CONSERVED QUANTITY PARTITION FOR DIRAC'S EQUATION*

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Abstract. Let M be the $(n+1)$-dimensional Minkowski space, $n \geqslant 3$. The energy of a solution ψ to Dirac's equation in M is a sum of n terms, the jth term depending on ψ and the space derivative $\partial\psi/\partial x_j$. We show that if the Cauchy datum for ψ is compactly supported, then each of these terms is eventually constant. Specifically, if ψ is initially supported in the closed ball of radius b about the origin in space ($\mathbf{R}^n$), then for times $|t| \geqslant b$, the jth term is equal to the energy of the jth Riesz transform $(-\Delta)^{-1/2}(\partial/\partial x_j)\psi$, which also solves Dirac's equation.

We give similar results for other conserved quantities, which arise due to the conformal invariance of Dirac's equation, in place of the energy. The Lorentz quantities are eventually partitioned into a number of linear functions of the time, while the zeroth inversional (or "conformal") quantity is partitioned into quadratic functions of the time.

0. Introduction. In [5], Duffin used the Paley-Wiener Theorem of Fourier analysis to prove energy equipartition for the wave equation, a result first explicitly stated and proved by Lax and Phillips in [7]. Specifically, the statement is that the energy of a classical solution to the wave equation, compactly supported in space at some fixed time, is partitioned in finite time into exactly equal kinetic and potential parts. Though the result is stated in [5] for 3 space dimensions, the same proof goes through for odd space dimension $n \geqslant 3$. In [4], Dassios uses the same method to prove finite time energy equipartition (into equal electric and magnetic parts) for Maxwell's equations. Again, the result can be stated and proved for odd space dimension $n \geqslant 3$ (the electromagnetic field being an $(n+1)/2$-form).

In [1] and [2], the author considered the problem of partitioning the "higher" conformal conserved quantities for the wave and Maxwell equations. It was shown that under the same compact support assumption on Cauchy data, the conserved quantities coming from Lorentz invariance are partitioned in finite time into a number of linear functions of the time t, and that the "zeroth inversional", or "conformal" conserved quantity is partitioned into quadratic functions of t. These theorems resulted from a partitioning of certain first and second moments of the energy into kinetic and potential (or electric and magnetic) parts.

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Dirac's equation is also conformally invariant, and our purpose here is to determine how the energy, Lorentz quantity, and zeroth inversional quantity (all of which can be calculated using the Lagrangian formalism) are partitioned. Dirac's equation differs from the wave and Maxwell equations in that there is no separation between kinetic and potential energy, and in that the energy is not positive. Instead of an equipartition of energy, we find a partition into n terms, each of which is constant after finite time. These eventual constant values are the energies of the various Riesz transforms $(-\Delta)^{-1/2}(\partial/\partial x_j)\psi$ of the original solution ψ. (These transforms also solve Dirac's equation.) It is important to note (see Remark 5.e below) that this result is not only different from the energy equipartition result given by Costa and Strauss in [3], but deals with an entirely different conserved quantity.

Longer calculations of the type used for the energy give the desired results on partition of the higher conserved quantities.

The approach used here for Dirac's equation can also be applied to the wave equation to further partition the potential energy. If $u = u(x, t)$ is a solution of the wave equation with $u(\,\cdot, 0)$ and $(\partial u/\partial t)(\cdot, 0)$ supported in $\{|x| \leqslant b\}$, then for $|t| \geqslant b$, each of the n terms $\frac{1}{2}\|\partial u/\partial x_j\|^2_{L^2}$ in the potential energy of u is constant. The same idea applied to moments of the energy leads to further partitioning of the Lorentz and zeroth inversional quantities.

The paper is organized as follows. In Sec. 1, we standardize notation and record the conformal conservation laws associated with Dirac's equation. In Sec. 2, we partition the energy, in Sec. 3 the Lorentz quantities, and in Sec. 4 the inversional quantity. In Sec. 5, we make some remarks and state some related results, including the new results for the wave equation described above.

1. Dirac's equation and associated conservation laws. Fix an odd integer $n \geqslant 3$, and let M be the $(n+1)$-dimensional Minkowski space, with coordinate functions $(x, t) = (x_1, \ldots, x_n, t)$. For our purposes, a *spinor* on M is a C^∞ function $\psi\colon M \to \mathbf{C}^N$, where $N = 2^{(n+1)/2}$. *Dirac's equation* is

$$\dot{\psi} = \sum_{j=1}^{n} \alpha_j \partial_j \psi, \tag{1.1}$$

where the dot denotes $\partial/\partial t$, $\partial_j = \partial/\partial x_j$, and the α_j are $N \times N$ complex matrices satisfying

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \tag{1.2}$$

$$\alpha_j^* = \alpha_j. \tag{1.3}$$

In terms of the so-called "γ-matrices" on M, $\alpha_j = \gamma_j\gamma_0$, where γ_0 is the matrix corresponding to the time coordinate. The particular representation of the relations (1.2) and (1.3), and even the dimension N of the representation space, will be immaterial to the results below.

