QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLIV APRIL · 1986 NUMBER 1

APRIL 1986, PAGES 1-12

SHEARING OF MATERIALS EXHIBITING THERMAL SOFTENING OR TEMPERATURE DEPENDENT VISCOSITY *

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Abstract. We consider the adiabatic shearing of an incompressible non-Newtonian liquid with temperature dependent viscosity, subjected to steady shearing of the boundary. Identical equations govern the plastic shearing of a solid exhibiting thermal softening and strain rate sensitivity with constitutive relation obeying a certain power law. We establish that every classical solution approaches a uniform shearing solution as $t \to +\infty$ at specific rates of convergence. Therefore, no shear bands formation is predicted for materials of this type.

1. Introduction. The balance laws of continuum thermomechanics, supplemented with constitutive relations characterizing the material, give rise to systems of partial differential equations that describe the evolution of thermomechanical processes. The nonlinear character of the constitutive relations induces, in general, a destabilizing mechanism on these processes, whereas the presence of dissipative mechanisms such as viscosity or thermal diffusion has the opposite effect. The outcome of the competition between the two opposing mechanisms depends on the nature of the material response.

Here we are interested in whether the dependence of viscosity upon temperature in viscous fluids or thermal softening in solids may destabilize the thermomechanical process. As a test problem we consider the adiabatic rectilinear shearing of an incompressible, non-Newtonian, liquid between two parallel plates. Identical equations govern the shearing in the plastic region of a solid exhibiting strain rate sensitivity and thermal softening but no strain hardening [4]. For this problem, the only dissipative mechanism present is viscosity which varies with temperature.

We assume the two parallel plates occupy the planes $x = 0$ and $x = 1$ in a Cartesian coordinate system. The thermomechanical process is described by the velocity field $v(x, t)$ in the direction of the flow, perpendicular to the x-axis, and the temperature field $\theta(x, t)$.

* Received September 13, 1984.

We assume that the referential density is $\rho_0 = 1$ and that the specific heat is $c = 1$, so that the internal energy e and the temperature coincide, $e = \theta$. We also assume that the flow is adiabatic. Then the equations of balance of momentum and energy read

$$v_t = \sigma_x, \tag{1.1}$$

$$\theta_t = \sigma v_x, \tag{1.2}$$

where σ is the shear stress.

We assume the constitutive law

$$\sigma = \mu(\theta)|v_x|^{n-1} v_x, \qquad n > 0, \tag{1.3}$$

with $\mu(\theta) > 0$, $\mu'(\theta) < 0$, $0 < \theta < \infty$ and $\mu(\theta) \to 0$ as $\theta \to +\infty$. This constitutive relation is appropriate for a solid in the plastic region exhibiting thermal softening and strain rate sensitivity but no strain hardening, as well as for an incompressible non-Newtonian liquid.

For simplicity, we shall consider here

$$\mu(\theta) = \theta^\nu, \qquad \nu < 0, \tag{1.4}$$

in which case the constitutive law becomes

$$\sigma = \theta^\nu |v_x|^{n-1} v_x, \qquad \nu < 0, n > 0. \tag{1.5}$$

Under the above constitutive assumption the field equations take the form:

$$v_t - \left(\theta^\nu |v_x|^{n-1} v_x\right)_x = 0, \tag{1.6}$$

$$\theta_t - \theta^\nu |v_x|^{n+1} = 0. \tag{1.7}$$

We assume that the material is subjected to steady shearing. Then the boundary conditions are

$$v(0,t) = 0, \quad v(1,t) = 1, \qquad 0 \leqslant t < +\infty. \tag{1.8}$$

We also impose initial conditions

$$v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \qquad 0 \leqslant x \leqslant 1, \tag{1.9}$$

which are to be compatible with (1.8) in that

$$v_0(0) = 0, \qquad v_0(1) = 1. \tag{1.10}$$

When $v_0(x) = x$, $\theta_0(x) = \Theta_0 =$ constant, we can write down explicitly the solution of (1.6), (1.7), (1.8), and (1.9) as

$$v(x,t) = x, \tag{1.11}$$

$$\theta(x,t) = \Theta(t), \tag{1.12}$$

where $\Theta(t)$ is the solution of the initial value problem

$$\frac{d\Theta}{dt}(t) = \Theta^\nu(t) \tag{1.13}$$

$$\Theta(0) = \Theta_0,$$

that is,

$$\Theta(t) = \left[(1-\nu)t + \Theta_0^{(1-\nu)}\right]^{1/(1-\nu)}. \tag{1.14}$$

The solution (1.11), (1.12) describes uniform shearing. It is the goal of this paper to examine the asymptotic stability of the uniform shearing solution.

