of class  $C^2$  and satisfy

$$u \leq 0 \quad \text{on } \partial\Omega, \quad Lu \leq 0 \quad \text{on } \Omega, \tag{II.10}$$

$$\lim_{A \rightarrow \infty} (\sup (u(x)/w(x); \quad w(x) = A, x \in \Omega)) \leq 0. \tag{II.11}$$

Then

$$u \leq 0 \quad \text{in } \Omega.$$

Notice that there is no assumption of smoothness on the boundary of  $\Omega$  or on the coefficients.

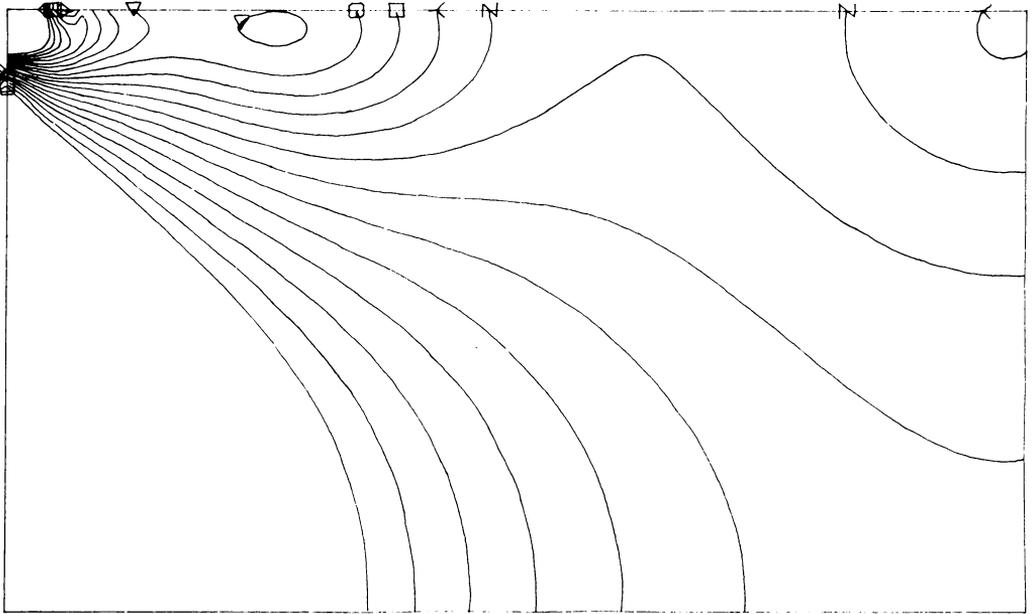
We deduce from Proposition 1 the following result, the proof of which is obvious:

LEMMA 2. Let  $K$  be a non-negative constant. The problem

$$-\Delta v + Kv = h \quad \text{in } \Omega = \mathbb{R} \times \mathbb{R}^+ \quad v = g \quad \text{on } \partial\Omega = \mathbb{R} \times \{0\},$$

possesses at most one classical bounded solution.

II.2b. Some Agmon-Douglis-Nirenberg estimates. We recall the definition of the spaces needed:  $L^p$  is the space of measurable functions whose  $p$ th power is integrable;  $L^\infty$

FIG. 11.  $N = 195$ .

is the space of essentially bounded measurable functions;  $W^{m,p}(\Omega)$ , where  $\Omega$  is an open set, is the space of functions which lie in  $L^p(\Omega)$  as well as their derivatives (in the sense of distributions) up to order  $m$ ;  $C^{m,\gamma}(\Omega)$  is the space of functions which are  $m$  times continuously differentiable on  $\Omega$  and whose  $m$ th derivatives satisfy a Hölder condition of order  $\gamma$ .

**PROPOSITION 3.** Let  $K$  be a positive number and let  $h$  be in  $L^\infty(\mathbb{R} \times \mathbb{R}^+)$  and  $g$  in  $W^{2,\infty}(\mathbb{R})$ . Then there exists a unique function  $u$  such that

$$u \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+) \cap W_{loc}^{2,p}(\mathbb{R} \times \mathbb{R}^+), \quad \forall p \in [1, \infty), \quad (\text{II.12})$$

$$-\Delta u + Ku = h \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \quad (\text{II.13})$$

$$u(x, 0) = g(x), \quad \forall x \in \mathbb{R}. \quad (\text{II.14})$$

If we assume that for some  $\gamma$  between 0 and 1  $g$  belongs to  $C^{2,\gamma}(\mathbb{R})$  and  $h$  belongs to  $C^{0,\gamma}(\mathbb{R} \times \mathbb{R}^+)$ , then  $u$  is in  $C^{2,\gamma}(\mathbb{R} \times \mathbb{R}^+)$ . If we assume that  $h$  and  $g$  are non-negative, so is  $u$ .

*Proof.* As this matter is very classical, we shall not give any details. First, under the hypotheses of Proposition 3, one proves by a variational argument that the problem

$$-\Delta u + ku = h \psi \quad \text{on } \Omega = \mathbb{R} \times \mathbb{R}^+ \quad u(x, 0) = g(x) \phi(x), \quad x \in \mathbb{R} \quad (\text{II.15})$$

possesses a unique solution in  $H^1(\Omega)$  if  $\phi$  is a truncation function belonging to  $\mathcal{D}^+(\mathbb{R})$ , with values in the interval  $[0, 1]$ , and  $\psi$  is a similar truncation function in  $\mathcal{D}(\bar{\Omega})$ . Then

$$\|u\|_{L^\infty(\Omega)} \leq \max(\|g\|_{L^\infty(\mathbb{R})}, \|h\|_{L^\infty(\mathbb{R})} \cdot K^{-1}),$$

and it is easy to show that  $u$  belongs to  $W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$ ,  $\forall p \in [1, \infty)$ , using the Agmon-Douglis-Nirenberg estimates to yield existence, as in Geymonat and Grisvard [9].

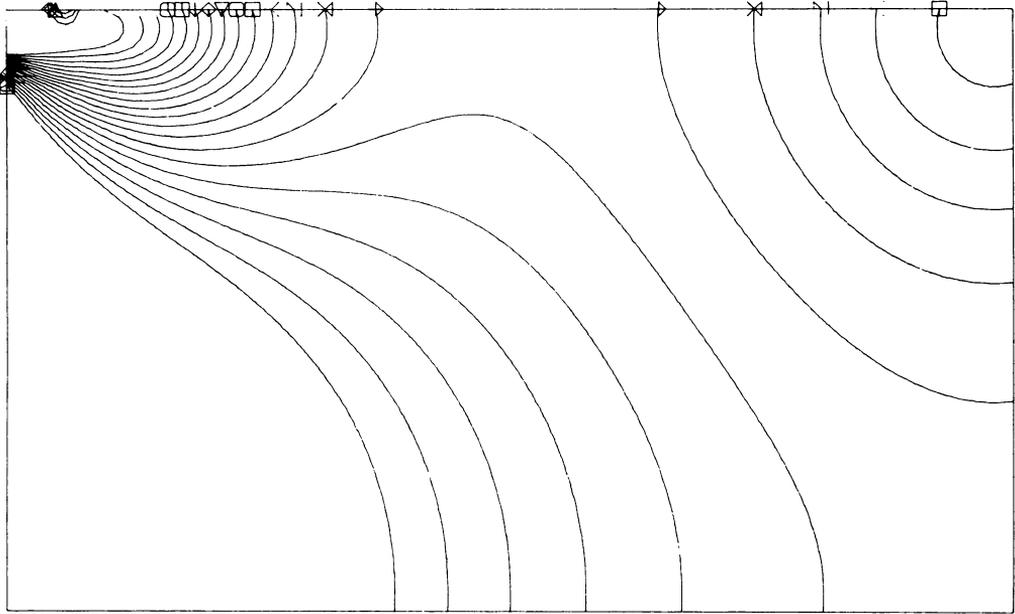


FIG. 12.  $N = 225$ .

