QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLIV JULY · 1986 NUMBER 2

JULY 1986, PAGES 201-212

A THIRD MIXED BOUNDARY VALUE PROBLEM ON A SPHERE*

By

HARRY L. JOHNSON

Virginia Polytechnic Institute and State University

Abstract. The paper is concerned with the existence of a classical solution of a mixed third boundary value problem on a sphere. The existence is proved by reducing the problem to a Fredholm integral equation that has a unique solution. Various consequences of the existence theorem are mentioned and some numerical results are given.

1. Introduction. Let D denote a solid, open sphere with center at 0 of radius R, and let the positive z-axis denote a specific direction through 0. Let ρ denote the distance from 0 along a ray that makes an angle ϕ with the z-axis. Let α be a specific angle from the z-axis, and set $I_1 = (0, \alpha]$, $J_1 = [0, \alpha]$, $I_2 = [\alpha, \pi)$, $J_2 = [\alpha, \pi]$,

$$H_p(I) = \{f : |f(x) - f(x_0)| \leqslant M(x_0)|x - x_0|^p,\ x \in I,\ x_0 \in I\},$$

$H(I) = \bigcup_{1/2 < p \leqslant 1} H_p(I)$, and $C(I) =$ the class of continuous functions on an interval I. Let

$$C^2H = \left\{w : w = w(\rho, \phi),\ w \in C^2(D),\ w(R, \cdot) \in H(I_1 \cup I_2) \cap C(I_1 \cup I_2),\ \frac{\partial w}{\partial \rho}(R, \cdot) \in H(I_1^0 \cup I_2^0)\right\}.$$

We seek a $w \in C^2H$ that satisfies the boundary value problem

$$\nabla^2 w = 0, \quad 0 \leqslant \rho < R,\ 0 \leqslant \phi \leqslant \pi, \tag{1}$$

$$\frac{\partial w}{\partial \rho}(R, \phi) + hw(R, \phi) = H(\phi), \quad \phi \in I_1^0 \cup I_2^0, \tag{2}$$

where

$$h = h(\phi) = \begin{cases} h_1, & \phi \in (0, \alpha) \\ h_2, & \phi \in (\alpha, \pi) \end{cases} \quad \text{and} \quad H(\phi) = \begin{cases} H_1(\phi), & \phi \in I_1^0 \\ H_2(\phi), & \phi \in I_2^0 \end{cases}$$

and h_1 and h_2 are nonnegative constants and $H_i \in H(I_i) \cap C(I_i)$, $i = 1, 2$.

* Received September 7, 1983.

Our interest in this problem is due to Robert B. Kelman [4]. Kelman points out that this boundary value problem is relevant to several physical problems: the packing of small spheres, the design of pebble-bed flow reactors, and the transportation of liquefied natural gas on oceangoing freighters in spherical tanks.

Existence theorems for third boundary value problems or oblique derivative problems and mixed third boundary value problems have a long history [6]. G. Giraud [2] proved an existence theorem for an oblique derivative problem with a boundary condition of the form $\alpha\, du/dl + \beta u = \phi$ with $\alpha > 0$ and β, ϕ continuous functions. Mixed third boundary value problems where β is discontinuous on a subset of the boundary have been studied by M. I. Visik and G. I. Eskin [7]. They have proved a normal solvability theorem in a certain function space for general linear elliptic equations subject to quite general discontinuous-type boundary conditions. The purpose of this paper is to provide a constructive existence theorem for Eqs. (1) and (2).

We prove the existence of a solution of the boundary value problem in C^2H by reducing the problem to a Fredholm integral equation of the second kind, Eq. (10), and showing that an operator generated by this integral equation is contracting in a maximum norm. This is shown for all values of $h_1 \geqslant 0$, $h_2 \geqslant 0$, $0 < \alpha < \pi$, and $R > 0$. Most of the analysis in the paper is concerned with transforming the kernel of the integral equation into various forms to bring out its properties. After the existence of a solution of the integral equation has been established, the existence of a solution $w \in C^2H$ is proved and some of its properties are listed as corollaries to the main theorem—Theorem 2.