As the material is being sheared, energy is pumped into the system. Since the flow is adiabatic, temperature will keep rising and will tend to infinity with time. The distribution of temperature could either go to infinity uniformly in x, or it could localize, thus destabilizing the system. For a solid, destabilization would possibly be manifested by shear band formation. From the point of view of analysis the asymptotic behavior of the solution will be determined by the competition between the degenerate parabolic equation (1.6) and the hyperbolic equation (1.7). It is conceivable that the asymptotic distribution of v_x, which is governed by (1.6), is not uniform. However, it turns out that if $-n < \nu < 0$, then the temperature of the material increases in an "orderly" fashion and v_x becomes asymptotically constant, so that the uniform shearing solution (1.11), (1.12) is asymptotically stable. More precisely, we shall prove the following theorem.

THEOREM 1. Assume that $v_0(x) \in W^{2,2}(0,1)$, $|v_{0_x}(x)|^{n-1}v_{0_x}(x) \in W^{1,2}(0,1)$, $\theta_0(x) \in W^{1,2}(0,1)$ and $\theta_0(x) > 0$, $0 \leqslant x \leqslant 1$. Then, if $-n < \nu < 0$, every classical solution of (1.6), (1.7), (1.8) and (1.9) on $[0,1] \times [0,\infty)$ has the following properties. As $t \to +\infty$,

$$|v_x(x,t)|^{n-1} v_x(x,t) = 1 + O(t^{-(\nu+n)/(n(1-\nu))}) \tag{1.15}$$

$$\int_{\theta_0(x)}^{\theta(x,t)} \frac{d\xi}{\xi^\nu} = t + O(t^{1-(\nu+n)/(n(1-\nu))}), \tag{1.16}$$

uniformly in x on $[0,1]$.

A similar result holds in the more general case of the constitutive law (1.3) provided that $\mu(\theta)$ goes to zero in an "orderly" fashion. Namely, under the same smoothness hypotheses on the initial data as in the above theorem and assuming that $\mu(\theta)$ is any twice continuously differentiable function in $(0, +\infty)$ which satisfies

$$\mu(\theta) > 0, \quad \mu'(\theta) < 0, \qquad 0 < \theta < +\infty,$$

and for some $\nu < 0$ and $N > 0$

$$1 + \frac{1}{n} < 1 - \frac{1}{\nu} \leqslant \frac{\mu(\theta)\mu''(\theta)}{[\mu'(\theta)]^2} \leqslant N, \qquad 0 < \theta < +\infty, \tag{1.17}$$

it can be shown that every classical solution of (1.1), (1.2), (1.3), (1.8), and (1.9) on $[0,1] \times [0,\infty)$ satisfies (1.15), while (1.16) is replaced by the obvious generalization

$$\int_{\theta_0(x)}^{\theta(x,t)} \frac{d\xi}{\mu(\xi)} = t + O(t^{1-(\nu+n)/(n(1-\nu))}). \tag{1.18}$$

Several studies of the effect of the dependence of viscosity upon temperature on the stabilization of thermomechanical processes have appeared recently in literature. All of them deal with shearing of Newtonian fluids, i.e., $n = 1$, caused by various external agencies. In Dafermos and Hsiao [6] the flow is caused by steady shearing of the boundary, while in Charalambakis [1, 2, 3] the flow is caused by a time-dependent inertial force [1, 2], or by periodic boundary shearing, or by a periodic boundary force [3].

The proof of Theorem 1 is presented in the following section and based on a priori estimates. The estimates are obtained with the help of identities for solutions of (1.6), (1.7), (1.8), and (1.9). The proof of the announced result for the more general constitutive relation (1.3) follows exactly the same pattern as the proof of Theorem 1. However, the analysis is more complicated and will not be presented here.

2. Proof of Theorem. Throughout this section we assume that $\{v(x,t), \theta(x,t)\}$ is a fixed classical solution of (1.6), (1.7), (1.8) and (1.9) on $[0,1]\times[0,+\infty)$ such that $v(\cdot,t)$, $v_x(\cdot,t)$, $v_{xx}(\cdot,t)$, $v_t(\cdot,t)$, $\theta(\cdot,t)$, $\theta_x(\cdot,t)$ are all in $C^0([0,\infty);\ L^2(0,1))$ while $v_{xt}(\cdot,t)$ is in $C^0((0,\infty);\ L^2(0,1))$ and $v_{tt}(\cdot,t)$ is in $L^2_{\mathrm{loc}}((0,\infty);\ L^2(0,1))$. We will establish a priori estimates that will yield the proof of the theorem stated in the introduction. In the sequel, K will stand for a generic constant that depends on ν and n in (1.5) and can be estimated from above in terms of lower bounds for $\min_{0\leqslant x\leqslant 1}\theta_0(x)$ and upper bounds for the $W^{2,2}(0,1)$ norm of $v_0(x)$ and the $W^{1,2}(0,1)$ norms of $|v_{0_x}(x)|^{n-1}v_{0_x}(x)$ and $\theta_0(x)$.