Let $\langle\ ,\ \rangle$ and $|\ |$ denote the usual inner product and norm in $\mathbf{C}^N$, and let $\langle\langle\ ,\ \rangle\rangle$ and $\|\ \|$ be the inner product and norm in $L^2(\mathbf{R}^n, \mathbf{C}^N)$;

$$\langle\langle \psi, \psi' \rangle\rangle = \int_{\mathbf{R}^n} \langle \psi(x), \psi'(x) \rangle \, dx.$$

Let $\mathscr{I}$ denote the taking of the imaginary part. As a consequence of its conformal invariance, Dirac's equation admits $(n+2)(n+3)/2$ independent conserved quantities. These are the integrals over space ($\mathbf{R}^n$) of the densities [6]

$$\begin{aligned}
\varepsilon &= \mathscr{I}\langle \sum_j \alpha_j \partial_j \psi, \psi\rangle && \text{(energy density)},\\
p_j &= \mathscr{I}\langle \partial_j \psi, \psi\rangle && \text{(linear momentum densities)},\\
\lambda_{jk} &= x_j p_k - x_k p_j - \tfrac{1}{2}\mathscr{I}\langle \alpha_j\psi, \alpha_k\psi\rangle && \text{(angular momentum densities)},\\
b_j &= t p_j + x_j \varepsilon && \text{(Lorentz densities)},\\
h &= t\varepsilon + \sum_j x_j p_j && \text{(dilational density)},\\
i_0 &= (r^2+t^2)\varepsilon + 2t\sum_j x_j p_j && \text{(zeroth inversional density)},\\
i_j &= -2t x_j \varepsilon - 2x_j \sum_k x_k p_k &&\\
&\quad +(r^2-t^2)p_j + \sum_k \mathscr{I}\langle x_k \alpha_j \psi, \alpha_k\psi\rangle && \text{(jth inversional density)}.
\end{aligned}$$

Here and below, the indices j, k, m, and q run from 1 to n, and $r^2 = \sum_j x_j^2$. The integrals are conserved (with suitable decay assumptions on ψ) because the time derivative of each density is an exact divergence:

$$\begin{aligned}
\dot{\varepsilon} &= \sum_k \partial_k\left(\mathscr{I}\left\langle \sum_j \alpha_j\partial_j\psi, \alpha_k\psi\right\rangle\right) \equiv \nabla\cdot\beta,\\
\dot{p}_j &= \sum_k \partial_k\big(\mathscr{I}\langle \partial_j\psi, \alpha_k\psi\rangle\big) \equiv \nabla\cdot\pi_j,\\
\dot{\lambda}_{jk} &= \nabla\cdot(x_j\pi_k - x_k\pi_j) + \frac{1}{2}\sum_m \partial_m\big(\mathscr{I}\langle \alpha_j\alpha_k\alpha_m\psi, \psi\rangle\big),\\
\dot{b}_j &= t\nabla\cdot\pi_j + \nabla\cdot(x_j\beta) + \frac{1}{2}\sum_k \partial_k\big(\mathscr{I}\langle \alpha_j\psi, \alpha_k\psi\rangle\big),\\
\dot{h} &= t\nabla\cdot\beta + \nabla\cdot\left(\sum_j x_j\pi_j\right),\\
\dot{i}_0 &= \nabla\cdot(r^2\beta) + t^2\nabla\cdot\beta + 2t\sum_j \nabla\cdot(x_j\pi_j) - \sum_j \partial_j\left(\mathscr{I}\sum_k \langle x_k\alpha_j\psi, \alpha_k\psi\rangle\right),\\
\dot{i}_j &= -2t\nabla\cdot(x_j\beta) - 2\sum_k \nabla\cdot(x_j x_k\pi_k) + \nabla\cdot(r^2\pi_j)\\
&\quad -t^2\nabla\cdot\pi_j - t\sum_k \partial_k\big(\mathscr{I}\langle \alpha_j\psi, \alpha_k\psi\rangle\big) - \sum_m \partial_m\big(\mathscr{I}\langle x_k\alpha_j\alpha_k\alpha_m\psi, \psi\rangle\big).
\end{aligned}$$

The conserved integrals will be denoted E, P_j, Λ_{jk}, B_j, H, I_0, I_j; for example, $E = \int_{\mathbf{R}^n} \varepsilon\, dx$. ($E$ and H re not to be confused with electric and magnetic fields.) When dealing with more than one Dirac spinor, we write $E = E(\psi)$, $P_j = P_j(\psi)$, etc. Our main results concern E (Sec. 2), B_j (Sec. 3), and I_0 (Sec. 4).

2. Energy partition. Let $\hat{}$ denote Fourier transformation in the x variables: if f is a C^∞ function of x and t, say Schwartz class in x for fixed t, then

$$\hat{f}(\xi, t) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{ix\cdot\xi} f(x, t)\, dx,$$

where dx is Lebesgue measure. (The dependence on t will sometimes be suppressed.) The Fourier transform of a spinor is defined component by component, i.e., by $(\hat{\psi})_\lambda = (\psi_\lambda)\hat{}$ for $\lambda = 1, \ldots, N$. We record the identities

$$(\partial_j\psi)\hat{} = \frac{1}{i}\xi_j\hat{\psi}, \qquad (x_j\psi)\hat{} = \frac{1}{i}\partial_j\hat{\psi}.$$

(We shall rely on the context to distinguish between $\partial_j = \partial/\partial x_j$ and $\partial_j = \partial/\partial\xi_j$.)

In ξ-space, let $\rho = \Sigma_j \xi_j^2$ and $\theta_j = \xi_j/\rho$. We define the *Riesz transforms*

$$R_j\psi = \frac{\partial_j}{\sqrt{-\Delta}}\psi = \left(\frac{1}{i}\theta_j\hat{\psi}\right)\check{}, \qquad \psi \in L^2.$$

Similarly, we have the iterated Riesz transforms

$$-R_jR_k\psi = \frac{\partial_j\partial_k}{\Delta}\psi = \left(\theta_j\theta_k\hat{\psi}\right)\check{}, \qquad \psi \in L^2.$$

Our energy partition result states that for Dirac spinors with compact support in space at time 0, the summand $\mathcal{I}\langle\langle\alpha_j\partial_j\psi, \psi\rangle\rangle$ (fixed j) in $E(\psi)$ eventually takes on the exact value $E(R_j\psi)$. (Note that a Riesz transform of a solution to (1.1) is again a solution.) Two remarks on notation are important: (1) Summation over repeated indices is *never* implied: all intended summation signs are explicitly supplied; (2) ψ_t will denote $\psi(\cdot, t)$, and *never* the time derivative of ψ, which is denoted $\dot{\psi}$.