Now, if  $\zeta$  is a smooth function on  $\Omega$ , supported by the half-disk

$$\Sigma(r) = \{y \geq 0, x^2 + y^2 \leq r^2\}$$

with

$$\zeta \Big|_{\Sigma(r/2)} = 1, \quad 0 \leq \zeta \leq 1,$$

one can write the Agmon-Douglis-Nirenberg estimates [1] for  $r$  small enough:

$$|u\zeta|_{H^2(\Omega)} \leq (C_2(|\Delta(u\zeta)|_{L^2(\Omega)} + |u\zeta|_{H^{3/2}(\mathbb{R})} + |u\zeta|_{L^2(\Omega)}),$$

so that

$$|\nabla u|_{L^p(\Sigma(r/2))} \leq C'_p |u\zeta|_{H^2(\Omega)}.$$

Iterating this process gives, for a suitable  $r_p$  and for any  $p < +\infty$ ,

$$|u|_{W^{2,p}(\Sigma(r_p/2))} \leq K(p, |h\psi|_{L^\infty(\Omega)}, |g\varphi|_{W^{2,\infty}(\mathbb{R})}).$$

As this kind of estimate is invariant by translation along the  $x$ -axis, we deduce from the above inequality that, for  $p > 2$ ,

$$|\nabla u|_{L^\infty(\mathbb{R} \times (0, r_p/2))} \leq K'_p K(p, |h\psi|_{L^\infty(\Omega)}, |g\varphi|_{W^{2,\infty}(\mathbb{R})}).$$

The interior estimates are obtained in the same fashion, by even simpler arguments, and they yield (II.12) for (II.15). But the estimates do not in fact depend on the truncations  $\varphi$  and  $\psi$ ; by a standard limiting process, we obtain (II.12) for problem (II.13), (II.14).

The next assertion of Proposition 2 is obtained by classical Hölder estimates, and the last is clear.

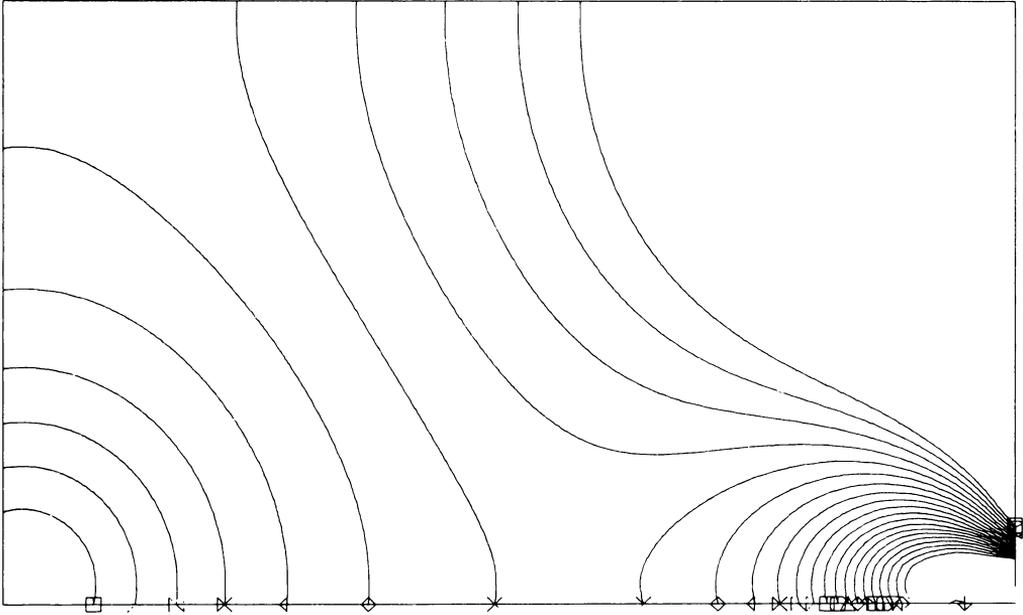


FIG. 13.  $N = 240$ .

II.3. *An a priori estimate.*

THEOREM 4. Any bounded non-negative classical solution of (II.1) satisfies the estimate

$$\sup_{\Omega} u(x, y) \leq \alpha_1 \stackrel{\text{def}}{=} \max\left(\alpha_0, \sup_{\mathbb{R}} g\right). \tag{II.16}$$

*Proof.* Let

$$\Omega' = \{(x, y) \in \Omega / u(x, y) > \alpha_1\},$$

and let  $\omega$  be a connected component of  $\Omega'$ . On  $\omega$ ,

$$-\Delta u = 0.$$

We choose  $w(x, y) = \log(x^2 + (y + 1)^2)$ ; obviously,  $w$  satisfies hypotheses (II.6)–(II.8) on  $\Omega \supset \omega$ , and  $u - \alpha_1|_{\omega}$  satisfies (II.10), because it is bounded. Moreover, by definition of  $\alpha_1$ ,  $u - \alpha_1|_{\partial\omega} \leq 0$ ; thus we may apply Proposition 1 and we get Theorem 4.

II.4. *Study of an associated ordinary differential equation.* We wish to obtain information concerning the  $x$ -independent case of (II.1). Thus we shall study the ordinary differential equation

$$-v'' = \lambda f(v) \quad \text{on } \mathbb{R}^+ \quad v(0) = 0 \tag{II.17}$$

and the relevant results are summarized by next proposition.

PROPOSITION 5. Let  $\lambda$  be positive. The problem (II.17) possesses a unique, bounded, non-negative, non-identically-zero solution of class  $C^2$ , and the zero solution. Moreover, the limit of the non-zero solution as  $y$  tends to  $+\infty$  is  $\alpha_c$ .

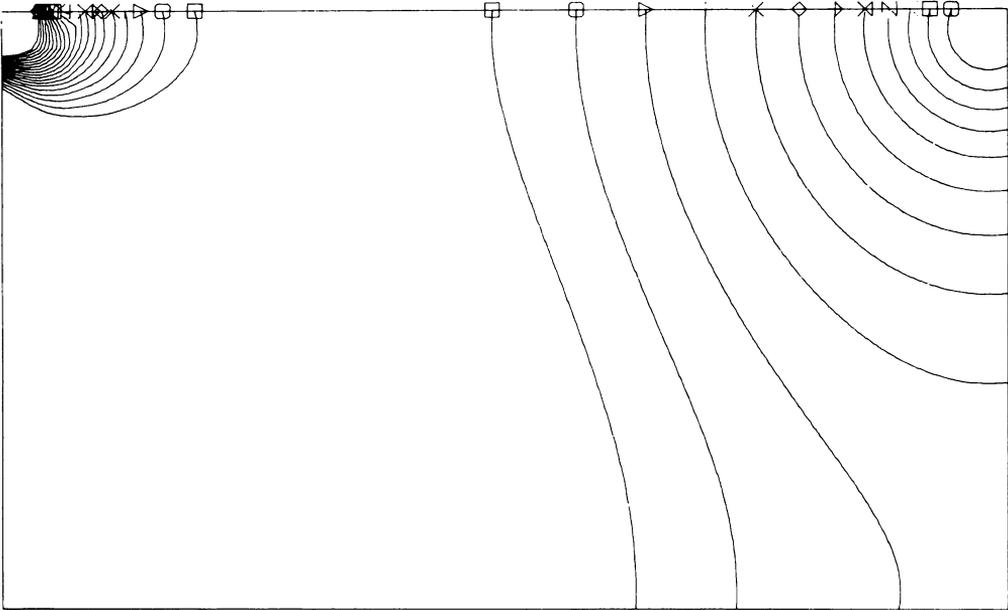


FIG. 14.  $N = 255$ .

*Proof.* Suppose that there exists a solution of (II.17) which is not identically zero; as  $f$  is non-negative, this solution, which we call  $v$ , is concave; in order for  $v$  to be bounded, the decreasing function  $v'$  must go to zero as  $y$  goes to infinity. So,  $v$  increases. The limit of  $v$  as  $y$  goes to infinity cannot be strictly smaller than  $\alpha_0$ ; if it were, then  $-v''(y)$  would converge to a strictly positive number, by (II.4); and this contradicts the boundedness of  $v$ . The limit of  $v$  cannot be larger than  $\alpha_0$  because there would exist in that case a  $y_0 > 0$  such that, for  $y > y_0$ ,  $v(y) > v(y_0)$  and  $v(y_0) = \alpha_0$ ; the speed  $v'(y)$  would be constant and positive for  $y > y_0$  and  $v$  could not stay bounded. Therefore,

$$\lim_{y \rightarrow \infty} v(y) = \alpha_0, \quad \lim_{y \rightarrow \infty} v'(y) = 0. \tag{II.18, .19}$$

If we multiply the equation of (II.17) by  $v'$  and integrate, we obtain

$$\frac{1}{2} |v'(0)|^2 + \lambda F(0) = \frac{1}{2} |v'(y)|^2 + \lambda F(v(y)) = \frac{1}{2} |v'(\infty)|^2 + \lambda F(v(\infty)), \tag{II.20}$$

where  $F$  is a primitive of  $f$ . Thanks to (II.18) and (II.19), we deduce that

$$v'(0) = \left( 2\lambda \int_0^{\alpha_0} f(r) dr \right)^{1/2} \tag{II.21}$$

is the only possible initial value for  $v'$ .