The first lemma already gives an indication that the L^2-norm of v_t decays with time.

LEMMA 2.1.

$$\int_0^t\int_0^1 v_t^2\,dx\,d\tau \leqslant K, \qquad 0\leqslant t<+\infty, \tag{2.1}$$

$$\int_0^t\int_0^1 |\sigma|^{1-n/n}v_t^2\,dx\,d\tau \leqslant K, \qquad 0\leqslant t<+\infty. \tag{2.2}$$

Proof. We multiply (1.1) by v_t, we integrate over $(0,1)\times(0,t)$, we integrate by parts with respect to x, we use (1.5) and (1.8), and then we integrate by parts with respect to t and use (1.7) to obtain

$$\int_0^t\int_0^1 v_t^2\,dx\,d\tau + \frac{1}{n+1}\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n+1}\,dx$$
$$-\frac{\nu}{n+1}\int_0^t\int_0^1 \theta^{2\nu-1}|v_x|^{2(n+1)}\,dx\,d\tau = \frac{1}{n+1}\int_0^1 \theta_0^\nu(x)|v_{0_x}(x)|^{n+1}\,dx. \tag{2.3}$$

Next, we multiply (1.1) by $|\sigma|^{(1-n)/n}v_t$ and perform the same steps, thus arriving at

$$\int_0^t\int_0^1 |\sigma|^{(1-n)/n}v_t^2\,dx\,d\tau + \frac{n}{2}\int_0^1 \theta^{\nu/n}(x,t)|v_x(x,t)|^2\,dx$$
$$-\frac{\nu}{2}\int_0^t\int_0^1 \theta^{\nu/n+\nu-1}|v_x|^{n+3}\,dx\,d\tau = \frac{n}{2}\int_0^1 \theta_0^{\nu/n}(x)|v_{0_x}(x)|^2\,dx. \tag{2.4}$$

(2.1), (2.2) follow from (2.3), (2.4), respectively. □

We set

$$E(t) = \left[1+\int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau\right]^{n/(\nu+n)}, \qquad 0\leqslant t<+\infty. \tag{2.5}$$

Note that $E(t)\geqslant E(0)=1$. Throughout the following we will express the time growth or decay of the various estimated quantities through the function $E(t)$.

LEMMA 2.2. *If* $-n<\nu<0$,

$$\frac{1}{K}E(t)\leqslant \theta(x,t)\leqslant KE(t), \qquad 0\leqslant x\leqslant 1, 0\leqslant t<+\infty. \tag{2.6}$$

Proof. On account of (1.5), (1.2) yields

$$\frac{n}{\nu+n}\frac{\partial}{\partial t}\theta^{(\nu+n)/n} = |\sigma|^{(n+1)/n}. \tag{2.7}$$

Using (1.1),

$$|\sigma(x,t)|^{(n+1)/n} = |\sigma(y,t)|^{(n+1)/n} + \frac{n+1}{n}\int_y^x |\sigma(\xi,t)|^{1/n}(\operatorname{sgn}\sigma(\xi,t))\sigma_x(\xi,t)\,d\xi$$

$$= \int_0^1 |\sigma(y,t)|^{(n+1)/n}\,dy + \frac{n+1}{n}\int_0^1\int_y^x |\sigma(\xi,t)|^{1/n}(\operatorname{sgn}\sigma(\xi,t))v_t(\xi,t)\,d\xi\,dy, \tag{2.8}$$

where, as usual, $\operatorname{sgn}\sigma = 1$ if $\sigma > 0$, 0 if $\sigma = 0$, -1 if $\sigma < 0$. By virtue of (2.8), (2.7) gives

$$\theta^{(\nu+n)/n}(x,t) = \theta_0^{(\nu+n)/n}(x) + \frac{\nu+n}{n}\int_0^t\int_0^1 |\sigma(y,\tau)|^{(n+1)/n}\,dy\,d\tau$$

$$+ \frac{(\nu+n)(n+1)}{n^2}\int_0^t\int_0^1\int_y^x |\sigma(\xi,\tau)|^{1/n}(\operatorname{sgn}\sigma(\xi,\tau))v_t(\xi,\tau)\,d\xi\,dy\,d\tau, \tag{2.9}$$

whence, upon using Schwarz's inequality and (2.2), we obtain

$$\theta^{(\nu+n)/n}(x,t) \leqslant K_1 + K_2\int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau$$

$$\leqslant (K_1+K_2)\left[1 + \int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau\right], \tag{2.10}$$

$$\theta^{(\nu+n)/n}(x,t) \geqslant \frac{1}{2}\frac{\nu+n}{n}\int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau - K_3$$

$$= \frac{1}{K_4}\left[1 + \int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau\right] - K_5. \tag{2.15}$$