THEOREM 2.1. If ψ solves (1.1) and $\psi_0 = \psi(\cdot, 0)$ is supported in $\{r \leqslant b\}$, then for $|t| \geqslant b$,

$$\mathcal{I}\left\langle\left\langle\alpha_j\partial_j\psi_t, \psi_t\right\rangle\right\rangle = \mathcal{I}\left\langle\left\langle\sum_k \alpha_k\partial_k\psi_0, \frac{\partial_j^2}{\Delta}\psi_0\right\rangle\right\rangle. \tag{2.1}$$

Thus the left side of (2.1) takes on the value of the constant of motion $E(R_j\psi)$ for $|t| \geqslant b$.

For the proof, we apply the Paley-Wiener Theorem according to the method of Duffin [5] (see also [4, 1, 2]).

Definition 2.2. A (possibly vector-valued) function $f(z)$ of one complex variable is of *exponential type* $a > 0$ if for each $\varepsilon > 0$, there is a constant A_ε with $|f(z)| \leqslant A_\varepsilon e^{(a+\varepsilon)|z|}$.

THEOREM 2.3 (PALEY-WIENER). Let $f(z)$ be an entire function (possibly vector-valued), of exponential type a, and of class L^1 on the real axis. Then the Fourier transform of the restriction of $f(z)$ to the real axis is zero outside $[-a, a]$.

Proof of Theorem 2.1. By (1.2), the components of ψ solve the wave equation: $\ddot{\psi} = \Delta\psi$. Fourier transforming, we find that $\hat{\psi}(\xi)$ is a solution of the ordinary differential equation

$$\ddot{u} + \rho^2 u = 0, \tag{2.2}$$

with initial data

$$u_0 = \hat{\psi}_0(\xi), \qquad \dot{u}_0 = \frac{1}{i}\sum_k \alpha_k \xi_k \hat{\psi}_0(\xi). \tag{2.3}$$

(The compact support assumption allows us to interchange $\hat{}$ and $\dot{}$. Since the wave equation has finite propagation speed, ψ has compact spatial support at *each* fixed time.) The problem (2.2), (2.3) has the explicit solution

$$u_t = \hat{\psi}_t(\xi) = \hat{\psi}_0(\xi)\cos\rho t + \frac{1}{i}\sum_k \alpha_k \theta_k \hat{\psi}_0(\xi)\sin\rho t. \tag{2.4}$$

Now $\mathcal{I}\langle\langle \alpha_j \partial_j \psi_t, \psi_t\rangle\rangle = \mathcal{I}((1/i)\langle\langle \alpha_j \xi_j \hat{\psi}_t, \hat{\psi}_t\rangle\rangle)$, but (2.4), together with the trigonometric identities

$$\begin{aligned} \cos^2\zeta &= \tfrac{1}{2}(1+\cos 2\zeta), \\ \sin^2\zeta &= \tfrac{1}{2}(1-\cos 2\zeta), \\ \sin\zeta\cos\zeta &= \tfrac{1}{2}\sin 2\zeta, \end{aligned} \tag{2.5}$$

yields

$$\mathcal{I}\big((1/i)\big\langle \alpha_j \xi_j \hat{\psi}_t, \hat{\psi}_t\big\rangle\big) = \tfrac{1}{2}(X_0 + X_2) + \tfrac{1}{2}(X_0 - X_2)\cos 2\rho t + \tfrac{1}{2}X_1 \sin 2\rho t,$$

where

$$\begin{aligned} X_0 &= \mathcal{I}\left(\frac{1}{i}\left\langle \alpha_j \xi_j \hat{\psi}_0, \hat{\psi}_0\right\rangle\right), \\ X_1 &= \mathcal{I}\left(2\left\langle \alpha_j \xi_j \hat{\psi}_0, \sum_k \alpha_k \theta_k \hat{\psi}_0\right\rangle\right), \\ X_2 &= \mathcal{I}\left(-\frac{1}{i}\left\langle \alpha_j \xi_j \hat{\psi}_0, \hat{\psi}_0\right\rangle + \frac{2}{i}\left\langle \sum_k \alpha_k \xi_k \hat{\psi}_0, \theta_j^2 \hat{\psi}_0\right\rangle\right). \end{aligned}$$

(The formula for X_2 involves some simplification using (1.2) and (1.3).) Integrating over ξ, we get

$$\begin{aligned} \mathcal{I}\left(\frac{1}{i}\left\langle\left\langle \alpha_j \xi_j \hat{\psi}_t, \hat{\psi}_t\right\rangle\right\rangle\right) &= \mathcal{I}\left(\frac{1}{i}\left\langle\left\langle \sum_k \alpha_k \xi_k \hat{\psi}_0, \theta_j^2 \hat{\psi}_0\right\rangle\right\rangle\right) \\ &\quad + \tfrac{1}{2}\int_{\mathbf{R}^n} (X_0 - X_2)\cos 2\rho t \, d\xi + \tfrac{1}{2}\int_{\mathbf{R}^n} X_1 \sin 2\rho t \, d\xi. \end{aligned} \tag{2.6}$$

We would like to show that the integral terms in (2.6) vanish for $|t| \geqslant b$.

