DYNAMIC BEHAVIOR FROM ASYMPTOTIC EXPANSIONS*

BY

JACK K. HALE† (*Brown University*)

AND

LUIZ CARLOS PAVLU‡ (*Universidade Federal de São Carlos*)

Abstract. The purpose of this paper is to discuss stability properties of solutions of periodic and almost-periodic differential equations containing a small parameter. The existence of the solution can be obtained in the first approximation but the stability only after k approximations. We obtain the results using asymptotic expansions, higher-order averaging and the concept of exponential hyperbolicity of order k.

1. Introduction. In the study of a certain class of models, for the spin/orbit resonance of the planet Mercury in particular and for nonlinear resonance in general, Murdoch [6] encountered an interesting problem in the stability of periodic solutions of periodic differential equations containing a small parameter. The existence of the periodic solution could be obtained from the first approximation, but the stability could not.

Murdoch and Robinson [7] resolved the difficulty through the introduction of the concept of "strong k-hyperbolicity" for the period map. It is the purpose of this paper to show that the same results are valid under the weaker hypothesis that the original vector field has an "exponential dichotomy of order k". Since the latter concept does not assume the vector field is periodic, it is possible to have applications to more general situations as in almost periodic cases, for example.

In Sec. 2, we introduce the concept of exponential dichotomy of order k and present a result relative to its "roughness". In Sec. 3, we present the main result in the periodic case showing that the stability properties of the periodic orbit can be solved through the latter concept; also, we give sufficient conditions to ensure "exponential dichotomy of order k" and some remarks and examples. Finally, in Sec. 4, we solve the almost-periodic case.

2. Exponential dichotomy of order k. Let $X(t, \varepsilon)$ be a fundamental matrix for the linear system:

$$\dot{x} = \varepsilon A(t, \varepsilon)x \tag{2.1}$$

where the $n \times n$ coefficient matrix $A(t, \varepsilon)$ is continuous on $t \in R_+$ and a sufficiently smooth function of the real parameter $\varepsilon \in (0, \varepsilon_0)$.

* Received May 24, 1982.

† This research has been supported in part by the Air Force Office of Scientific Research under contract #AF-AFOSR 81-0198, in part by the National Science Foundation under grant #MCS 79-05774-05 and in part by the U.S. Army Research Office under contract #ARO-DAAG-29-79-C-0161.

‡ This research has been supported in part by FAPESP, Proc. No. 80/1608-7 and in part by C.N.P.q., Proc. No. 200.919-81-MA.

Definition 2.1. The equation (2.1) has an *exponential dichotomy of order k* if there exists a projection P_ε continuous for $\varepsilon \in (0, \varepsilon_0)$, a positive constant K and a function $\alpha(\varepsilon) = c\varepsilon^k$, $c > 0$, such that

$$\begin{aligned} |X(t, \varepsilon)P_\varepsilon X^{-1}(s, \varepsilon)| &\leq Ke^{-\alpha(\varepsilon)(t-s)}, \qquad t \geq s \geq 0, \\ |X(t, \varepsilon)(I - P_\varepsilon)X^{-1}(s, \varepsilon)| &\leq Ke^{-\alpha(\varepsilon)(s-t)}, \qquad s \geq t \geq 0. \end{aligned} \tag{2.2}$$

An important property of exponential dichotomies is their roughness; that is, they are not destroyed by small perturbations. More precisely, we can state the following result:

THEOREM 2.1. *Suppose that the linear system* (2.1) *has an exponential dichotomy of order k. Let $B_\varepsilon = B(t, \varepsilon)$ be a continuous matrix function, bounded uniformly on $t \in R_+$ for each fixed $\varepsilon \in (0, \varepsilon_0)$. If $|B_\varepsilon| = \sup_{t \in R_+} |B(t, \varepsilon)| = O(|\varepsilon|^N)$ for $N \geq k$, then the perturbed system*

$$\dot{y} = \varepsilon(A(t, \varepsilon) + B(t, \varepsilon))y \tag{2.3}$$

also possesses an exponential dichotomy of the same order.