At the same time, on account of (2.7),

$$\theta^{(\nu+n)/n}(x,t) \geqslant \min_{0\leqslant x\leqslant 1}\theta_0^{(\nu+n)/n}(x) = K_6. \tag{2.12}$$

Combining (2.11) and (2.12), we deduce

$$\theta^{(\nu+n)/n}(x,t) \geqslant \frac{K_6}{K_4(K_5+K_6)}\left(1 + \int_0^t\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau\right). \tag{2.13}$$

Combining (2.5) with (2.10) and (2.13) we arrive at (2.6). □

We want to estimate the growth of $E(t)$ with time. To this end, we differentiate (2.5) to obtain

$$\frac{dE}{dt}(t) = \frac{n}{\nu+n}E(t)^{-\nu/n}\int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx. \tag{2.14}$$

We thus need an estimate of $\int_0^1|\sigma(x,t)|^{(n+1)/n}\,dx$, and this is provided by the following lemma.

LEMMA 2.3. If $-n < \nu < 0$,

$$\frac{1}{K}E(t)^{\nu(n+1)/n} \leqslant \int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx \leqslant KE(t)^{\nu(n+1)/n}, 0 \leqslant t < +\infty. \tag{2.15}$$

Proof. Using (1.5), (2.6), and (1.8),

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx \geqslant \frac{1}{K_1}E(t)^{\nu(n+1)/n}\int_0^1 |v_x(x,t)|^{n+1}\,dx$$

$$\geqslant \frac{1}{K_1}E(t)^{\nu(n+1)/n}\left|\int_0^1 v_x(x,t)\,dx\right|^{n+1} = \frac{1}{K_1}E(t)^{\nu(n+1)/n}, \quad (2.16)$$

we arrive at the left-hand inequality in (2.15).

To establish the right-hand inequality, we distinguish two cases, namely $t \geqslant 1$ and $0 \leqslant t \leqslant 1$. Using (1.5) and (2.6) we deduce

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx \leqslant K_2 E(t)^{\nu/n}\int_0^1 \theta^{\nu}(x,t)|v_x(x,t)|^{n+1}\,dx. \quad (2.17)$$

First we consider the case $t \geqslant 1$. In order to estimate the right-hand side of (2.17), we multiply (1.6) by tv_t, integrate over $(0,t)\times(0,1)$, integrate twice by parts and use (1.8) and (1.7), thus arriving at

$$\int_0^t\int_0^1 \tau v_t^2\,d\tau + \frac{1}{n+1}t\int_0^1 \theta^{\nu}(x,t)|v_x(x,t)|^{n+1}\,dx + \frac{1}{n+1}\int_0^1 \theta_0(x)\,dx$$

$$-\frac{\nu}{n+1}\int_0^t\int_0^1 \tau\theta^{2\nu-1}|v_x|^{2(n+1)}\,dx\,d\tau = \frac{1}{n+1}\int_0^1 \theta(x,t)\,dx. \quad (2.18)$$

By virtue of (2.18) and (2.6), (2.17) yields

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx \leqslant K_3\frac{E(t)^{(\nu/n)-n}}{t}\left[\int_0^1 \theta(x,t)\,dx\right]^{n+1}. \quad (2.19)$$

Next, we multiply (1.6) by v, integrate over $(0,1)\times(0,t)$, integrate by parts and use (1.7), (1.8), and (1.5) to obtain the integral form of balance of energy:

$$\frac{1}{2}\int_0^1 v^2(x,t)\,dx + \int_0^1 \theta(x,t)\,dx$$

$$= \int_0^t \sigma(1,\tau)\,d\tau + \frac{1}{2}\int_0^1 v_0^2(x)\,dx + \int_0^1 \theta_0(x)\,dx. \quad (2.20)$$