Switching to spherical coordinates, the integral terms become

$$\begin{aligned} &\tfrac{1}{2}\int_0^\infty \rho^{n-1}\left(\int_{S^{n-1}} (X_0 - X_2)\, d\theta\right)\cos 2\rho t \, d\rho \\ &\quad + \tfrac{1}{2}\int_0^\infty \rho^{n-1}\left(\int_{S^{n-1}} X_1 \, d\theta\right)\sin 2\rho t \, d\rho, \end{aligned}$$

where $d\theta$ is the usual measure on the unit sphere S^{n-1}. Let $\theta = (\theta_1, \ldots, \theta_n)$, so that $\xi = \rho\theta$. (The θ_j are not angular variables, but components of the unit vector in the direction ξ.) By definition,

$$\hat{\psi}_0(\xi) = \hat{\psi}_0(\rho\theta) = (2\pi)^{-n/2} \int_{\{r \leqslant b\}} e^{i\rho x \cdot \theta} \psi_0(x)\, dx.$$

With θ fixed, this integral converges for all *complex* ρ, yielding an entire extension of $\hat{\psi}_0$, of exponential type b, and of Schwartz class in real ρ. The same is true for $\bar{\hat{\psi}}_0$, though the extension of $\bar{\hat{\psi}}_0$ is not the complex conjugate of the extension of $\hat{\psi}_0$; in the integral formula, ψ_0 is conjugated and the i in the exponent is changed to $-i$, but the ρ is not conjugated. Since $\xi_k = \rho\theta_k$, the X_p extend, for fixed θ, to entire functions of ρ of exponential type $2b$, of Schwartz class in real ρ. (Multiplication by a polynomial in ρ does not change the exponential type.) Furthermore,

$$X_p(-\rho, -\theta) = X_p(\rho, \theta), \qquad p = 0, 2;$$
$$X_1(-\rho, -\theta) = -X_1(\rho, \theta).$$

Thus

$$Y_0 = \rho^{n-1} \int_{S^{n-1}} (X_0 - X_2)\, d\theta,$$

$$Y_1 = \rho^{n-1} \int_{S^{n-1}} X_1\, d\theta$$

are entire, of exponential type $2b$, and of class L^1 in real ρ. Because $n - 1$ is even, Y_0 is even and Y_1 is odd. This means that the integral terms in (2.5) are equal to

$$\frac{\sqrt{2\pi}}{2} \left(\hat{Y}_0(2t) + \hat{Y}_1(2t) \right),$$

since the Fourier transform of an even (odd) function is a cosine (sine) transform. By the Paley-Wiener Theorem, the integral terms vanish for $|t| \geqslant b$, and (2.6) becomes (2.1). □

3. Partition of the Lorentz quantities. We show next that under the same compact support assumption on ψ_0, each term $\mathfrak{I}\langle\langle x_m \alpha_j \partial_j \psi, \psi \rangle\rangle$ (fixed j) in the Lorentz quantity

$$B_m = \mathfrak{I}\left\langle\left\langle x_m \sum_j \alpha_j \partial_j \psi, \psi \right\rangle\right\rangle + t\mathfrak{I}\langle\langle \partial_m \psi, \psi \rangle\rangle$$

is eventually a constant-coefficient linear function of t, with coefficients depending only on ψ_0.

THEOREM 3.1. If ψ solves (1.1) and ψ_0 is supported in $\{r \leqslant b\}$, then for $|t| \geqslant b$,

$$\mathfrak{I}\langle\langle x_m \alpha_j \partial_j \psi_t, \psi_t \rangle\rangle = \mathfrak{I}\left\langle\left\langle x_m \sum_k \alpha_k \partial_k \psi_0, \frac{\partial_j^2}{\Delta} \psi_0 \right\rangle\right\rangle$$
$$-t\mathfrak{I}\left\langle\left\langle \partial_m \psi_0 \frac{\partial_j^2}{\Delta} \psi_0 \right\rangle\right\rangle + \tfrac{1}{2} \sum_k \mathfrak{I}\left\langle\left\langle \alpha_j \alpha_k \alpha_m \psi_0, \frac{\partial_j \partial_k}{\Delta} \psi_0 \right\rangle\right\rangle. \quad (3.1)$$

Proof. Proceeding as in the proof of Theorem 2.1, we get

$$\mathscr{I}\left\langle\left\langle x_m\alpha_j\partial_j\psi_t, \psi_t\right\rangle\right\rangle = -\mathscr{I}\left\langle\left\langle \partial_m\left(\alpha_j\xi_j\hat{\psi}_t\right), \hat{\psi}_t\right\rangle\right\rangle$$
$$= -\mathscr{I}\left\langle\left\langle \alpha_j\xi_j\partial_m\hat{\psi}_t, \hat{\psi}_t\right\rangle\right\rangle,$$

since $\langle \alpha_j\hat{\psi}, \hat{\psi}\rangle$ is real. Now

$$\partial_m\hat{\psi}_t = \left\{\partial_m\hat{\psi}_0 + \frac{t}{i}\sum_k \alpha_k\theta_k\theta_m\hat{\psi}_0\right\}\cos\rho t$$
$$+\left\{-t\theta_m\hat{\psi}_0 + \frac{1}{i}\sum_k \alpha_k\theta_k\partial_m\hat{\psi}_0 + \frac{1}{i}\alpha_m\frac{\hat{\psi}_0}{\rho} - \frac{1}{i}\sum_k \alpha_k\theta_k\theta_m\frac{\hat{\psi}_0}{\rho}\right\}\sin\rho t. \tag{3.2}$$