The proof can be accomplished by applying the contraction mapping principle to the operator

$$TY(t) = X(t, \varepsilon)P_\varepsilon + \int_0^t X(t, \varepsilon)P_\varepsilon X^{-1}(s, \varepsilon)\varepsilon B(s, \varepsilon)Y(s)\, ds - \int_t^\infty X(t, \varepsilon)(I - P_\varepsilon)X^{-1}(s, \varepsilon)\varepsilon B(s, \varepsilon)Y(s)\, ds.$$

Elementary estimates yield:

$$\begin{aligned} |TY(t)| &\leq K + 2\alpha(\varepsilon)^{-1}K\varepsilon|B_\varepsilon|\, \|Y\|, \\ |TY_1(t) - TY_2(t)| &\leq 2\alpha(\varepsilon)^{-1}K\varepsilon|B_\varepsilon|\, \|Y_1 - Y_2\| \end{aligned}$$

where $\|Y\| = \sup_{t \geq 0} |Y(t)|$ and $\alpha(\varepsilon) = c\varepsilon^k$. If $\alpha(\varepsilon)^{-1}K\varepsilon|B_\varepsilon| < \frac{1}{2}$, the mapping T has a unique fixed point. Since $|B_\varepsilon| = O(|\varepsilon|^N)$, there exists an $\varepsilon_1 > 0$ such that the latter estimate is valid for $\varepsilon \in (0, \varepsilon_1)$. The remainder of the proof is easily supplied by following the techniques in Coppel [1].

Corresponding results for the half-line R_- may be obtained by the change of variable $t \to -t$ and the same question for the whole line R can be answered in terms of the results for the two half-lines or directly using the operator:

$$TY(t) = \int_{-\infty}^t X(t, \varepsilon)P_\varepsilon X^{-1}(s, \varepsilon)\varepsilon B(s, \varepsilon)Y(s)\, ds - \int_t^\infty X(t, \varepsilon)(I - P_\varepsilon)X^{-1}(s, \varepsilon)\varepsilon B(s, \varepsilon)Y(s)\, ds.$$

COROLLARY 2.3. *Consider the perturbed system*

$$\dot{y} = \varepsilon A(t, \varepsilon)y + \varepsilon^{N+1}f(t, y, \varepsilon) \tag{2.4}$$

where $f: R \times \Omega \subset R^n \times [0, \varepsilon_0] \to R^n$ is uniformly continuous and bounded in $t \in R$ for each (y, ε) fixed in $\Omega \times [0, \varepsilon_0]$ and $f(t, 0, \varepsilon) = 0$.

If the unperturbed system (2.1) has an exponential attraction of order $k \leq N$ then, for small ε, the solution $y = 0$ of (2.4) is uniformly asymptotically stable.

More generally, results of this kind can be extended to a system of the form

$$\dot{x} = \varepsilon A(t, \varepsilon)x + \varepsilon^{N+1}f(t, x, \varepsilon) + g(x, \varepsilon) \tag{2.5}$$

where $g(0, \varepsilon) = 0$, $(\partial g/\partial x)\,(0, \varepsilon) = 0$. The reader is referred to Hale [4] and Coppel [1] for details.

Remark. In [7], Murdoch and Robinson consider a system $\dot{x} = \varepsilon f(t, x, \varepsilon)$, where $\varepsilon \geq 0$ is a small parameter, f is continuous, ω-periodic in t and smooth in x and ε, and the Poincaré map $U_\varepsilon\colon R^n \to R^n$ is given by $U_\varepsilon(x) = \phi(\omega, x, \varepsilon)$ where $\phi(t, x, \varepsilon)$ is the general solution of the system above satisfying $\phi(0, x, \varepsilon) = x$.

Suppose that the Taylor series of U_ε is available; that is, $U_\varepsilon(x) = V_\varepsilon(x) + \varepsilon^{k+1}\tilde{U}_\varepsilon(x)$ where $V_\varepsilon(x) = x + \varepsilon U_1(x) + \cdots + \varepsilon^k U_k(x)$. If there is a x_0 such that $U_1(x_0) = 0$ and $U_1'(x_0)$ is nonsingular, the implicit function theorem gives us fixed points $x^*(\varepsilon)$ of $V_\varepsilon(x)$ and $\bar{x}(\varepsilon)$ of $U_\varepsilon(x)$ with $x^*(\varepsilon), \bar{x}(\varepsilon) \to x_0$ as $\varepsilon \to 0$.

The question is: if $x^*(\varepsilon)$ is a hyperbolic fixed point for V_ε, will $\bar{x}(\varepsilon)$ be a hyperbolic fixed point of U_ε? This, in general, is not true, except when the first approximation U_1 is hyperbolic, as is well known.