We employ (2.20) and use Hölder's inequality together with (2.8), (2.2), and (2.5) to deduce

$$\left[\int_0^1 \theta(x,t)\,dx\right]^{n+1} \leqslant K_4 + K_5\left(\int_0^t \sigma(1,\tau)\,d\tau\right)^{n+1}$$

$$\leqslant K_4 + K_5 t\left(\int_0^t |\sigma(1,\tau)|^{(n+1)/n}\,d\tau\right)^n$$

$$\leqslant K_4 + K_5 t\left(K_6 + K_7\int_0^1\int_0^1 |\sigma|^{(n+1)/n}\,dx\,d\tau\right)^n$$

$$\leqslant K_4 + K_8 t E(t)^{\nu+n}. \quad (2.21)$$

Since $t \geqslant 1$ and $E(t) \geqslant 1$, combining (2.19) with (2.21) we get

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}\,dx \leqslant K_9 E(t)^{\nu(n+1)/n}, \qquad t \geqslant 1. \tag{2.22}$$

It remains to establish the right half of (2.15) for the case $0 \leqslant t \leqslant 1$. To this end observe that, by account of (2.3) and the fact that $E(t) \geqslant 1$, (2.17) yields

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}dx \leqslant K_{10}E(t)^{\nu/n} \leqslant K_{10}, \qquad 0 \leqslant t \leqslant 1. \tag{2.23}$$

By virtue of (2.5), $E(t) \leqslant K_{11}$, $0 \leqslant t \leqslant 1$. Hence,

$$\int_0^1 |\sigma(x,t)|^{(n+1)/n}dx \leqslant K_{10} \leqslant K_{12}E(t)^{\nu(n+1)/n}, \qquad 0 \leqslant t \leqslant 1, \tag{2.24}$$

and this completes the proof of the lemma. □

Combining (2.14) with (2.15) we arrive at the following relations, which identify the time evolution of $E(t)$ and $dE(t)/dt$.

COROLLARY 2.1. *If* $-n < \nu < 0$,

$$\frac{1}{K}E(t)^{\nu} \leqslant \frac{dE(t)}{dt} \leqslant KE(t)^{\nu}, \qquad 0 \leqslant t < \infty, \tag{2.25}$$

$$\frac{1}{K}(t+1)^{1/(1-\nu)} \leqslant E(t) \leqslant K(t+1)^{1/(1-\nu)}, \qquad 0 \leqslant t < +\infty. \tag{2.26}$$

Our next objective is to establish (2.46). For that we need to estimate the L^2-norm of v_t and the L^1-norm of θ_x. These estimates are provided by Lemmas 2.4 and 2.5 respectively.

LEMMA 2.4. *If* $-n < \nu < 0$,

$$\int_0^1 v_t^2(x,t)\,dx \leqslant KE(t)^{2\nu-1}, \qquad 0 \leqslant t < \infty, \tag{2.27}$$

$$|v_x(x,t)| \leqslant K, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t < +\infty. \tag{2.28}$$

Proof. We differentiate (1.6) with respect to t and use (1.7) to get

$$v_{tt} - \left(n\theta^{\nu}|v_x|^{n-1}v_{xt} + \nu\theta^{2\nu-1}|v_x|^{2n}v_x\right)_x = 0 \tag{2.29}$$

and then we multiply by v_t, integrate over $(0,1)$ and integrate by parts, thus arriving at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2(x,t)\,dx + n\int_0^1 \theta^{\nu}(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx \\ = -\nu\int_0^1 \theta^{2\nu-1}(x,t)|v_x(x,t)|^{2n}v_{xt}(x,t)\,dx,\end{aligned} \tag{2.30}$$

which, upon using Schwarz's inequality, yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v_t^2(x,t)\,dx + n\int_0^1 \theta^{\nu}(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx \\ \leqslant \frac{n}{2}\int_0^1 \theta^{\nu}(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx \\ + K_1\int_0^1 \theta^{3\nu-2}(x,t)|v_x(x,t)|^{3(n+1)}\,dx,\end{aligned} \tag{2.31}$$

combining (2.31) with (1.5), (2.6), and (2.15).

$$\frac{d}{dt}\int_0^1 v_t^2(x,t)\,dx + n\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx$$
$$\leqslant K_2\int_0^1 \theta^{-2-3\nu/n}(x,t)|\sigma(x,t)|^{3(n+1)/n}\,dx$$
$$\leqslant E(t)^{\nu-2-2\nu/n}\max_{0\leqslant x\leqslant 1}|\sigma(x,t)|^{2(n+1)/n}. \tag{2.32}$$