It is now straightforward to show that the initial conditions  $v(0) = 0$  and  $v'(0)$  given by (II.21) provide a bounded non-negative solution of (II.17) which is not identically zero and is unique by (II.21).

*II.5 Proof of existence by supersolutions and subsolutions.* To prove existence, we use a general algorithm which is described in Courant and Hilbert [7, p. 369] and has been

used by many authors. Suppose that we know two functions  $u^0$  and  $u_0$  such that

$$u^0 \geq u_0, \quad (\text{II.22})$$

$$-\Delta u^0 \geq f(u^0) \quad \text{in } \Omega, \quad u^0 \Big|_{\partial\Omega} \geq g, \quad (\text{II.23})$$

$$-\Delta u_0 \leq f(u_0) \quad \text{in } \Omega, \quad u_0 \Big|_{\partial\Omega} \leq g. \quad (\text{II.24})$$

The function  $u^0$  is called a supersolution and the function  $u_0$  is called a subsolution. Then, we shall define two monotonic sequences  $u^n$  and  $u_n$  by

$$\begin{aligned} -\Delta u^{n+1} + \lambda K u^{n+1} &= \lambda f(u^n) + \lambda K u^n \quad \text{in } \Omega, & u^{n+1} \Big|_{\partial\Omega} &= g, \\ -\Delta u_{n+1} + \lambda K u_{n+1} &= \lambda f(u_n) + \lambda K u_n \quad \text{in } \Omega, & u_{n+1} \Big|_{\partial\Omega} &= g. \end{aligned}$$

where  $K$  is chosen such that  $K + f'(r) > 0$ ,  $\forall r$ .

We have to check only that in the case of an infinite domain, and with our choice of functional spaces, the standard argument of convergence still holds. We take  $u^0$  and  $u_0$  in  $C^{2,\gamma}(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ . Assume that  $u^n$  is in this same space; then  $\lambda f(u^n) + \lambda K u^n$  is in  $C^\gamma(\bar{\Omega}) \cap L^\infty(\Omega)$  and, by Proposition 3,  $u^{n+1}$  is in  $W^{1,\infty}(\Omega) \cap C^{2,\gamma}(\bar{\Omega})$ . Thus  $u^n$  is in  $C^{2,\gamma}(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$  for all  $n$ , and the same result holds for  $u_n$ .

From hypothesis (II.23) and the definition of  $u^1$ , we get

$$-\Delta(u^1 - u^0) + \lambda K(u^1 - u^0) \leq 0, \quad u^1 - u^0 \Big|_{\partial\Omega} \leq 0$$

so that, by Proposition 3, we have  $u^1 \leq u^0$ . Similarly, if we assume that  $u^{n-1} \geq u^n$ , we can deduce from Proposition 3 that  $u^{n+1} \leq u^n$  with the hypothesis  $K + f'(r) < 0$  for all  $r$ . Thus the sequence  $(u^n)$  decreases, and analogously the sequence  $(u_n)$  increases, and it is easy to see that

$$u^0 \geq u^1 \geq \dots \geq u^n \geq u^{n+1} \geq \dots \geq u_{n+1} \geq u_n \geq \dots \geq u_1 \geq u_0.$$

Let  $u^\infty(x) = \inf_n u^n(x)$ , and  $u_\infty(x) = \sup_n u_n(x)$ . Proposition 3 gives uniform  $W_{\text{loc}}^{2,p}$  bounds on  $u^n$  and  $u_n$ , for any finite  $p$ ; thus

$$u^n \rightharpoonup u^\infty \quad \text{in } W_{\text{loc}}^{2,p} \text{ weakly,} \quad u_n \rightharpoonup u_\infty \quad \text{in } W_{\text{loc}}^{2,p} \text{ weakly,}$$

and

$$u^n \rightarrow u^\infty \quad \text{uniformly on compact sets,} \quad u_n \rightarrow u_\infty \quad \text{uniformly on compact sets,}$$

and in the limit

$$-\Delta u^\infty = \lambda f(u^\infty), \quad -\Delta u_\infty = f(u_\infty).$$

Finally, we conclude from Proposition 3 that  $u^\infty$  and  $u_\infty$  are in  $C^{2,\gamma}(\bar{\Omega})$ .

The point now is to exhibit explicitly supersolutions and subsolutions. We shall give two couples of these, in order to prove that there exist two branches of solutions.

*Upper supersolution.* Let

$$\bar{w}_\lambda = \alpha_1,$$

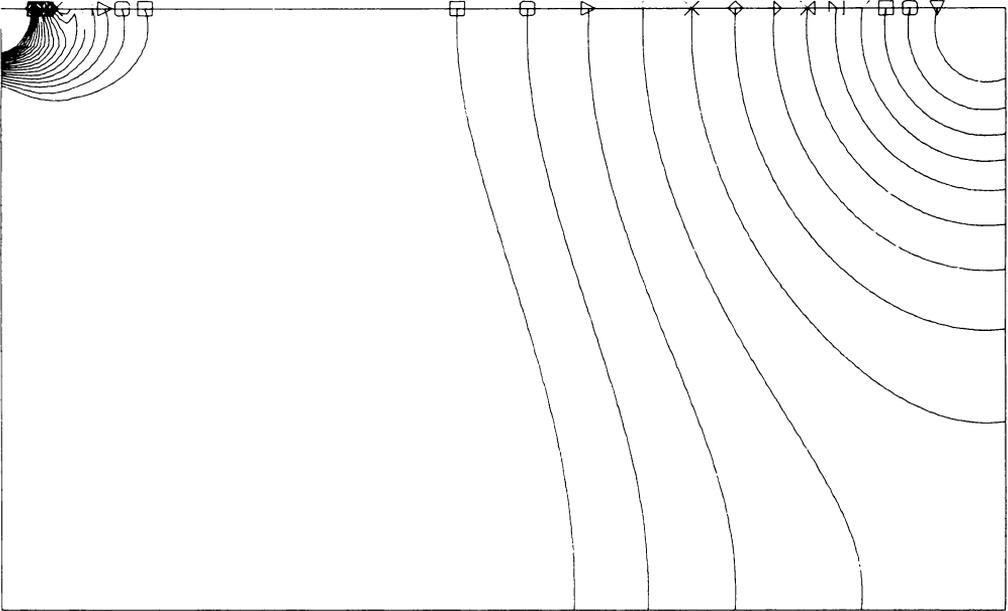


FIG. 15.  $N = 300$ .

which is obviously a supersolution and estimates from above any solution of (II.1) (see Theorem 4).

*Upper subsolution.* Let  $w_\lambda$  be the solution of (II.17) which is not identically zero; it is obviously a subsolution, and  $w_\lambda \leq \bar{w}_\lambda$ ; therefore we can apply the algorithm defined above. Let  $\bar{u}_\lambda$  denote the limit of the sequence  $\bar{w}_\lambda^n$  of supersolutions defined from  $\bar{w}_\lambda$ . This limit  $\bar{u}_\lambda$  is the maximum solution of (II.1), by virtue of the estimate (II.16), and moreover,

$$\lim_{y \rightarrow \infty} \bar{u}_\lambda(x, y) = \alpha_0, \quad \text{uniformly in } x. \tag{II.25}$$

To prove this fact, we need a refinement of estimate (II.16). Let  $\hat{w}$  be the bounded solution of

$$-\Delta \hat{w} = 0 \text{ in } \Omega, \quad \hat{w}(x, 0) = \max(\alpha_0, g(x)).$$

Obviously,  $\hat{w}(\cdot, 0) - \alpha_0$  is compactly supported, thanks to hypothesis (II.6), and  $\hat{w}$  is given explicitly by

$$\hat{w}(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y(g(x') - \alpha_0)^+}{(x - x')^2 + y^2} dx' + \alpha_0,$$

which has the limit  $\alpha_0$  as  $y$  tends to infinity uniformly in  $x$ , and is greater than  $\alpha_0$ .