Using the double angle identities (2.5), we get

$$-\mathscr{I}\left\langle \alpha_j\xi_j\partial_m\hat{\psi}_t, \hat{\psi}_t\right\rangle = \tfrac{1}{2}(X_0 + X_2) + \tfrac{1}{2}(X_0 - X_2)\cos 2\rho t + \tfrac{1}{2}X_1\sin 2\rho t, \tag{3.3}$$

where (after some simplification of X_0 and X_2)

$$X_0 = -\mathscr{I}\left\langle \alpha_j\xi_j\partial_m\hat{\psi}_0, \hat{\psi}_0\right\rangle - \mathscr{I}\left(\frac{t}{i}\left\langle \xi_m\hat{\psi}_0, \theta_j^2\hat{\psi}_0\right\rangle\right),$$

$$X_1 = -\mathscr{I}\left\langle \alpha_j\xi_j\left(\partial_m\hat{\psi}_0 + \frac{t}{i}\sum_k \alpha_k\theta_k\theta_m\hat{\psi}_0\right), \frac{1}{i}\sum_q \alpha_q\theta_q\hat{\psi}_0\right\rangle$$
$$-\mathscr{I}\left\langle \alpha_j\theta_j\left(-t\xi_m\hat{\psi}_0 + \frac{1}{i}\sum_k \alpha_k\xi_k\partial_m\hat{\psi}_0 + \frac{1}{i}\alpha_m\hat{\psi}_0 - \frac{1}{i}\sum_k \alpha_k\theta_k\theta_m\hat{\psi}_0\right), \hat{\psi}_0\right\rangle,$$

$$X_2 = \mathscr{I}\left\langle \alpha_j\xi_j\partial_m\hat{\psi}_0, \hat{\psi}_0\right\rangle - 2\mathscr{I}\left\langle \sum_k \alpha_k\xi_k\partial_m\hat{\psi}_0, \theta_j^2\hat{\psi}_0\right\rangle$$
$$+\sum_k \mathscr{I}\left\langle \alpha_j\alpha_k\alpha_m\hat{\psi}_0, \theta_j\theta_k\hat{\psi}_0\right\rangle - \mathscr{I}\left(\frac{t}{i}\left\langle \xi_m\hat{\psi}_0, \theta_j^2\hat{\psi}_0\right\rangle\right).$$

Integrating (3.3) in ξ, the method of Section 2 works perfectly to show that the integrals involving the trigonometric functions vanish for $|t| \geqslant b$. ($\partial_m\hat{\psi}_0 = (ix_m\psi_0)\hat{}$ has an entire extension with the same properties as that of $\hat{\psi}_0$.) The other integrals simplify to

$$-\mathscr{I}\left\langle\left\langle \sum_k \alpha_k\partial_m(\xi_k\hat{\psi}_0), \theta_j^2\hat{\psi}_0\right\rangle\right\rangle - \mathscr{I}\left(\frac{t}{i}\left\langle\left\langle \xi_m\hat{\psi}_0, \theta_j^2\hat{\psi}_0\right\rangle\right\rangle\right)$$
$$+\tfrac{1}{2}\sum_k \mathscr{I}\left\langle\left\langle \alpha_j\alpha_k\alpha_m\hat{\psi}_0, \theta_j\theta_k\hat{\psi}_0\right\rangle\right\rangle.$$

Applying the Plancherel Theorem, we get (3.1). □

Thus $B_m(\psi)$ is eventually partitioned into $n + 1$ linear functions of t, the coefficients of which may be calculated from the Cauchy datum ψ_0.

Remark 3.2. The interpretation of (3.1) in terms of constants of motion of the solution $R_j\psi$ is not so clear-cut as for the energy. (3.1) implies that

$$\mathcal{I}\langle\langle x_m\alpha_j\partial_j\psi_t, \psi_t\rangle\rangle = B_m(R_j\psi) - tP_m(R_j\psi) + \tfrac{1}{2}\sum_k \mathcal{I}\left\langle\left\langle \alpha_j\alpha_k\alpha_m\psi_0, \frac{\partial_j\partial_k}{\Delta}\psi_0\right\rangle\right\rangle, \qquad |t| \geqslant b.$$

It appears that the coefficients of this linear function of t, though constant, cannot be expressed entirely as constants of motion of Riesz transforms of ψ.

4. Partition of the zeroth inversional quantity. The zeroth inversional quantity may be written

$$I_0 = \mathcal{I}\left\langle\left\langle x\sum_j \alpha_j\partial_j\psi, x\psi\right\rangle\right\rangle + t^2E + 2t\mathcal{I}\langle\langle x\cdot\nabla\psi, \psi\rangle\rangle$$

Here $x\psi$, for example, is to be thought of as a "vector of spinors" $(x_1\psi, \ldots, x_n\psi)$, and $x\cdot\nabla\psi$ denotes $\Sigma_k x_k\partial_k\psi$. We would like to show that each $\mathcal{I}\langle\langle x\alpha_j\partial_j\psi, x\psi\rangle\rangle$ (fixed j) is eventually a constant-coefficient quadratic polynomial in t.

THEOREM 4.1. *If ψ solves* (1.1) *and ψ_0 is supported in $\{r \leqslant b\}$, then for $|t| \geqslant b$,*

$$\begin{aligned}\mathcal{I}\langle\langle x\alpha_j\partial_j\psi_t, x\psi_t\rangle\rangle = {} & \mathcal{I}\left\langle\left\langle x\sum_k\alpha_k\partial_k\psi_0, x\frac{\partial_j^2}{\Delta}\psi_0\right\rangle\right\rangle \\ & + t^2\mathcal{I}\left\langle\left\langle\sum_k\alpha_k\partial_k\psi_0, \frac{\partial_j^2}{\Delta}\psi_0\right\rangle\right\rangle \\ & - 2t\mathcal{I}\left\langle\left\langle x\cdot\nabla\psi_0, \frac{\partial_j^2}{\Delta}\psi_0\right\rangle\right\rangle \\ & + \sum_k\sum_m \mathcal{I}\left\langle\left\langle x_m\alpha_j\alpha_k\alpha_m\psi_0, \frac{\partial_j\partial_k}{\Delta}\psi_0\right\rangle\right\rangle. \end{aligned} \tag{4.1}$$