They resolved the problem by introducing the concept of "strong k-hyperbolicity" and giving sufficient conditions to obtain "strong k-hyperbolicity". The concept of exponential dichotomy of order k is more general and can be applied to "the almost-periodic problem", as we will show in the last section.

3. The periodic case. Suppose $x \in R^n$, $\varepsilon \geq 0$ a small parameter, $f\colon R \times R^n \times [0, \varepsilon_0) \to R^n$ analytic in $\varepsilon \in (0, \varepsilon_0)$, ω-periodic in t for each fixed (x, ε) and sufficiently smooth in $x \in R^n$. Consider the system:

$$\dot{x} = \varepsilon f(t, x, \varepsilon). \tag{3.1}$$

Suppose the asymptotic expansion, in powers of ε, of the system (3.1) up to order N is known:

$$\dot{x} = \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \cdots + \varepsilon^N f_N(f, x) + \varepsilon^{N+1}\tilde{f}(t, x, \varepsilon) \tag{3.2}$$

where each f_i, $i = 1, 2, \ldots N$ is an ω-periodic function in t and $\tilde{f}$ is ω-periodic for each (x, ε) fixed.

Using averaging up to order N, we may choose a suitable change of coordinates $x \to y$, ω-periodic in t, which eliminates t from the first N terms in the right-hand side of (3.2). The resulting system has the form

$$\dot{y} = \varepsilon \bar{f}_1(y) + \cdots + \varepsilon^N \bar{f}_N(y) + \varepsilon^{N+1}\hat{f}(t, y, \varepsilon) \tag{3.3}$$

where $\hat{f}$ has the same properties as $\tilde{f}$ before. If there is a y_0 such that $\bar{f}_1(y_0) = 0$ and $(\partial f_1/\partial y)\,(y_0)$ is nonsingular and, furthermore, if λ_i is an eigenvalue of $\partial \bar{f}_1(y_0)/\partial y$ and Re $\lambda_i[\partial \bar{f}_1(y_0)/\partial y] \neq 0$, $i = 1, 2, \ldots n$, then we can conclude existence, uniqueness and stability properties of the periodic solution $x^*(t, \varepsilon)$ of (3.1) from known results (see Hale [4]).

Actually, in this case, we have an exponential dichotomy of order one. In what follows, we discuss a more general situation. That is, suppose at least for some i, $1 \leq i \leq n$, we have Re $\lambda_i[\partial \bar{f}_1(y_0)/\partial y] = 0$ and consider the equations

$$\dot{x} = \varepsilon F(x, \varepsilon), \tag{3.4}$$

where $F(x, \varepsilon) = \bar{f}_1(x) + \varepsilon \bar{f}_2(x) + \cdots + \varepsilon^N \bar{f}_N(x)$, and

$$\dot{x} = \varepsilon F(x, \varepsilon) + \varepsilon^{N+1} \hat{f}(t, x, \varepsilon). \tag{3.5}$$

THEOREM 3.1. Suppose f_i, $i = 1, 2, \ldots N$, $\hat{f}$ satisfies the conditions enumerated in this section. If there is an x_0 such that $F(x_0, 0) = 0$ and $(\partial F/\partial x)(x_0, 0)$ is nonsingular, then there exist an $\varepsilon_1 > 0$ and functions $x_N(\varepsilon)$ and $x^*(t, \varepsilon)$, both analytic in ε, $x^*(t, \varepsilon)$ continuous in t for each fixed $\varepsilon \in [0, \varepsilon_1]$, $x^*(t + \omega, \varepsilon) = x^*(t, \varepsilon)$, $x_N(\varepsilon)$ is an equilibrium point of (3.4), $x^*(t, \varepsilon)$ satisfies (3.5), $x_N(0) = x^*(t, 0) = x_0$ and $\| x^*(t, \varepsilon) - x_N(\varepsilon) \| = O(\varepsilon^N)$.

Furthermore, if the linear variational equation of (3.4) at the equilibrium point $x_N(\varepsilon)$ has an exponential dichotomy of order k, $k \leq N$, then the linear variational equation of (3.5) at $x^*(t, \varepsilon)$ also has an exponential dichotomy of the same order.

To prove the theorem we need the following result:

LEMMA 3.2. Consider the system

$$\dot{x} = \varepsilon A(\varepsilon) x + f(t) \tag{3.6}$$

where $x \in R^n$, $A(\varepsilon)$ is a continuous matrix function of the parameter $\varepsilon \geq 0$ and $f \in \mathscr{P}\omega$, the Banach space of the continuous ω-periodic functions.