From an integration by parts there follows the identity:

$$\int_0^1 v_t^2(x,t)\,dx = -\int_0^1 |\sigma(x,t)|^{(n+1)/2n}\left\{|\sigma(x,t)|^{(n-1)/2n}(\operatorname{sgn}\sigma(x,t))v_{xt}(x,t)\right\}dx, \tag{2.33}$$

whence, by virtue of Schwarz's inequality together with (2.6) and (2.15), we obtain

$$\left(\int_0^1 v_t^2(x,t)\,dx\right)^2 \leqslant K_4E(t)^\nu\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx. \tag{2.34}$$

Next, we use (1.1), Schwarz's inequality, Hölder's inequality, (2.34) and (2.15) to deduce

$$|\sigma(x,t)|^{(2n+1)/2n} = \int_0^1 |\sigma(y,t)|^{(2n+1)/2n}\,dy$$
$$+\frac{2n+1}{2n}\int_0^1\int_y^x |\sigma(\xi,t)|^{1/2n}(\operatorname{sgn}\sigma(\xi,t))v_t(\xi,t)\,d\xi\,dy$$
$$\leqslant \int_0^1|\sigma(y,t)|^{(2n+1)/2n}\,dy + \frac{2n+1}{2n}\left(\int_0^1 |\sigma(\xi,t)|^{1/n}\,d\xi\right)^{1/2}\left(\int_0^1 v_t^2(\xi,t)\,d\xi\right)^{1/2}$$
$$\leqslant \left(\int_0^1 |\sigma(y,t)|^{(n+1)/n}\,dy\right)^{(2n+1)/2(n+1)}$$
$$+K_5\left(\int_0^1 |\sigma(\xi,t)|^{(n+1)/n}\,d\xi\right)^{1/2(n+1)}E(t)^{\nu/4}\left(\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx\right)^{1/4}$$
$$\leqslant K_6E(t)^{\nu(2n+1)/2n} + K_7E(t)^{\nu(n+2)/4n}\left(\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx\right)^{1/4}. \tag{2.35}$$

Using the inequality $a^rb^{1-r} \leqslant r\varepsilon a + (1-r)\varepsilon^{-r/(1-r)}b$, $0 \leqslant r \leqslant 1$, $\varepsilon > 0$ and (2.35) we get

$$K_3E(t)^{\nu-2-2\nu/n}\max_{0\leqslant x\leqslant 1}|\sigma(x,t)|^{2(n+1)/n}$$
$$\leqslant K_3E(t)^{\nu-2-2\nu/n}\Big\{K_8E(t)^{2\nu(n+1)/n}$$
$$+K_9E(t)^{\nu(n+1)(n+2)/n(2n+1)}\left(\int_0^1 \theta^\nu|v_x|^{n-1}v_{xt}^2\,dx\right)^{(n+1)/(2n+1)}\Big\}$$
$$\leqslant K_{10}E(t)^{3\nu-2} + K_{11}E(t)^{3\nu-2-2(n+1)/n}$$
$$+n\int_0^1 \theta^\nu(x,t)|v_x(x,t)|^{n-1}v_{xt}^2(x,t)\,dx. \tag{2.36}$$

Combining (2.32) with (2.36),

$$\frac{d}{dt}\int_0^1 v_t^2(x,t)\,dx \leqslant K_{12}E(t)^{3\nu-2}. \tag{2.37}$$

On account of (2.6), (2.18) yields

$$\int_0^t\int_0^1 \tau v_t^2\,dx\,d\tau \leqslant \frac{1}{n+1}\int_0^1 \theta(x,t)\,dx \leqslant K_{13}E(t). \tag{2.38}$$

By virtue of the identity

$$\begin{aligned}(t+1)^2\int_0^1 v_t^2(x,t)\,dx - \int_0^1 v_t^2(x,0)\,dx \\ = 2\int_0^t\int_0^1 (\tau+1)v_t^2(x,\tau)\,dxd\tau + \int_0^t (\tau+1)^2\left\{\frac{d}{d\tau}\int_0^1 v_t^2(x,\tau)\,dx\right\}d\tau\end{aligned} \tag{2.39}$$

together with (2.37), (2.38), (2.1) and (2.26),

$$\begin{aligned}\int_0^1 v_t^2(x,t)\,dx &\leqslant \frac{1}{(t+1)^2}\left\{K_{14} + K_{15}(t+1)^{1/(1-\nu)} + K_{16}\int_0^t (\tau+1)^{2+(3\nu-2)/(1-\nu)}\,d\tau\right\} \\ &= K_{17}(t+1)^{-2} + K_{18}(t+1)^{(2\nu-1)/(1-\nu)} \leqslant K_{19}E(t)^{2\nu-1}.\end{aligned} \tag{2.40}$$