Let  $u$  be any classical, bounded, non-negative solution of (II.1),

$$\Omega' = \{(x, y) \in \Omega / u(x, y) > \hat{w}(x, y)\},$$

and let  $\omega$  be a connected component of  $\Omega'$ . On  $\omega$ ,

$$-\Delta u = 0,$$

and on the boundary of  $w$ ,  $u - \hat{w}$  is nonpositive. If we choose  $w$  as in Theorem 4, we may apply Proposition 1 and get for any solution  $u$

$$u(x, y) \leq \hat{w}(x, y) \quad \text{on } \Omega. \quad (\text{II.26})$$

Now  $\bar{u}_\lambda$  is between  $w_\lambda$  and  $\hat{w}$  which both converge to  $\alpha_0$  as  $y$  goes to infinity, uniformly in  $x$ , which proves the claim (II.25).

*Lower supersolution* (for  $\lambda$  small enough). We utilize hypothesis (II.6), from which we deduce that there exist  $\bar{L}$  and  $a$  such that  $g(\alpha) \leq \bar{L}(a/(x^2 + a^2))^{n(s)}$ , and we contend that

$$\bar{z}_\lambda(x, y) = \bar{L} h(x, y)^{n(s)}, \quad \text{with } h(x, y) = \frac{y + a}{x^2 + (y + a)^2} \quad (\text{II.27})$$

is a supersolution for small enough  $\lambda$ . First, notice that  $h$  is harmonic and that

$$-\Delta \bar{z}_\lambda = -\bar{L} \Delta(h^{n(s)}) = -n(s)(n(s) - 1)h^{n(s)-2} |\nabla h|^2 \bar{L};$$

we shall have

$$-\Delta \bar{z}_\lambda \geq \lambda f(\bar{z}_\lambda)$$

if

$$\inf_{x, y} \{ -n(s)(n(s) - 1)h(x, y)^{n(s)-2} |\nabla h(x, y)|^2 \} \geq \lambda C(\bar{L})^{s-1} h^{sn(s)},$$

which is possible if

$$(x^2 + (y + a)^2)^{(s-1)n(s)} / (y + a)^{2+(s-1)n(s)} \quad (\text{II.28})$$

is bounded away from zero and if  $n(s) < 1$ .

But (II.28) is bounded away from zero if and only if  $2(s - 1)n(s) \geq 2 + (s - 1)n(s)$ , that is  $n(s) \geq 2/(s - 1)$ . Then, for

$$\lambda \leq n(s)(s-1)^{-2} (\bar{L})^{s-1} C^{-1} n(s)(1 - n(s)),$$

the function  $\bar{z}_\lambda$  defined by (II.27) is a supersolution.

*Lower subsolution.* There is the obvious subsolution

$$z_\lambda = 0.$$

We summarize this section with the following theorem which is now clear.

**THEOREM 6.** Under assumptions (II.1)–(II.6), problem (II.1) admits at least two branches of non-negative, classical, bounded solutions: a branch of maximum solutions  $\bar{u}_\lambda$  which satisfy for any  $\lambda$  positive the relation

$$\lim_{y \rightarrow \infty} \bar{u}_\lambda(x, y) = \alpha_0; \quad (\text{II.29})$$

a branch of minimum solutions  $u_\lambda$  which for small enough  $\lambda$  satisfy

$$\lim_{|x| + y \rightarrow \infty} u_\lambda(x, y) = 0. \quad (\text{II.30})$$

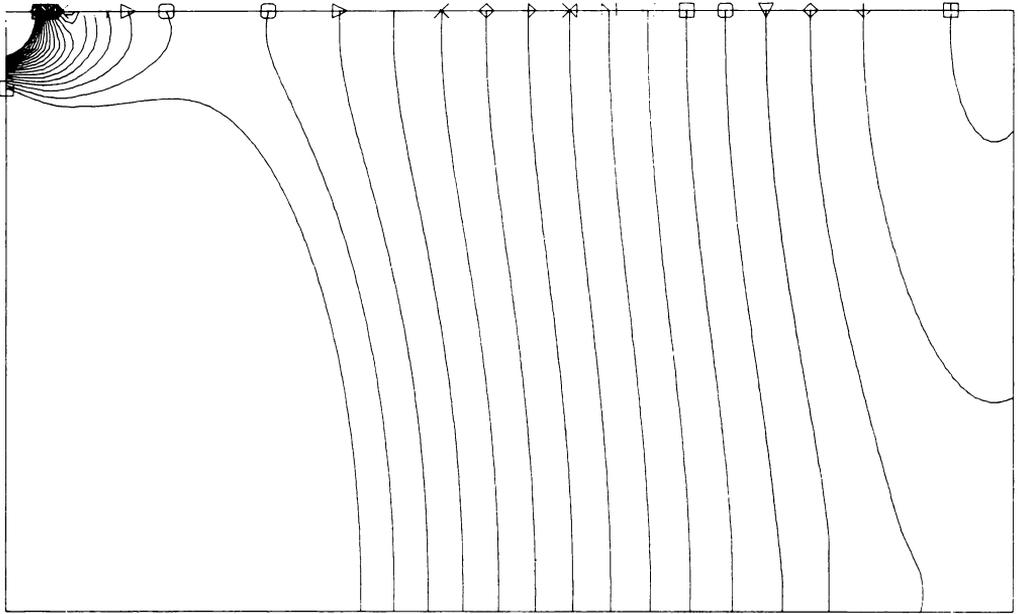


FIG. 16.  $N = 315$ .

**II.6. Behavior of the solutions at infinity.** We shall prove in this section that the solutions of (II.1) can either tend to zero or to  $\alpha_0$  as  $y$  tends to infinity; moreover, as  $x$  goes to infinity, the behavior of  $u$  is completely known, and we can have  $\lim_{y \rightarrow \infty} u(x, y) = 0$  only for a finite interval of  $\lambda$ .

**PROPOSITION 7.** Any solution of (II.1) has a limit as  $y$  goes to infinity; this limit is either 0 or  $\alpha_0$ , and is uniform in  $x$ .

*Proof.* Let  $B$  be a ball of center  $(x_0, y_0)$  and of radius  $r_0$ , ( $y_0 > r_0$ ), and let  $1_B$  be the function equal to 1 in  $B$  and to zero outside. Set

$$t = \inf_B u(x, y), \tag{II.31}$$

and let  $w$  be the unique bounded solution of

$$-\Delta w = 1_B \text{ in } \Omega, \quad w(x, 0) = 0. \tag{II.32}$$

We can compute  $w$  explicitly; it is given by

$$w(x, y) = \left( r_0^2 - (x - x_0)^2 - (y - y_0)^2 + r_0^2 \log \left( \frac{(x - x_0)^2 + (y - y_0)^2}{r_0^2} \right) \right) / 4$$

$$\text{if } (x - x_0)^2 + (y - y_0)^2 \leq r_0^2;$$

$$w(x, y) = \frac{r_0^2}{4} \log \frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2} \text{ if } (x - x_0)^2 + (y - y_0)^2 \geq r_0^2.$$

From the maximum principle,

$$u \geq \lambda t w \text{ in } \Omega, \tag{11.33}$$

and if we take into account the estimate (11.16) and take (11.33) at the center of  $B$ ,

$$\alpha_0 \geq \min_B f(u(x, y)) \frac{r_0^2}{4} \left( 1 + \log \frac{2y}{r_0} \right). \quad (11.34)$$

We deduce from (11.16) and Proposition 3 the estimate

$$|\Delta u|_{L^\infty(\Omega)} \leq C_1(1 + \lambda),$$

where  $C_1$  depends only on  $g$  and  $f$ .