If $\det A(0) \neq 0$, then there exist $K > 0$, $\varepsilon_0 > 0$ such that (3.6) has a unique solution

$$\mathscr{K}(\varepsilon) f \in \mathscr{P}\omega \qquad \text{and} \qquad |\mathscr{K}(\varepsilon) f| < K/\varepsilon |f|, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

Proof. Since $\det A(0) \neq 0$, there exists an $\varepsilon_0 > 0$ such that the unperturbed system $\dot{x} = \varepsilon A(\varepsilon) x$ is noncritical with respect to $\mathscr{P}_\omega$. This implies the existence of a continuous linear operator $\mathscr{K}(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$, defined by

$$(\mathscr{K}(\varepsilon) f)(t) = \int_0^\omega \varepsilon [e^{-\varepsilon A(\varepsilon)\omega} - I]^{-1} e^{-\varepsilon A(\varepsilon)s} 1/\varepsilon f(t + s)\, ds.$$

Furthermore, $\lim_{\varepsilon \to 0^+} \varepsilon [e^{-\varepsilon A(\varepsilon)\omega} - I]^{-1} = -1/\omega A(0)^{-1}$. Thus, $\mathscr{K}(\varepsilon)$ has a uniform bound on $(0, \varepsilon_0]$ given by $|\mathscr{K}(\varepsilon) f| \leq K/\varepsilon |f|$. More details can be found in Hale [4].

Proof of Theorem 3.1. Consider the system

$$\dot{x} = \varepsilon F(x, \varepsilon) + \varepsilon^{N+1} \hat{f}(t, x, \varepsilon). \tag{3.5}$$

If there is an x_0 such that $F(x_0, 0) = 0$ and $\det (\partial F/\partial x)(x_0, 0) \neq 0$, by the implicit function theorem, for ε small, there is a unique $x_N(\varepsilon)$ such that $x_N(0) = x_0$ and $F(x_N(\varepsilon), \varepsilon) = 0$. $x_N(\varepsilon)$ is the equilibrium point of the autonomous system

$$\dot{x} = \varepsilon F(x, \varepsilon). \tag{3.4}$$

Using the transformation $x \to y$ defined by $x = x_N(\varepsilon) + y$, (3.5) becomes

$$\begin{aligned} \dot{y} &= \varepsilon F(x_N(\varepsilon) + y, \varepsilon) + \varepsilon^{N+1} \hat{f}(t, x_N(\varepsilon) + y, \varepsilon) \\ &= \varepsilon \frac{\partial F}{\partial x}(x_N(\varepsilon), \varepsilon) y + \varepsilon G(y, \varepsilon) + \varepsilon^{N+1} \hat{f}(t, x_N(\varepsilon) + y, \varepsilon) \end{aligned} \tag{3.7}$$

where $G(0, \varepsilon) = 0$, $(\partial G/\partial y)(0, \varepsilon) = 0$.

Eq. (3.7) can be written in the form

$$\dot{y} = \varepsilon A(\varepsilon) y + \varepsilon [G(y, \varepsilon) + \varepsilon^N \hat{f}(t, x_N(\varepsilon) + y, \varepsilon)], \tag{3.8}$$

where $A(\varepsilon) = (\partial F/\partial x)(x_N(\varepsilon), \varepsilon) = (\partial F/\partial x)(x_0, 0) + \varepsilon \tilde{F}(\varepsilon) = A(0) + \varepsilon \tilde{F}(\varepsilon)$.

A solution y of Eq. (3.8) in $\mathscr{P}_\omega$ must satisfy

$$H(y, \varepsilon) = y - \varepsilon\mathscr{K}(\varepsilon)\mathscr{F}(y, \varepsilon) = 0 \tag{3.9}$$

where $\mathscr{F}: \mathscr{P}_\omega \times (0, \varepsilon_0] \to \mathscr{P}_\omega$. Lemma 3.2 and the properties of $\mathscr{F}$ and $\mathscr{K}(\varepsilon)$ give us a unique ω-periodic solution y^* of (3.8), for ε sufficiently small, defined by

$$y^* = \varepsilon\mathscr{K}(\varepsilon)[G(y^*, \varepsilon) + \varepsilon^N \hat{f}(\cdot, x_N(\varepsilon) + y^*, \varepsilon)].$$

Since $|\varepsilon\mathscr{K}(\varepsilon)| \le K$, using successive approximations with $y_0 = 0$, we obtain $|y^*| \le \bar{K}\varepsilon^N$; this shows that there exists an ω-periodic solution $x^*(t, \varepsilon)$ of (3.5), $x^*(\cdot, 0) = x_N(0) + y^*(\cdot, 0) = x_0$ such that $\| x^*(\cdot, \varepsilon) - x_N(\varepsilon)\| = O(\varepsilon^N)$.