Proof. Since $\langle\alpha_j\psi, x_j\psi\rangle$ is real, we have

$$\mathcal{I}\langle\langle x\alpha_j\partial_j\psi, x\psi\rangle\rangle = \mathcal{I}\langle\langle\alpha_j\partial_j(x\psi), x\psi\rangle\rangle = \mathcal{I}\left(\frac{1}{i}\langle\langle\alpha_j\xi_j\nabla\hat{\psi}, \nabla\hat{\psi}\rangle\rangle\right),$$

where the gradient is in the ξ variables. Rewriting (3.2) in vector notation, we get

$$\begin{aligned}\nabla\psi_t = {} & \left\{\nabla\hat{\psi}_0 + \frac{t}{i}\sum_k\alpha_k\theta_k\theta\hat{\psi}_0\right\}\cos\rho t \\ & + \left\{-t\theta\hat{\psi}_0 + \frac{1}{i}\sum_k\alpha_k\theta_k\nabla\hat{\psi}_0 + \frac{1}{i}\sum_k\alpha_k e_k\frac{\hat{\psi}_0}{\rho} - \frac{1}{i}\sum_k\alpha_k\theta_k\theta\frac{\hat{\psi}_0}{\rho}\right\}\sin\rho t,\end{aligned}$$

where e_k is the kth standard basis vector in $\mathbf{R}^n$.

In analogy with Secs. 2 and 3, we write

$$\mathcal{I}\left(\frac{1}{i}\left\langle \alpha_j \xi_j \nabla \hat{\psi}_t, \nabla \hat{\psi}_t \right\rangle\right) = \tfrac{1}{2}(X_0 + X_2) + \tfrac{1}{2}(X_0 - X_2)\cos 2\rho t + \tfrac{1}{2} X_1 \sin 2\rho t, \tag{4.2}$$

where (after a very tedious calculation to simplify X_2)

$$\begin{aligned} X_0 = {} & \mathcal{I}\left(\frac{1}{i}\left\langle \alpha_j \xi_j \nabla \hat{\psi}_0, \nabla \hat{\psi}_0 \right\rangle\right) \\ & + t^2 \mathcal{I}\left(-\frac{1}{i}\left\langle \alpha_j \xi_j \hat{\psi}_0, \hat{\psi}_0 \right\rangle + \frac{2}{i}\left\langle \sum_k \alpha_k \xi_k \hat{\psi}_0, \theta_j^2 \hat{\psi}_0 \right\rangle\right) \\ & + 2t\mathcal{I}\left(-\left\langle \alpha_j \theta_j \sum_k \alpha_k \theta_k \xi \cdot \nabla \hat{\psi}_0, \hat{\psi}_0 \right\rangle + 2\left\langle \xi \cdot \nabla \hat{\psi}_0, \theta_j^2 \hat{\psi}_0 \right\rangle\right), \end{aligned}$$

$$\begin{aligned} X_2 = {} & \mathcal{I}\left\{-\frac{1}{i}\left\langle \alpha_j \xi_j \nabla \hat{\psi}_0, \nabla \hat{\psi}_0 \right\rangle + \frac{2}{i}\sum_k \left\langle \alpha_k \nabla(\xi_k \hat{\psi}_0), \nabla(\theta_j^2 \hat{\psi}_0) \right\rangle \right. \\ & + \frac{2}{i}\sum_{k,m} \left\langle \alpha_j \alpha_k \alpha_m \partial_m \hat{\psi}_0, \theta_j \theta_k \hat{\psi}_0 \right\rangle \\ & - \frac{1}{i}\sum_k \left(\left\langle \alpha_j \theta_j \theta_k \hat{\psi}_0, \partial_k \hat{\psi}_0 \right\rangle + \left\langle \partial_k(\alpha_j \theta_j \theta_k \hat{\psi}_0), \hat{\psi}_0 \right\rangle\right) \\ & \left. - \frac{1}{i}\sum_k \left(\left\langle \alpha_k \hat{\psi}_0, \partial_k(\theta_j^2 \hat{\psi}_0) \right\rangle + \left\langle \partial_k(\alpha_k \hat{\psi}_0), \theta_j^2 \hat{\psi}_0 \right\rangle\right)\right\} \\ & + t^2 \mathcal{I}\left(\frac{1}{i}\left\langle \alpha_j \xi_j \hat{\psi}_0, \hat{\psi}_0 \right\rangle\right) + 2t\mathcal{I}\left\langle \alpha_j \theta_j \sum_k \alpha_k \theta_k \xi \cdot \nabla \hat{\psi}_0, \hat{\psi}_0 \right\rangle. \end{aligned}$$

In deriving the expression for X_2, we have repeatedly used the formula

$$\partial_k \theta_m = (\delta_{km} - \theta_k \theta_m)/\rho. \tag{4.3}$$

The proof that the integrals

$$\int_{\mathbf{R}^n} (X_0 - X_2)\cos 2\rho t \, d\xi, \qquad \int_{\mathbf{R}^n} X_1 \sin 2\rho t \, d\xi \tag{4.4}$$

vanish for $|t| \geqslant b$ now goes through as in Secs. 2 and 3, with one slight modification. Because of (4.3), the extensions to complex ρ of functions like $\partial_k(\alpha_j \theta_j \theta_k \hat{\psi}_0)$, and thus the extension of $X_2(\cdot, \theta)$, may have simple poles at $\rho = 0$. To get around this, we note that for fixed θ, $Z = \rho^2 X_2$ is entire, of exponential type $2b$, and of Schwartz class in real ρ. Furthermore, $Z(-\rho, -\theta) = Z(\rho, \theta)$. Since $n \geqslant 3$, $\rho^{n-3}\int_{S^{n-1}} Z \, d\theta$ is entire, of exponential type $2b$, of class L^1 in real ρ, and even. Thus the integrals (4.4) vanish for $|t| \geqslant b$.