Therefore, for any  $(x, y)$  in  $B$ ,

$$f(u(x, y)) \leq \min_B f(u(x', y)) + kC_1(1 + \lambda)2r_0, \quad (11.35)$$

where  $k$  is the Lipschitz constant of  $f$  (see condition (II.3)). From (II.34) and (II.35),

$$f(u(x, y)) \leq \frac{4C}{r_0^2(1 + 2 \log(2y/r_0))} + 2kC_1(1 + \lambda)r_0, \quad \forall (x, y) \in B. \quad (11.36)$$

Notice that  $x_0$  does not appear in (II.36), so that this inequality is fulfilled as soon as  $r_0 < y$ ; we can consider  $r_0$  as a parameter that we can choose at our convenience, provided it is not larger than  $y$ ; taking for  $r_0(y)$  some decreasing function of  $y$  such that, however,  $r_0^2(y) \log [2y/r_0(y)]$  increases infinitely, like  $r_0(y) = (\log(1 + 2y))^{-1/4}$ , we deduce that

$$\lim_{y \rightarrow \infty} f(u(x, y)) = 0, \quad (11.37)$$

and this limit is uniform in  $x$ . We conclude now from Theorem 6 and (II.37) that Proposition 7 is true.

We are now able to give the asymptotic behavior as  $x$  goes to infinity of all the solutions of (II.1).

**PROPOSITION 8.** Let  $u$  be a classical bounded non-negative solution of (II.1). Then either

$$\lim u(x, y) = 0 \quad \text{or} \quad \lim u(x, y) = \underline{w}_\lambda(y),$$

where  $\underline{w}_\lambda$  is the upper solution defined in Sec. II.5.

*Proof.* Let  $x_n$  be an arbitrary sequence of real numbers whose absolute value increases infinitely; set

$$u_n(x, y) = u(x - x_n, y), \quad g_n(x) = g(x - x_n)$$

which obviously satisfy

$$-\Delta u_n = \lambda f(u_n) \text{ in } \Omega, \quad u_n(x, 0) = g_n(x).$$

According to Proposition 3 and Theorem 4, the sequence is bounded in  $W^{1, \infty}(\Omega)$  and in  $W_{\text{loc}}^{2, p}(\Omega)$  for every finite  $p$ ; therefore we can extract a sequence, still denoted by  $(u_n)_n$ , which converges to a certain  $\hat{u}$ , weakly\* in  $W^{1, \infty}(\Omega)$ , and weakly in  $W_{\text{loc}}^{2, p}(\Omega)$ . Thus  $u_n$  converges to  $\hat{u}$  uniformly on compact subsets of  $\bar{\Omega}$ , and in the limit  $\hat{u}$  can be shown to be a classical solution of

$$-\Delta \hat{u} = \lambda f(\hat{u}), \quad \hat{u}(x, 0) = 0,$$

because, under assumption (II.6),  $g_n$  tends to zero as  $n$  tends to infinity. Let us prove that all solutions of problem (II.38) are independent of  $x$ ; this will allow us to finish, because

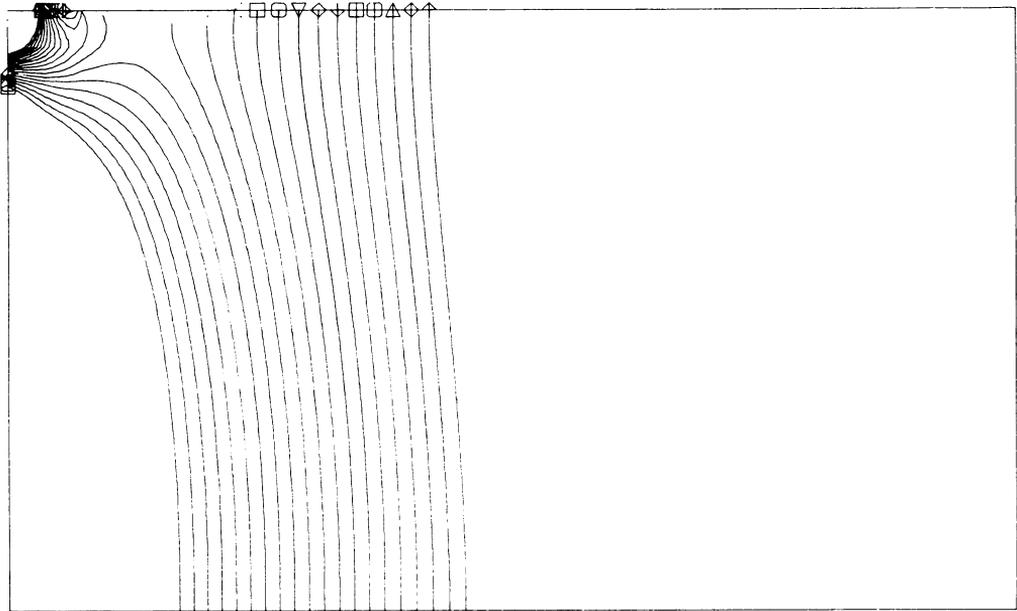


FIG. 17.  $N = 330$ .

Proposition 5 tells us that there exist only two such solutions which are selected by their asymptotic behavior. Then, thanks to proposition 7, we shall be able to select which of these two solutions is the good one and by a uniqueness argument we shall obtain Proposition 8.

To prove that the solutions of (II.38) are indeed independent of  $x$ , we use an argument due to Gidas and Spruck [11]. We perform an inversion about the point  $(0, -1)$ . Setting

$$X = x/(x^2 + (y + 1)^2), \quad Y = ((y + 1)/(x^2 + (y + 1)^2)) - 1,$$

we note that the open upper half-plane is transformed into the disk  $X^2 + (Y + \frac{1}{2})^2 \leq \frac{1}{4}$ , and that the boundary of the half-plane goes to the circumference of this disk save for the point  $(0, -1)$  which is the image of the point at infinity of the upper half-plane. We define

$$U(X, Y) = u(x, y);$$

we can immediately check that  $U$  satisfies

$$\begin{aligned} -\Delta U &= \lambda f(U) \quad \text{in } \Omega' = \{(X, Y)/X^2 + (Y + \frac{1}{2})^2 \leq \frac{1}{4}\} \\ U &= 0 \quad \text{on } \partial\Omega' \setminus \{(0, -1)\}; \end{aligned}$$

moreover,  $U$  is twice differentiable on  $\overline{\Omega'}$ , except at the point  $(0, -1)$  where it may be singular. We now use the method of moving planes as described in [10], with the simplification that as we are in dimension two, we do not need the restriction on the growth of  $f$  which is imposed in [11]. As in theorem 1' of [10], we conclude that  $U$  is invariant by a reflexion on the  $Y$  axis; therefore  $u$  is invariant by a reflexion on the  $y$  axis; but the same argument holds if the center of the inversion is  $(x_0, -1)$  with an arbitrary  $x$ ; therefore,  $u$  does not depend on  $x$ .

We now prove a lemma which describes the behavior as  $\lambda$  goes to infinity of a class of supersolutions of (1) taken on a *bounded domain*.

LEMMA 9. Let  $f$  satisfy assumptions (II.3)–(II.4) and let  $\Omega_1$  be a bounded open set of  $\mathbb{R}^2$  with piecewise smooth boundary. Assume that  $h$  is measurable, bounded, non-negative, not almost everywhere zero, and that  $w_\lambda$  belongs to  $H^1(\Omega_1) \cap C^2(\Omega_1)$  and satisfies

$$-\Delta w_\lambda \geq \lambda f(w_\lambda) \quad \text{in } \Omega_1, \quad w_\lambda \Big|_{\partial\Omega_1} \geq h. \quad (\text{II.41})$$

Then, for every relatively compact open set  $\omega$ , with  $\bar{\omega} \subset \Omega_1$ ,

$$\lim_{\lambda \rightarrow \infty} (\inf_{\omega} w_\lambda(x, y)) = \alpha_0. \quad (\text{II.42})$$

*Proof.* We deduce from (II.41) that

$$-\Delta w_\lambda \geq 0 \quad \text{in } \Omega_1, \quad w_\lambda \Big|_{\partial\Omega_1} \geq h,$$

so that  $w_\lambda$  is greater than or equal to the solution  $u_0$  of

$$-\Delta u_0 = 0 \quad \text{in } \Omega_1, \quad u_0 \Big|_{\partial\Omega_1} = h, \quad (\text{II.43})$$

which is known to be strictly positive in the interior of  $\Omega_1$ , thanks to our hypotheses on  $h$ . Now let  $B(\rho)$  be a ball of radius  $\rho$  centered at 0 and let  $\varepsilon$  be an arbitrary positive number. Then the problem

$$-\Delta v = \lambda f(v) = B(\rho), \quad v \Big|_{\partial B(\rho)} = \varepsilon \quad (\text{II.44})$$

possesses a minimum solution  $v_\lambda$  which is radial, because it is constructed as the limit of the increasing sequence of radial functions  $v_n$  defined by

$$\begin{aligned} v_0 &= 0, & -\Delta v_{n+1} + \lambda K v_{n+1} &= \lambda f(v_n) + \lambda K v_n \quad \text{in } B(\rho), \\ v_{n+1} &\Big|_{\partial B(\rho)} &= \varepsilon. \end{aligned}$$