The linear variational equation of the system (3.5) at the ω-periodic solution $x^*(t, \varepsilon)$ is given by

$$\begin{aligned} \dot{z} &= \varepsilon \frac{\partial F}{\partial x}(x^*(t, \varepsilon), \varepsilon)z + \varepsilon^{N+1} \frac{\partial \hat{f}}{\partial x}(t, x^*(t, \varepsilon), \varepsilon)z \\ &= \varepsilon\left(\frac{\partial F}{\partial x}(x_N(\varepsilon), \varepsilon) + O(\varepsilon^N)\right)z + \varepsilon^{N+1} \frac{\partial \hat{f}}{\partial x}(t, x^*(t, \varepsilon), \varepsilon)\, z \\ &= \varepsilon \frac{\partial F}{\partial x}(x_N(\varepsilon), \varepsilon)z + O(\varepsilon^{N+1}). \end{aligned} \tag{3.10}$$

This means that the linear variational equations of (3.4) at $x_N(\varepsilon)$ and (3.5) at $x^*(t, \varepsilon)$ coincide up to order N, and a simple application of Theorem 2.1 completes the proof.

Sufficient conditions for exponential dichotomy of order $k \le N$, equivalent to those given by Murdoch and Robinson for strong k-hyperbolicity, can be given as follows.

LEMMA 3.3. Consider the system

$$\dot{x} = \varepsilon A(\varepsilon)x \tag{3.11}$$

where $x \in R^n$ and $A(\varepsilon)$ is a $n \times n$ matrix such that $\varepsilon A(\varepsilon) = \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^N A_N + O(\varepsilon^{N+1})$. If the eigenvalues of A_1 are distinct and if the eigenvalues $\varepsilon\lambda_i(\varepsilon)$ of $\varepsilon A_1 + \cdots + \varepsilon^N A_N$, suitably numbered, satisfy:

$$\operatorname{Re} \varepsilon\lambda_i(\varepsilon) < -c\varepsilon^k, \qquad i = 1, 2, \dots r$$

$$\operatorname{Re} \varepsilon\lambda_i(\varepsilon) > c\varepsilon^k, \qquad i = r+1, \dots, n$$

for some $k \le N$ and some positive constant c, then the equation

$$\dot{x} = (\varepsilon A_1 + \cdots + \varepsilon^N A_N)x \tag{3.12}$$

has an exponential dichotomy of order $k \le N$. In fact, consider $A(\varepsilon) = A_1 + O(\varepsilon)$. If A_1 has distinct eigenvalues, $A(\varepsilon) = A_1 + \cdots + \varepsilon^{N-1}A_N$ has distinct eigenvalues $\lambda_i(\varepsilon)$ for ε small. The matrix $\mathscr{C}_\varepsilon$ of eigenvalues is nonsingular (even for $\varepsilon = 0$) and $\mathscr{C}_\varepsilon^{-1}A(\varepsilon)\mathscr{C}_\varepsilon = \operatorname{Diag}(\lambda_i(\varepsilon))$, $i = 1, 2, \dots n$. If $x = \mathscr{C}_\varepsilon y$, Eq. (3.10) becomes

$$\dot{y} = \operatorname{Diag}(\varepsilon\lambda_i(\varepsilon))y \tag{3.13}$$

with fundamental matrix $Y(t) = \operatorname{Diag}(\exp(\varepsilon\lambda_i(\varepsilon))t$.