To establish (2.28) we employ the relation:

$$\sigma(x,t) = \int_0^1 \sigma(y,t)\,dy + \int_0^1\int_y^x \sigma_x(\xi,t)\,d\xi\,dy \tag{2.41}$$

and use Hölder's inequality together with (1.1), (2.15), and (2.40) to obtain

$$\begin{aligned}|\sigma(x,t)| &\leqslant \left(\int_0^1 |\sigma(y,t)|^{(n+1)/n}\,dy\right)^{n/(n+1)} + \left(\int_0^1 v_t^2(\xi,t)\,d\xi\right)^{1/2} \\ &\leqslant K_{20}E(t)^{\nu}.\end{aligned} \tag{2.42}$$

Combining (2.42) with (1.5) and (2.6) we arrive at (2.28). □

LEMMA 2.5. If $-n < \nu < 0$,

$$\int_0^1 |\theta_x(x,t)|dx \leqslant KE(t)^{\max\{-\nu/n,1/2\}}, \quad 0 \leqslant t < +\infty. \tag{2.43}$$

Proof. Differentiating (2.9) with respect to x,

$$\begin{aligned}\theta^{\nu/n}(x,t)\theta_x(x,t) = \theta_0^{\nu/n}(x)\theta_{0_x}(x) \\ + \frac{n+1}{n}\int_0^t |\sigma(x,\tau)|^{1/n}(\operatorname{sgn}\sigma(x,\tau))v_t(x,\tau)\,d\tau.\end{aligned} \tag{2.44}$$

On account of (2.42), Schwarz's inequality, (2.27) and (2.25) we obtain

$$\begin{aligned}&\int_0^1 \left|\int_0^t |\sigma(x,\tau)|^{1/n}\operatorname{sgn}\sigma(x,\tau)v_t(x,\tau)\,d\tau\right|dx \\ &\leqslant \int_0^t\int_0^1 |\sigma(x,\tau)|^{1/n}|v_t(x,\tau)|dx\,d\tau \\ &\leqslant K_1\int_0^t E(\tau)^{\nu/n}\left(\int_0^1 v_t^2(x,\tau)\,dx\right)^{1/2}d\tau \leqslant K_2\int_0^t E(\tau)^{\nu/n-1/2}\frac{dE}{d\tau}(\tau)\,d\tau \\ &= K_2\int_{E(0)}^{E(t)} E(\tau)^{\nu/n-1/2}\,dE(\tau) = K_2\frac{2n}{2\nu+n}\left(E(t)^{\nu/n+1/2} - 1\right).\end{aligned} \tag{2.45}$$

Upon using (2.6) and (2.45), (2.4) yields (2.43). □

LEMMA 2.6. If $-n < \nu < 0$,

$$\left| |v_x(x,t)|^n \operatorname{sgn} v_x(x,t) - 1 \right| \leqslant KE(t) \max\{-1/2, -(\nu+n)/n\},$$
$$0 \leqslant x \leqslant 1, 0 \leqslant t < +\infty.$$

Proof. Since $\int_0^1 v_x(x,t) = 1$, for each fixed t there exists $y(t)$ such that $v_x(y(t),t) = 1$. Thus,

$$\left| |v_x(x,t)|^n \operatorname{sgn} v_x(x,t) - 1 \right| = \left| \int_{y(t)}^x \frac{\partial}{\partial \xi}\left(|v_x(\xi,t)|^n \operatorname{sgn} v_x(\xi,t)\right) d\xi \right|$$
$$\leqslant \int_0^1 \left| \frac{\partial}{\partial \xi}\left(|v_x(\xi,t)|^n \operatorname{sgn} v_x(\xi,t)\right) \right| d\xi. \tag{2.47}$$

From (1.6),

$$\left(|v_x|^n \operatorname{sgn} v_x\right)_x = \theta^{-\nu} v_t - \nu\theta^{-1}\theta_x |v_x|^n \operatorname{sgn} v_x. \tag{2.48}$$

Substituting (2.48) in (2.47) and using (2.6), (2.27), (2.28), and (2.43) we arrive at (2.46). □

Our next project is to establish sharper estimates in Lemmas 2.4, 2.5, and 2.6.