Each of the  $v_n$ 's is lesser than or equal to  $\max(\varepsilon, \alpha_0)$  by the maximum principle, so that

$$v_\lambda \leq \max(\varepsilon, \alpha_0) \quad \text{on } B(\rho).$$

Moreover, if  $\mu \geq \lambda$ ,  $v_\mu$  is a supersolution with respect to the  $\lambda$ -problem. This implies that  $v_\lambda$  increases as a function of  $\lambda$ , and as

$$\int_{\Omega_1} v_\lambda(x, y) \, dx \, dy \leq \max(\varepsilon, \alpha_0) \text{ meas } B(\rho),$$

the sequence  $v_\lambda$  converges almost everywhere, and strongly in  $L^1(B(\rho))$  to a  $v_\infty$  which satisfies, in the sense of distributions,

$$f(v_\infty) = 0.$$

On the other hand,

$$v_\lambda \geq \varepsilon > 0 \quad \text{on } B(\rho),$$

so that

$$v_\infty \geq \alpha_0,$$

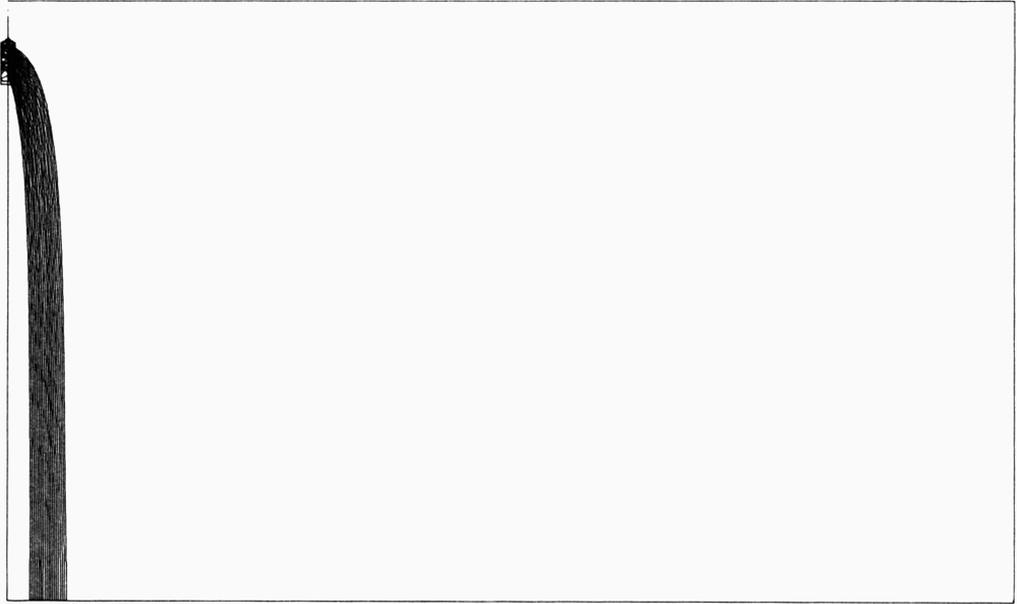


FIG. 18.  $N = 350$ .

thanks to hypothesis (II.4).

On the other hand, if we write (II.44) in polar coordinates, we obtain

$$-v''_\lambda - v'_\lambda/r \geq 0, \quad v'_\lambda(0) = 0, \quad v_\lambda(\rho) = \varepsilon,$$

from which we deduce that  $v_\lambda$  is a continuous decreasing function of  $r$ . Thus  $v_\lambda$  converges uniformly to  $v_\infty$  on the smaller ball  $B(\rho/2)$ , and

$$\liminf_{\lambda \rightarrow \infty} \inf_{B(\rho/2)} v_\lambda(x, y) \geq \alpha_0. \tag{II.45}$$

Let now  $\omega$  be a relatively compact, open subset of  $\Omega_1$  such that  $\bar{\omega} \subset \Omega_1$ , and set

$$2\rho = \inf\{|x - x'|^2 + |y - y'|^2\}^{1/2} / (x, y) \in \omega \text{ and } (x', y') \notin \Omega_1\},$$

$$\varepsilon = \min\{u_0(x, y) / d((x, y), \omega) \leq \rho\}.$$

We know that  $\rho$  and  $\varepsilon$  are positive and we can see that, for  $(x_0, y_0)$  in  $\omega$ ,

$$w_\lambda(x, y) \geq v_\lambda(x - x_0, y - y_0),$$

because  $w(\cdot + x_0, \cdot + y_0)|_{B(\rho)}$  is a non-negative supersolution for problem (II.44), from which we can construct by the usual algorithm a solution of (II.44) which is greater than or equal to the minimum solution  $v_\lambda$ .

From (II.45), it follows that

$$\liminf_{\lambda \rightarrow \infty} \inf_{B(\rho/2)} w(x + x_0, y + y_0) \geq \alpha_0,$$

and we obtain immediately (II.42).

We can prove the following result.

**THEOREM 10.** The interval  $(0, \lambda_0)$  in which the minimal solution tends to zero as  $|x| + y$  tends to infinity is bounded, if  $g$  is not identically zero.

*Proof.* Suppose that  $g(0) > 0$ . If not, by a translation along the  $x$ -axis, we can reduce ourselves to this case. We may even suppose that

$$g(x) \geq \alpha_2 \text{ on } [-a, a], \text{ with } \alpha_2 < \alpha_0.$$

Let  $u$  be a solution of (II.1). Then, on the square  $\Omega_1 = (-1, 1) \times (10, 2)$   $u$  satisfies

$$-\Delta u \geq \lambda f(u) \text{ in } \Omega_1, \quad u \Big|_{\partial\Omega_1} \geq h,$$

with  $h$  defined by

$$h = \alpha_2 \text{ on } [-a, a], \quad h = 0 \text{ elsewhere on } \partial\Omega_1.$$

Let  $\omega_1 = (-\frac{1}{2}, \frac{1}{2}) \times (\frac{1}{2}, \frac{3}{2})$ . According to Lemma 9, there exists a  $\lambda_1$  such that for  $\lambda \geq \lambda_1$ ,

$$u \Big|_{\omega_1} \geq \alpha_2.$$

Let  $\Omega_n$  and  $\omega_n$  be given by

$$\Omega_n = (-1, 1) \times (n-1, n+1), \quad \omega_n = (-\frac{1}{2}, \frac{1}{2}) \times (n-\frac{1}{2}, n+\frac{1}{2}).$$

If we assume that

$$u \Big|_{\omega_n} \geq \alpha_2, \text{ for } \lambda \geq \lambda_1,$$

then

$$u(x, n) \geq \alpha_2 \text{ on } [-a, a],$$

i.e.

$$u \Big|_{\partial\Omega_{n+1}} \geq h_n,$$

with  $h_n$  defined obviously by a translation, so that, according to Lemma 9,

$$u \Big|_{\Omega_{n+1}} \geq \alpha_2 \text{ for } \lambda \geq \lambda_1.$$

Now we can see that

$$\lim_{n \rightarrow \infty} u(0, n) \geq \alpha_2 > 0.$$

and thus it follows from Proposition 7 that

$$\lim_{y \rightarrow \infty} u(x, y) = \alpha_0 \text{ for } \lambda \geq \lambda_1.$$

As the minimum solution  $\underline{u}_\lambda$  is an increasing function of  $\lambda$ , there is a maximal interval  $[0, \lambda^*)$  such that

$$\lim_{y \rightarrow \infty} \underline{u}_\lambda(x, y) = 0, \quad \forall \lambda < \lambda^*,$$

and moreover, for any solution of (II.1),

$$\lim_{y \rightarrow \infty} u(x, y) = \alpha_0, \quad \forall \lambda > \lambda^*.$$

The behavior of  $\underline{u}_{\lambda^*}$  is not known.