Using the hypotheses, reordering if necessary and taking the projection $P_\varepsilon = (I_{r\times r}, 0)_{n\times n}$, we obtain

$$|Y(t)P_\varepsilon Y^{-1}(s)| = |\operatorname{Diag}(\exp(\varepsilon\lambda_i(\varepsilon))(t-s)| < e^{-c\varepsilon^k(t-s)}, \quad t \ge s, \quad i = 1, 2, \dots r$$

and

$$|Y(t)(I-P_\varepsilon)Y^{-1}(s)| < e^{-c\varepsilon^k(s-t)}, \quad s \geq t, \quad i = r+1, \ldots, n.$$

Since (3.13) is similar to (3.12), the fundamental matrix $X(t)$ of (3.12) satisfies

$$|X(t)P_\varepsilon X^{-1}(s)| \leq Me^{-c\varepsilon^k(t-s)}, \qquad t \geq s,$$

$$|X(t)(I-P_\varepsilon)X^{-1}(s)| \leq Me^{-c\varepsilon^k(s-t)}, \qquad s \geq t,$$

with same projection and same order.

Remarks. (1) Actually, to obtain an exponential dichotomy we do not need to have distinct eigenvalues, as the following example shows:

$$\dot{x} = \begin{pmatrix} -\varepsilon^2 & \varepsilon^n \\ 0 & -\varepsilon^2 \end{pmatrix} x \tag{3.14}$$

where $x \in R^2$, $\varepsilon \ll 1$, $n \geq 2$.

The fundamental matrix $X(t)$ of (3.14) satisfies $|X(t)X^{-1}(s)| \leq e^{-1/2\varepsilon^2(t-s)}$, $t \geq s$ where $P_\varepsilon = I_{2\times 2}$ and we have an "exponential attraction of order 2." Observe, if $n = 2$, that $A_1 = 0$, $A_2 = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and the matrix $\mathscr{C}_\varepsilon = \left(\begin{smallmatrix} 1 & 1/\varepsilon \\ 0 & 0 \end{smallmatrix}\right)$ of eigenvectors becomes unbounded when $\varepsilon \to 0$.

(2) The example given by Murdoch and Robinson to show that the hyperbolicity present at order ε^2 may be destroyed by a perturbation of order ε^3, can be obtained up to order ε^2 considering the solution operator $e^{\varepsilon A(\varepsilon)t}$ at $t = 1$ of the system (3.14), when $n = 1$:

$$e^{\varepsilon A(\varepsilon)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In fact, in this case the hyperbolicity present at order ε^2, $(\lambda_{1,2} = 1 - \varepsilon^2)$, is destroyed by $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right) \varepsilon^3$, for example. However, the fundamental matrix $X(t)$, given by

$$X(t) = \begin{pmatrix} e^{-\varepsilon^2 t} & \varepsilon t e^{-\varepsilon^2 t} \\ 0 & e^{-\varepsilon^2 t} \end{pmatrix},$$

does not have an exponential dichotomy of order 2.

The same problem may occur if we have, in the first approximation, a double eigenvalue in the imaginary axis.

4. The almost-periodic case.

Consider the system

$$\dot{x} = \varepsilon f(t, x, \varepsilon) \tag{4.1}$$

where f is uniformly almost periodic in $t \in R$, analytic in $\varepsilon \geq 0$ and sufficiently smooth in x, $x \in R^n$.

Consider the expansion

$$\dot{x} = \varepsilon f_1(t, x) + \cdots + \varepsilon^N f_N(t, x) + \varepsilon^{N+1}\tilde{f}(t, x, \varepsilon) \tag{4.2}$$

where the f_i, $i = 1, 2, \ldots N$, and $\tilde{f}$ are almost periodic in t with the same properties as f.

Under some nonresonance hypothesis on the frequencies of the f_i, we can average up through order N to obtain the following system:

$$\dot{x} = \varepsilon \bar{f}_1(x) + \varepsilon^2 \bar{f}_2(x) + \cdots + \varepsilon^N \bar{f}_N(x) + \varepsilon^{N+1}\hat{f}(t, x, \varepsilon). \tag{4.3}$$

This system can be seen as a perturbation (almost periodic) of the autonomous equation

$$\dot{x} = F(x, \varepsilon) \tag{4.4}$$

where $F(x, \varepsilon) = \varepsilon \bar{f}_1(x) + \cdots + \varepsilon^N \bar{f}_N(x)$. The method used in the proof of Theorem 3.1 does not work in this case, but with a slight modification we can establish the following:

THEOREM 4.1. Suppose f_i, $i = 1, 2, \ldots N, \hat{f}$ satisfy the conditions enumerated in the beginning of this section.