By (4.2), then, the eventual value of $\mathcal{I}((1/i)\langle\langle \alpha_j \xi_j \nabla \hat{\psi}_t, \nabla \hat{\psi}_t\rangle\rangle$ is

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbf{R}^n}(X_0 + X_2)\,d\xi = \mathcal{I}\Big\{&\frac{1}{i}\Big\langle\Big\langle \sum_k \alpha_k \nabla(\xi_k\psi_0), \nabla\big(\theta_j^2\psi_0\big)\Big\rangle\Big\rangle \\ &+ \frac{t^2}{i}\Big\langle\Big\langle \sum_k \alpha_k\xi_k\hat{\psi}_0, \theta_j^2\hat{\psi}_0\Big\rangle\Big\rangle + 2t\big\langle\big\langle \nabla\cdot(\xi\psi_0), \theta_j^2\psi_0\big\rangle\big\rangle \\ &+ \frac{1}{i}\sum_k\sum_m \big\langle\big\langle \alpha_j\alpha_k\alpha_m\partial_m\hat{\psi}_0, \theta_j\theta_k\hat{\psi}_0\big\rangle\big\rangle \\ &- \frac{1}{i}\sum_k \big\langle\big\langle \alpha_j\theta_j\theta_k\hat{\psi}_0, \partial_k\hat{\psi}_0\big\rangle\big\rangle - \frac{1}{i}\sum_k\big\langle\big\langle \partial_k\big(\alpha_j\theta_j\theta_k\hat{\psi}_0\big), \hat{\psi}_0\big\rangle\big\rangle \qquad (4.5) \\ &- \frac{1}{i}\sum_k\big\langle\big\langle \alpha_k\hat{\psi}_0, \partial_k\big(\theta_j^2\hat{\psi}_0\big)\big\rangle\big\rangle - \frac{1}{i}\sum_k\big\langle\big\langle \partial_k\big(\alpha_k\hat{\psi}_0\big), \theta_j^2\hat{\psi}_0\big\rangle\big\rangle\Big\}.\end{aligned}$$

By the skew-adjointness of ∂_k, the last four terms above sum to 0. ($\theta_j\theta_k\hat{\psi}_0$ is in the L^2 Sobolev class $H^{(n-1)/2} \subseteq H^1$, since each differentiation introduces one new factor of ρ^{-1}.) Applying the Plancherel Theorem to (4.5) without these terms, we get exactly (4.1). □

Formula (4.1), the energy partition result, and the identity

$$i_0 = (r^2 - t^2)\varepsilon + 2th$$

show that I_0 is eventually partitioned into $2n+1$ constant-coefficient quadratic or linear functions of t.

5. Remarks.

(a) The energy partition result trivially gives a partition result for the dilational quantity H, since $h = t\varepsilon + \sum_k x_k p_k$. The result on partition of the Lorentz quantities gives a partition result for the mth inversional quantity I_m, since

$$i_m = -2tb_m - 2x_m\sum_k x_k p_k + (r^2+t^2)p_m + \sum_k \mathcal{I}\langle x_k\alpha_m\psi, \alpha_k\psi\rangle.$$

H is partitioned into $n+1$ linear functions,

$$\begin{aligned} t\mathcal{I}\big\langle\big\langle \alpha_j\partial_j\psi, \psi\big\rangle\big\rangle &= tE\big(R_j\psi\big), && |t| \geqslant b; \\ \sum_k \mathcal{I}\big\langle\big\langle x_j\partial_j\psi, \psi\big\rangle\big\rangle &= H(\psi) - tE(\psi), && \text{all } t.\end{aligned}$$

I_m is partitioned into $n+3$ quadratic functions, $n+1$ of them making up the $-2tB_m(\psi)$ term, using Theorem 3.1; and the other two being $t^2P_m(\psi)$ and $I_m(\psi) - t^2P_m(\psi) + 2tB_m(\psi)$.

(b) It is natural to ask whether the present approach gives new information about solutions of the wave equation $\ddot{u} = \Delta u$. (Again we assume odd space dimension $n \geqslant 3$.)

Let $v = \dot{u}$. Conserved quantities of interest are [8]

$$
\begin{aligned}
&E(u) = \tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\|\nabla u\|^2 && \text{(energy)},\\
&P_m(u) = \langle\langle \partial_m u, v \rangle\rangle && \text{(momenta)},\\
&B_m(u) = \tfrac{1}{2}\langle\langle x_m v, v \rangle\rangle + \tfrac{1}{2}\langle\langle x_m \nabla u, \nabla u \rangle\rangle + tP_m(u) && \text{(Lorentz quantities)},\\
&H(u) = tE(u) + \langle\langle x \cdot \nabla u, v \rangle\rangle + \frac{n-1}{2}\langle\langle u, v \rangle\rangle && \text{(dilational quantity)},\\
&I_0(u) = \tfrac{1}{2}\|rv\|^2 + \tfrac{1}{2}\|r\nabla u\|^2 + t^2 E(u) &&\\
&\qquad +2t\langle\langle x \cdot \nabla u, v \rangle\rangle - \frac{n-1}{2}\|u\|^2 &&\\
&\qquad + (n-1)t\langle\langle u, v \rangle\rangle && \text{(zeroth inversional quantity)}.
\end{aligned}
$$

We would first like to know whether the constituents $\frac{1}{2}\|\partial_j u\|^2$ of the potential energy eventually take on constant values, assuming compact support for u_0 and v_0. The answer is yes, as we can see formally, without doing the Paley-Wiener calculation, by applying energy equipartition to the wave equation solution $R_j u$. This solution has potential energy $\frac{1}{2}\nabla(\partial_j/\sqrt{-\Delta})u\|^2 = \frac{1}{2}\|\partial_j u\|^2$, which should eventually equal $\frac{1}{2}E(R_j u)$. In fact, this approach can be made rigorous, even though the Cauchy data for $R_j u$ are not, in general, supported in $\{r \leqslant b\}$ when u_0 and v_0 are. What matters is that the extensions of $\theta_j \hat{u}_0$ and $\theta_j \hat{v}_0$ to complex ρ have the correct exponential type, and that their inner products have the correct total parity in ρ and θ. We have

Theorem 5.1. If u is a solution of the wave equation with u_0 and v_0 supported in $\{r \leqslant b\}$, then

$$\tfrac{1}{2}\|\partial_j u\|^2 = \tfrac{1}{2}E(R_j u), \qquad |t| \geqslant b.$$

Of course, one could also prove Theorem 5.1 (and Theorems 5.2 and 5.3 below) by applying Duffin's original method.