*II.7 Bifurcation in a Frechet space.* As we noted before,  $u_\lambda$  and  $\bar{u}_\lambda$  are increasing functions of  $\lambda$ . Let  $\lambda_n$  be a sequence decreasing to zero. For any relatively compact open set  $\omega$  in  $\mathbb{R} \times \mathbb{R}^+$ , we have, for  $p < \infty$ ,

$$|\bar{u}_{\lambda_n}|_{W^{2,p}(\omega)} \leq C(p, \omega).$$

Therefore we can extract a convergent subsequence  $\bar{u}_{\lambda_m}$  such that

$$\bar{u}_{\lambda_m} \rightarrow \bar{u}_0 \text{ in } W^{2,p}(\omega) \text{ weakly, } \quad \bar{u}_{\lambda_m} \rightarrow \bar{u}_0 \text{ in } C^1(\bar{\omega}).$$

Thus  $\bar{u}_{\lambda_m}$  converges to  $\bar{u}_0$  in  $W^{2,p}(\omega')$ , with  $\omega' \subset \omega \cup \partial\Omega$ , and by the diagonal process we can extract a new sequence, denoted by  $\bar{u}_{\lambda_l}$ , such that

$$\bar{u}_{\lambda_l} \rightarrow \bar{u}_0 \text{ in } W^{2,p}(\omega) \text{ strongly, } \quad \text{for any } \omega \text{ and any } p < \infty.$$

In the limit,

$$-\Delta \bar{u}_0 = 0, \quad \bar{u}_0 \Big|_{\partial\Omega} = g.$$

As  $\bar{u}_0$  is bounded by  $\alpha_1$  (see Theorem 4),  $\bar{u}_0$  is the unique solution of (II.1) for  $\lambda = 0$ . As  $\bar{u}_0$  is unique, all the sequence  $\bar{u}_\lambda$  converges to  $\bar{u}_0$  in  $W_{loc}^{2,p}(\omega)$ .

Similarly, we can see that the lower branch  $u_\lambda$  has  $\bar{u}_0$  as its limit when  $\lambda$  goes to zero. Thus the lower branch and the upper branch of solutions meet at  $\lambda = 0$ . There is bifurcation in the following sense: for any  $p$  in  $[1, \infty)$ , for any relatively compact open set  $\omega$  included in  $\mathbb{R} \times \mathbb{R}^+$ , and for any neighborhood of  $\bar{u}_0$  in  $W^{2,p}(\omega)$ , there exists a  $\tilde{\lambda}$  such that for  $\lambda$  less than  $\tilde{\lambda}$ ,  $\bar{u}_\lambda|_\omega$  and  $u_\lambda|_\omega$  belong to this neighborhood.

Note that we allow  $\bar{\omega}$  to intersect the boundary of  $\Omega$ , because  $u_\lambda$  and  $\bar{u}_\lambda$  behave well at the boundary, but that  $\omega$  must be bounded, because  $\bar{u}_\lambda$  and  $u_\lambda$  have different behaviors at infinity. An intuitive way of describing this bifurcation is the following. Let  $C$  be an arbitrary open bounded subset of  $\Omega$ , then, for  $\lambda$  close enough to zero,  $\bar{u}_\lambda$  and  $u_\lambda$  can be made arbitrarily close on  $C$ .

*II.8 Solutions in a bounded domain.* To perform numerical computations of the solutions of problem (II.1), it is necessary to work in a bounded domain so as to apply finite differences or finite element methods, together with some curve continuation algorithm. Moreover, we need information on the behavior of the solutions of the problem in a finite domain when this domain increases infinitely.

Let  $\Omega'$  be the bounded domain which we consider, and let

$$\Gamma' = \bar{\Omega}' \cap \partial\Omega;$$

obviously, we must take the boundary condition

$$u \Big|_{\Gamma'} = g.$$

The choice of the condition on  $\Gamma'' = \partial\Omega \setminus \Gamma'$  is not crucial from the theoretical point of view, because we can conjecture that for very large domains, the effect of different boundary conditions is small close to  $(0, 0)$ . On the other hand, this choice is essential from the numerical point of view. We chose the condition

$$\frac{\partial u}{\partial n} \Big|_{\Gamma''} = 0.$$

This choice of Neumann condition has the following advantages:

(1) It is almost satisfied for large  $|x| + y$ , for arbitrary solutions of (II.1); in particular, if  $\Omega'$  is a rectangle, we know from Sec. II.6 that  $\lim_{|x|+y \rightarrow \infty} (\partial u / \partial n)(x, y)|_{\Gamma''} = 0$ .

(2) We can therefore work on a smaller domain than the one which would have been needed for a fixed Dirichlet condition, or avoid changing the Dirichlet condition according to the kind of solutions that we are looking for.

For simplicity, we particularize  $\Omega'$  to be a rectangle  $\Omega' = \Omega_{l, h} = (-l, l) \times (0, h)$  with

$$\Gamma'_{l, h} = (-l, l) \times \{0\}, \quad \Gamma''_{l, h} = \partial\Omega_{l, h} \setminus \Gamma'_{l, h}.$$

The problem studied in this section can be summarized as

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega_{l, h}; \quad u \Big|_{\Gamma'_{l, h}} = g, \quad \frac{\partial u}{\partial n} \Big|_{\Gamma''_{l, h}} = 0$$

It is well known that for small enough  $\lambda$  depending on  $(l, h)$ , (II.46) admits a unique solution. To prove the existence of distinct minimum and maximum solutions for large enough  $\lambda$ , we prove first that if  $l$  and  $h$  satisfy some supplementary condition then

$$\bar{z}_\lambda(x, y) \Big|_{\Omega_{l, h}} = L \left( \frac{y+a}{x^2 + (y+a)^2} \right)^{n(s)} \Big|_{\Omega_{l, h}}$$

is a supersolution. The function  $\bar{z}_\lambda$  satisfies the right inequality in  $\Omega_{l, h}$  according to Sec. II.5, and we must check that

$$\frac{\partial \bar{z}_\lambda}{\partial n} \Big|_{\Gamma''_{l, h}} \leq 0. \quad (\text{II.47})$$

But a direct calculation gives

$$\begin{aligned} \frac{\partial \bar{z}_\lambda}{\partial y}(x, h) &= Ln(s) \left( \frac{h+a}{x^2 + (h+a)^2} \right)^{n(s)-1} \frac{x^2 - (h+a)^2}{(x^2 + (h+a)^2)^2}, \\ \frac{\partial \bar{z}_\lambda}{\partial x}(\varepsilon l, y) &= Ln(s) \left( \frac{y+a}{l^2 + (y+a)^2} \right)^{n(s)-1} (-2\varepsilon l(y+a)) / (l^2 + (y+a)^2)^2, \end{aligned}$$

with  $\varepsilon = +1$  or  $-1$ . Thus, if

$$l^2 \leq (h+a)^2, \quad (\text{II.48})$$

the relation (II.47) holds, and by the subsolution algorithm starting from 0, there is an  $\underline{u}_{\lambda, l, h}$  which is the minimum solution of (II.46) and satisfies

$$\underline{u}_{\lambda, l, h} \leq \bar{z}_\lambda \Big|_{\Omega_{l, h}}, \quad \text{for small enough } \lambda. \quad (\text{II.49})$$

On the other hand, for any  $l, h$ , the function  $\underline{w}_\lambda$  defined in Sec. II.5 satisfies

$$\frac{\partial \underline{w}_\lambda}{\partial n} \Big|_{\Gamma''_{l, h}} \geq 0.$$

Therefore  $\underline{w}_\lambda|_{\Omega_{l, h}}$  is a subsolution for problem (II.46), and we can construct the maximum solution  $\bar{u}_{\lambda, l, h}$  of (II.46) by the supersolution algorithm, starting from  $\alpha_1 = \max(\alpha_0, \max g)$  which is obviously an estimate of the solutions of (II.46).