If there is an x_0 such that $\bar{f}_1(x_0) = 0$ and $(\partial \bar{f}_1/\partial x)(x_0)$ is nonsingular, then there exist an $\varepsilon_0 > 0$ and a function $x_N(\varepsilon)$ analytic in ε, $0 < \varepsilon < \varepsilon_0$, $x_N(0) = x_0$, $x_N(\varepsilon)$ is an equilibrium point of (4.4).

Furthermore, if the linear variational equation of (4.4) at $x_N(\varepsilon)$ has an exponential dichotomy of order $k \leq N/2$ then, in a small neighborhood of $x_N(\varepsilon)$, Eq. (4.3) has a unique almost-periodic solution $x^*(t, \varepsilon)$, analytic in ε, $x^*(\cdot, 0) = x_0$ and the linear variational equation of (4.3) at $x^*(t, \varepsilon)$ also has an exponential dichotomy of order k for ε small enough (possibly with a positive constant $\bar{c}$ smaller than c).

To prove the theorem, we need the following:

LEMMA 4.2. Consider the system

$$\dot{x} = \varepsilon A(\varepsilon)x + f(t) \tag{4.5}$$

where $x \in R^n$, $A(\varepsilon)$ is a continuous matrix function of $\varepsilon \geq 0$ and $f \in \mathscr{AP}$, the Banach space of almost-periodic functions. If the autonomous system $\dot{x} = \varepsilon A(\varepsilon)x$ has an exponential dichotomy of order k, then there exist $K > 0$, $\varepsilon_0 > 0$ such that (4.5) has a unique almost-periodic solution $\mathscr{K}(\varepsilon)f \in \mathscr{AP}$ for $0 < \varepsilon < \varepsilon_0$ and $|\mathscr{K}(\varepsilon)f| \leq K/\varepsilon^k |f|$. In fact, there exist a projection P_ε and positive constants M and c such that the fundamental matrix $X(t)$ of the autonomous system $\dot{x} = \varepsilon A(\varepsilon)x$ satisfies

$$|X_\varepsilon(t)P_\varepsilon X_\varepsilon^{-1}(s)| \leq Me^{-c\varepsilon^k(t-s)}, \qquad t \geq s,$$

$$|X_\varepsilon(t)(I - P_\varepsilon)X_\varepsilon^{-1}(s)| \leq Me^{-c\varepsilon^k(s-t)}, \qquad s \geq t.$$

Since $\mathscr{K}(\varepsilon)f$ is given by

$$\mathscr{K}(\varepsilon)f = \int_{-\infty}^{t} X_\varepsilon(t)P_\varepsilon X_\varepsilon^{-1}(s)f(s)\, ds - \int_{t}^{\infty} X_\varepsilon(t)(I - P_\varepsilon)X_\varepsilon^{-1}(s)f(s)\, ds,$$

elementary estimates yield $|\mathscr{K}(\varepsilon)f| \leq K/\varepsilon^k |f|$.

Proof of Theorem 4.1. The first part of the proof is similar to the periodic case and, making the transformation of variables $x = x_N(\varepsilon) + y$, we can consider directly the equation

$$\dot{y} = \varepsilon A(\varepsilon)y + \varepsilon[G(y, \varepsilon) + \varepsilon^N \hat{f}(t, x_N(\varepsilon) + y, \varepsilon)] \tag{4.6}$$

where $A(\varepsilon) = (\partial F/\partial x)(x_N(\varepsilon), \varepsilon)$, $G(0, \varepsilon) = 0$, $(\partial G/\partial y)(0, \varepsilon) = 0$ and $\hat{f}$ as before. If there is a solution $y^* \in \mathscr{AP}$ of (4.6), this solution must satisfy the equation

$$y - \varepsilon\mathscr{K}(\varepsilon)[G(y, \varepsilon) + \varepsilon^N f(\cdot, x_N(\varepsilon) + y, \varepsilon)] = H(y, \varepsilon) = 0 \tag{4.7}$$

where $\mathscr{K}(\varepsilon)$ is the linear operator defined in Lemma 4.2.