LEMMA 2.7. If $-n < \nu < 0$,

$$\int_0^1 v_t^2(x,t)\, dx \leqslant KE(t)^{2\nu-2}, \qquad 0 \leqslant t < +\infty, \tag{2.49}$$

$$\int_0^1 |\theta_x(x,t)| dx \leqslant KE(t)^{-\nu/n}, \qquad 0 \leqslant t < +\infty, \tag{2.50}$$

$$\left| |v_x(x,t)|^n \operatorname{sgn} v_x(x,t) - 1 \right| \leqslant KE(t)^{-(\nu+n)/n}, \qquad 0 \leqslant x \leqslant 1, 0 \leqslant t < +\infty. \tag{2.51}$$

Proof. On account of (2.46) there exists T such that

$$\tfrac{1}{2} \leqslant v_x(x,t) \leqslant 2, \qquad 0 \leqslant x \leqslant 1, t \geqslant T. \tag{2.52}$$

Upon using (2.6), (2.42), and Poincaré's inequality,

$$\int_0^1 v_t^2(x,t)\, dx \leqslant \int_0^1 v_{xt}^2(x,t)\, dx, \tag{2.53}$$

(2.32) yields

$$\frac{d}{dt}\int_0^1 v_t^2(x,t)\, dx + \frac{1}{K_1} E(t)^\nu \int_0^1 v_t^2(x,t)\, dx \leqslant K_2 E(t)^{3\nu-2}, \qquad t \geqslant T. \tag{2.54}$$

Integrating the above differential inequality,

$$\int_0^1 v_t^2(x,t)\, dx \leqslant \left(\int_0^1 v_t^2(x,T)\, dx\right) e^{-(1/K_1)\int_T^t E(\tau)^\nu d\tau}$$
$$+ K_2 \int_T^t E(s)^{3\nu-2} e^{-(1/K_1)\int_s^t E(\tau)^\nu d\tau} ds, \qquad t \geqslant T. \tag{2.55}$$

Recalling (2.25), (2.55) yields

$$\int_0^1 v_t^2(x,t)\,dx \leqslant \left(\int_0^1 v_t^2(x,T)\,dx\right) e^{-(1/K_3)(E(t)-E(T))} + K_2 e^{-(1/K_3)E(t)} \int_{E(T)}^{E(t)} \xi^{2\nu-2} e^{(1/K_3)\xi}\,d\xi, \qquad t \geqslant T. \tag{2.56}$$

Applying L'Hopital's rule we get

$$\limsup_{\eta\to\infty} \frac{\int_{E(T)}^{\eta} \xi^{2\nu-2} e^{(1/K_3)\xi}\,d\xi}{\eta^{2\nu-2} e^{(1/K_3)\eta}} = \limsup_{\eta\to\infty} \frac{\eta^{2\nu-2} e^{(1/K_3)\eta}}{\left[\frac{1}{K_3}\eta^{2\nu-2} + (2\nu-2)\eta^{2\nu-3}\right] e^{(1/K_3)\eta}} = K_3. \tag{2.57}$$

Combining (2.56), (2.57) and taking into acount (2.26) and (2.27) we arrive at (2.49). To prove (2.50) and (2.51) we follow the same steps as in Lemmas 2.5, 2.6, using the estimate (2.43) in place of (2.27). □

LEMMA 2.8. If $-n < \nu < 0$,

$$\left|\int_{\theta_0(x)}^{\theta(x,t)} \frac{d\xi}{\xi^\nu} - t\right| \leqslant KE(t)^{-\nu(n+1)/n}, \qquad 0 \leqslant x \leqslant 1, 0 \leqslant t < +\infty. \tag{2.58}$$

Proof. Using (1.7), (2.51), (2.25), and the inequality

$$\left| |\eta|^{n+1} - 1\right| \leqslant \frac{n+1}{n}(1+|\eta|)\left| |\eta|^n \operatorname{sgn}\eta - 1\right|, \tag{2.59}$$

we obtain

$$\begin{aligned}\left|\int_{\theta_0(x)}^{\theta(x,t)} \frac{d\xi}{\xi^\nu} - t\right| &= \left|\int_0^t \left(|v_x(x,\tau)|^{n+1} - 1\right) d\tau\right| \\ &\leqslant K_1 \int_0^t \left| |v_x(x,\tau)|^n \operatorname{sgn} v_x(x,\tau) - 1\right| d\tau \\ &\leqslant K_2 \int_{E(0)}^{E(t)} E(\tau)^{-(\nu+n)/n-\nu}\,dE(\tau) \\ &\leqslant K_3 E(t)^{-\nu(n+1)/n}. \quad \square\end{aligned} \tag{2.60}$$

Next, by virtue of (2.26), (2.51) and (2.58) yield (1.15) and (1.16), respectively. This concludes the proof of the theorem.

Acknowledgement. I would like to thank Professor Constantine M. Dafermos who suggested this problem and provided many illuminating discussions during the preparation of this work.

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