By the procedure of Sec. (II.7), we can extract from the family  $(u_{\lambda, l, h})$  defined for  $l^2 \leq h^2 + a^2$  a converging subsequence such that for any  $\omega$  included in  $\bigcap_{n \geq m} \Omega_{l_n, h_n}$ ,  $u_{\lambda, l_n, h_n}$  converges to some  $\underline{u}_\lambda$  in  $W^{2,p}(\omega)$ , and  $\underline{u}_\lambda$  is a solution of (II.1). We can similarly extract a converging subsequence of  $(\underline{u}_{\lambda, l, h})$  whose limit is a solution  $\mathbf{u}_\lambda$  of (II.1). Moreover,

$$u_\lambda \leq \bar{z} \quad \text{and} \quad \tilde{u}_\lambda \geq \underline{w}_\lambda.$$

Thus  $\tilde{u}_\lambda$  and  $\mathbf{u}_\lambda$  have the correct asymptotic behavior, but we cannot at present prove that  $\tilde{u}$  and  $\mathbf{u}_\lambda$  are respectively equal to  $\bar{u}_\lambda$  and  $\underline{u}_\lambda$ . If we had assumed a homogeneous Dirichlet condition on  $\Gamma''_{l, h}$ , then it would have been trivial to remark that  $u_\lambda|_{\Omega_{l, h}}$  is a supersolution with respect to (II.46) and that  $u_{\lambda, l, h}$  increases when  $\Omega_{l, h}$  increases; therefore, the limit of the  $u_{\lambda, l, h}$  in this case is precisely  $\underline{u}_\lambda$ .

Unfortunately, we do not possess a similar proof device for the mixed problem. However, we conjecture that  $u_\lambda = \underline{u}_\lambda$  and that  $\tilde{u}_\lambda = \bar{u}_\lambda$ .

*II.9. Summary of numerical results.* Several numerical computations were performed in a rectangle and gave unexpected information on the solutions of (II.46), and therefore of (II.1).

We used a kind of continuation method to follow branches of solutions. For other methods of continuation see [12, 13, 19, 20, 22]. This method is of interest in itself, and will be the object of a separate paper.

Let us merely outline the main features of the numerical analysis of (II.46): we consider the pair  $(u, \lambda)$  as a function of the arc length  $s$  along the branches of solutions and differentiate (II.46) with respect to  $s$ :

$$\begin{aligned} -\Delta \dot{u}(s) - \lambda(s)f'(u)\dot{u}(s) &= \dot{\lambda}(s)f(u(s)) \quad \text{in } \Omega_{l, h} \\ \dot{u}(s) \Big|_{\Gamma'_{l, h}} &= 0, \quad \partial u / \partial n \Big|_{\Gamma''_{l, h}} = 0. \end{aligned} \tag{II.50}$$

The normalization that will ensure that  $s$  is the arc length is

$$|\dot{u}(s)|^2 + \dot{\lambda}(s)^2 = 1. \tag{II.51}$$

Now, (II.50)–(II.51) is considered as an ordinary differential equation and is integrated by a Runge-Kutta method, with (II.51) suitably discretized, so as to make sure that there will be no change of direction when following the branch of solutions.

Discretization in space is by finite differences. The matrix that we obtain has the structure where the upper left, coming from (II.50) has a band structure and the rest of the matrix comes essentially from (II.51) discretized.

The upper left part of the matrix can be singular or very ill-conditioned close to the eventual turning points of the curve  $s \mapsto (u(s), \lambda(s))$ , i.e. points where  $\dot{\lambda}(s) = 0$ ,  $\dot{u}(s) \neq 0$ . To avoid losing the benefits of the band structure, the domain  $\Omega_{l, h}$  is cut into  $N$  overlapping smaller domains, where the matrix assumes the same structure as (II.52) and the upper left part is certainly well conditioned, because the smaller the domain, the larger the minimum parameter  $\lambda$  at which  $-\Delta - \lambda f'(u)$  ceases to be invertible. We decompose the matrices (II.52) on the smaller domains and, in addition, we are left with a much smaller matrix which expresses the coupling conditions between the subdomains; this matrix is full, and is decomposed by Gauss' method with pivot selection. In all the computations,

we assume that there are no bifurcations of the branches of solutions, which implies that the above smaller matrix is invertible.

$$\begin{pmatrix} \circ & \circ \\ \circ & \circ \end{pmatrix} \quad (II.52)$$

As a result, we followed a branch of solutions for different sizes of  $\Omega_{l,h}$  and observed that the larger  $\Omega_{l,h}$ , the more complex the pattern of the branch of solutions; in particular, the number of turning points increases (we even reached eight turning points in the last of our numerical experiments), and the solutions go through more different states, from the topological point of view: more states, with more and more critical points, such as saddles, hills and basins.

We display here the results obtained in the largest domain, which was  $\Omega_{1,2}$ ; the boundary value was assumed symmetric, so that we computed only the symmetric solutions (with respect to  $x = 0$ ). The function  $f$  is given by

$$f(x) = (x - 0.2)^2(1 - x)^2 \quad \text{for } x \in [0.2, 1]; \quad f(x) = 0 \text{ elsewhere};$$

and the function  $g$  by

$$g(x) = (1 + 400x^2)^{-1}$$

Observe that in this case, though hypothesis (II.4) is not fulfilled, the results of Sec. II.6 concerning the upper branch of solutions still hold, and the results concerning the lower branch of solutions would hold if we could prove that  $u_\lambda$  decreases fast enough at infinity in  $x$ . But this fact is very probable, so that we can expect that the pattern is the same that the one displayed for an  $f$  which satisfies all of (II.4).

In Fig. 4  $u(A)$  is plotted against  $\lambda$ , where  $A$  is a fixed point in  $\Omega_{l,h}$ . Note that the crossings are *not* bifurcations, but are the effects of the projection from infinite-dimensional space to one-dimensional space. Figs. 5–18 give the level lines of the solutions for several values of the number  $N$  of iterations of the numerical method (indicated along the curve of Fig. 4).

*III. Conclusions.* We have from Sec. (II.5) and (II.6) the following information:

(1) If  $\lim_{y \rightarrow \infty} u(x, y) = \alpha_0$ , then the level lines of  $u$  (which, as we saw in Sec. I, are the projections on the  $x - y$  plane of the lines of force of the magnetic field) certainly cannot all connect to the boundary, and the general behavior that we can expect is represented in Fig. 18.

(2) For  $\lambda$  greater than or equal to  $\lambda^*$ , no solution can have all of its level lines connected to the boundary; in this case, the astrophysicists claim that such a solution is certainly not stable. They argue indeed that all the lines of force, which are current lines too, must be connected to the boundary  $y = 0$ ; if some of them were to be closed on themselves or to go to infinity, then electric current would flow out, and this is an unstable situation.

(3) The branch of lower solutions satisfies for  $\lambda$  small enough a necessary condition for its level lines to connect to the boundary:  $\lim_{|x|+y \rightarrow \infty} u(x, y) = 0$ .

(4) If we assume a quasistatic evolution of  $\lambda$  in time, when  $\lambda$  reaches  $\lambda^*$ , the evolution of the solution becomes dynamical and is in fact catastrophic: a solar flare takes place.

Many questions still remain open from the astrophysical and mathematical points of

view; we shall list some of them below:

(1) The model (II.1) is very simplified, even with respect to Eqs. (I.5) and (I.7), and the results that we obtain are phenomenologically related to observation. It would be interesting to show that a more complete model yields the same catastrophic evolution, starting from quasistatic evolution of the data.

(2) Stability of solutions should be studied in a more quantitative fashion; the present state of the ideas is that the solution is not stable if all the current lines do not connect to the boundary; the mathematical study of this stability involves a large set of dynamical equations, i.e. the equations of magnetohydrodynamics.

(3) Trying to explain the increase in the complexity of the solution pattern with the size of the domain seems to be a challenging mathematical question. There does not seem to be an obvious relation in our model between the appearance of bubbles (see Fig. 8) and the crossing of a turning point. For instance, we plotted no level lines pattern before iteration 65, which is past the first turning point, because their aspect was exactly the same before that. Perhaps this increase of complexity is related to the topological phenomena occurring in nonlinear Sturm-Liouville problems (see [6]) which appear for increasing values of  $\lambda$ . But increasing  $\lambda$  in a fixed domain is not very different from increasing the domain with a fixed  $\lambda$ , and the explanation might be somewhere on this side of knowledge.

*Acknowledgements.* Proposition 8, in its present form, is due to B. Gidas; it is a pleasure to thank him here, as this result is not only interesting in itself, but puts on a firm basis our intuitive choice of the Neumann condition for the numerical resolution of the problem in a bounded domain.

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