Suppose the linear system $\dot{x} = \varepsilon A(\varepsilon)x$ has an exponential dichotomy of order k. Scale

$y = \varepsilon^k z$ in formula (4.7) and, if $k \le N/2$, H, $\partial H/\partial y$ are continuous in a small neighborhood of the origin, $H(0, 0) = 0$ and $(\partial H/\partial y)(0, 0) = I$. Using the implicit function theorem, we obtain a unique almost-periodic solution $y^* = y^*(t, \varepsilon)$ of (4.6), analytic in $\varepsilon \in (0, \varepsilon_1)$ for some $\varepsilon_1 < \varepsilon_0$, $y^*(\cdot, 0) = 0$. This means that Eq. (4.3) has a continuous almost-periodic solution $x^*(t, \varepsilon) = x_N(\varepsilon) + y^*(t, \varepsilon)$, analytic in ε, $x^*(\cdot, 0) = x_0$. To estimate $|y^*(\cdot, \varepsilon)| = |x^*(\cdot, \varepsilon) - x_N(\varepsilon)|$, consider $y^* = y^*(\cdot, \varepsilon)$ given by the formula

$$y^* = \varepsilon \mathscr{K}(\varepsilon)[G(y^*, \varepsilon) + \varepsilon^N \hat{f}(\cdot, x_N(\varepsilon) + y^*, \varepsilon)] \tag{4.8}$$

where $|\mathscr{K}(\varepsilon)| \le K/\varepsilon^k$. We can proceed by iterations, taking $y_0 = 0$, and if $k \le N/2$ the estimates yield $|y^*| \le \bar{K}\varepsilon^{N/2+1}$ and the linear variational equation of (4.3) at $x^*(t, \varepsilon)$ and of (4.4) at $x_N(\varepsilon)$ coincide up to order $N/2$. Theorem 2.1 is applied to complete the proof.

As a special case consider the linear system:

$$\dot{x} = L_\varepsilon(t)x = \varepsilon A_1(t)x + \cdots + \varepsilon^k A_k(t)x + \varepsilon^{k+1}C(t, \varepsilon)x \tag{4.9}$$

where $A_i(t)$, $i = 1, 2, \ldots k$, are matrices whose elements are trigonometrical polynomials and $C(t, \varepsilon)$ is an almost-periodic matrix, continuous in $\varepsilon \in [0, \varepsilon_0]$, uniformly in $t \in R$.

Consider the averaged system:

$$\dot{y} = \varepsilon B(\varepsilon)y + \varepsilon^{k+1}D(t, \varepsilon)y \tag{4.10}$$

where $B(\varepsilon) = B_1 + \varepsilon B_2 + \cdots + \varepsilon^{k-1}B_k$, $0 < \varepsilon < \varepsilon_0$, and $D(t, \varepsilon)$ has the same properties as $C(t, \varepsilon)$.

Suppose that B_1 has a simple eigenvalue zero, all others lying in the left half-plane. Then $B(\varepsilon)$ has, for ε small, a simple real analytic eigenvalue $\lambda(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \cdots$ and the stability properties of (4.9) depend only on the sign of the first nonvanishing coefficient a_{j_0} of $\lambda(\varepsilon)$ provided that $j_0 \le k - 1$. Actually, the following result is valid:

THEOREM 4.3. Let $j_0 \le k - 1$; if $a_{j_0} \ne 0$, Eq. (4.9) has an exponential dichotomy of order $j_0 + 1$ and for each function $f \in \mathscr{AP}$ the equation

$$\dot{x} = L_\varepsilon(t)x + f(t) \tag{4.11}$$

has a unique solution $x^* = x(t, f, \varepsilon) \in \mathscr{AP}$, stable if $a_{j_0} < 0$ and unstable if $a_{j_0} > 0$.

This result is given by Krasnosels'ki [5].

REFERENCES

[1] W. A. Coppel, *Dichotomies in stability theory*, Springer-Verlag, 1978
[2] W. A. Coppel, *Dichotomies and reducibility*, J. Diff. Eqs. **3**, 500–521 (1973)
[3] A. M. Fink, *Almost periodic differential equations*, Springer-Verlag, 1974
[4] J. K. Hale, *Ordinary differential equations*, 2nd ed., Robert E. Krieger Publishing Co., New York, 1980
[5] M. A. Krasnosels'ki, V. S. Burd and Y. S. Kolesov, *Nonlinear almost periodic oscillations*, John Wiley & Sons, New York, 1973
[6] J. A. Murdock, *Some mathematical aspects of spin orbit resonance*, Cel. Mech. **8**, 237–253 (1978)
[7] J. A. Murdock and C. Robinson, *Qualitative dynamics from asymptotic expansions—local theory*, J. Diff. Eqs. **36**, 425–441 (1980)