For the Lorentz quantities, the partition theorem analogous to energy equipartition was given by the author in [1]:

$$\tfrac{1}{2}\langle\langle x_m \nabla u, \nabla u \rangle\rangle = \tfrac{1}{2}B_m(u) - \tfrac{1}{2}tP_m(u), \qquad |t| \geqslant b.$$

Just as for the energy, we can extend the eventual partition argument for B_m to cover the non-compactly supported $R_j u$. This, along with a short calculation in the ξ domain (recall (4.3)) which shows

$$\langle\langle x_m \partial_j u, \partial_j u \rangle\rangle = \langle\langle x_m \nabla(R_j u), \nabla(R_j u) \rangle\rangle,$$

gives

Theorem 5.2. If u is a solution of the wave equation with u_0 and v_0 supported in $\{r \leqslant b\}$, then

$$\tfrac{1}{2}\langle\langle x_m \partial_j u, \partial_j u \rangle\rangle = \tfrac{1}{2}B_m(R_j u) - \tfrac{1}{2}tP_m(R_j u), \qquad |t| \geqslant b.$$

The situation for the inversional quantity I_0 is similar. By a result of Zachmanoglou [9], $\|u\|^2$ is constant for $|t| \geq b$ if u_0 and v_0 are supported in $\{r \leq b\}$, say

$$\|u\|^2 = C(u), \qquad |t| \geq b.$$

This result can also be extended to Riesz transforms of solutions with Cauchy data in $\{r \leq b\}$. The partition result for I_0, proved by the author in [1], says that

$$\frac{1}{2}\|r\nabla u\|^2 = \frac{1}{2}t^2E(u) - tH(u) + \frac{1}{2}I_0(u) + \frac{n-1}{2}C(u), \qquad |t| \geq b.$$

A calculation in the ξ domain using (4.3) gives

$$\frac{1}{2}\|r\partial_j u\|^2 = \frac{1}{2}\|r\nabla(R_j u)\|^2 - \frac{n-1}{2}\|R_j u\|^2.$$

Since the I_0 partition result can be extended to $R_j u$, we have

THEOREM 5.3. If u is a solution of the wave equation with u_0 and v_0 supported in $\{r \leq b\}$, then

$$\tfrac{1}{2}\|r\partial_j u\|^2 = \tfrac{1}{2}t^2E(R_j u) - tH(R_j u) + \tfrac{1}{2}I_0(R_j u), \qquad |t| \geq b.$$

(c) For the Maxwell equations, one can apply the results of Remark b to a *vector potential* (see [2, Remark 3.2]) to get new partition results. It seems, however, that these new formulas must involve the vector potential explicitly, and cannot be written in terms of the electromagnetic field alone.

(d) The above finite time partition results (for both the wave and Dirac equations), true for solutions with compactly supported Cauchy data, can be extended to asymptotic results for Cauchy data with suitable spatial decay. Essentially, one applies the Riemann-Lebesgue Lemma in place of the Paley-Wiener Theorem (see, e.g., [4]).

(e) The energy partition result of Sec. 2 is quite different from the equipartition result given by Costa and Strauss [3]. Their result concerns the conserved quantity $\|\psi\|^2$. They show that if a choice of α-matrices is made with

$$\alpha_j = \left(\begin{array}{c|c} 0 & \sigma_j \\ \hline \sigma_j & 0 \end{array}\right),$$

where the σ_j are $N/2 \times N/2$ matrices satisfying

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}, \qquad \sigma_j^* = \sigma_j,$$

then eventually,

$$\|\psi_1\|^2 + \cdots + \|\psi_{N/2}\|^2 = \|\psi_{N/2+1}\|^2 + \cdots + \|\psi_N\|^2.$$

Formally, this is wave equation energy equipartition for the potential

$$(-\Delta)^{-1/2}(\psi_1, \ldots, \psi_{N/2}).$$

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REFERENCES

[1] T. Branson, *Eventual partition of conserved quantities in wave motion*, J. Math. Anal. Appl., **96**, 54–62 (1983)
[2] T. Branson, *Eventual partition of conserved quantities for Maxwell's equations*, Arch. Rational Mech. Anal., to appear
[3] D. Costa and W. Strauss, *Energy splitting*, Quart. Appl. Math. **38**, 351–361 (1981)
[4] G. Dassios, *Equipartition of energy for Maxwell's equations*, Quart. Appl. Math. **37**, 465–469 (1980)
[5] R. Duffin, *Equipartition of energy in wave motion*, J. Math. Anal. Appl. **32**, 386–391 (1970)
[6] R. Glassey, unpublished communication (1982)
[7] P. Lax and R. Phillips, *Scattering theory*, Academic Press, New York, 1967
[8] W. Strauss, *Nonlinear invariant wave equations*, in Lecture Notes in Physics, vol. 73, 197–249, Springer-Verlag, Berlin, 1978
[9] E. Zachmanoglou, *Integral constants in wave motion*, J. Math. Anal. Appl. **39**, 296–297 (1972)