Numerical calculations are carried out by forming an associated infinite system of linear algebraic equations from the integral equation of the form $X = F + AX$. The contraction property of the integral equation implies that the sequence $X_0 = F$, $X_{n+1} = F + AX_n$ has a limit X. The solution w is obtained by using the vector X. Numerical values of w are given for various settings of the parameters α, h_1, h_2 and variables ρ and ϕ.

2. The integral equation. Initially, we proceed in a formal manner and seek a solution $w \in C^2H$ in the form

$$w = w(\rho,\phi) = \sum_{n=0}^{\infty} A_n(\rho/R)^n P_n(\cos(\phi)), \quad 0 \leqslant \rho < R, 0 \leqslant \phi \leqslant \pi, \tag{3}$$

where $P_n(\cos(\phi))$ are Legendre polynomials.

Since

$$\nabla^2 w(\rho,\phi) = \frac{1}{\rho^2}\left(\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right) + \csc(\phi)\frac{\partial}{\partial\phi}\sin(\phi)\frac{\partial w}{\partial\phi}\right)$$

is invariant under the transformation $\phi \to \pi - \phi$, it is without loss of generality to assume that $h_1 > h_2 \geqslant 0$. From (3), one formally obtains

$$\frac{\partial w}{\partial\rho}(R,\phi) + h_1 w(R,\phi) = H_1(\phi) = \sum_{n=0}^{\infty} A_n(n/R + h_1)P_n(\cos(\phi)), \quad \phi \in I_1^0. \tag{4}$$

Let

$$\mu(\phi) = \frac{\partial w}{\partial\rho}(R,\phi) + h_1 w(R,\phi) = \sum_{n=0}^{\infty} A_n(n/R + h_1)P_n(\cos(\phi)), \quad \phi \in I_2^0, \tag{5}$$

and

$$u = u(\phi) = \begin{cases} H_1(\phi), & \phi \in I_1^0, \\ \mu(\phi), & \phi \in I_2^0, \\ (H_1(\alpha) + \mu(\alpha^+))/2, & \phi = \alpha. \end{cases}$$

It follows that

$$A_n = \frac{R(n + \frac{1}{2})}{(n + h_1 R)} \int_0^\pi u(\theta) P_n(\cos(\theta)) \sin(\theta)\, d\theta, \tag{6}$$

and that

$$H_2(\phi) = \sum_{n=0}^{\infty} \frac{(n + h_2 R)(n + \frac{1}{2})}{(n + h_1 R)} \int_0^\pi u(\theta) P_n(\cos(\theta)) P_n(\cos(\phi)) \sin(\theta)\, d\theta, \quad \phi \in I_2^0.$$

One should note that Eqs. (2) and (5) imply that

$$\mu(\phi) - H_2(\phi) = (h_1 - h_2) w(R, \phi), \quad \phi \in I_2^0. \tag{8}$$

Our working assumption now is that $\mu \in H(I_2^0) \cap C(J_2)$.

The classical representation theorem [1] of a function f with piecewise continuous derivatives in terms of Legendre polynomials states that

$$\frac{f(\phi + 0) + f(\phi - 0)}{2} = \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) \int_0^\pi f(\theta) P_n(\cos(\theta)) \sin(\theta)\, d\theta\, P_n(\cos(\phi)). \tag{9}$$

This theorem can be extended to functions $f \in H((0, \pi))$.

Writing

$$\frac{n + h_2 R}{n + h_1 R} = 1 + \frac{(h_2 - h_1) R}{n + h_1 R}$$

and applying (8) to u on the interval I_2^0 as expressed by (7) yields the integral equation

$$\mu(\phi) = F(\phi) + \lambda \int_\alpha^\pi \mu(\theta) K_a(\theta, \phi) \sin(\theta)\, d\theta, \quad \phi \in I_2^0 \tag{10}$$

where

$$F(\phi) = H_2(\phi) + \lambda \int_0^\alpha H_1(\theta) K_a(\theta, \phi) \sin(\theta)\, d\theta, \tag{11}$$

$$\lambda = (1 - h_2/h_1), \qquad a = h_1 R - \tfrac{1}{2},$$

and

$$K_a = K_a(\theta, \phi) = \sum_{n=0}^{\infty} \frac{(a + \frac{1}{2})(n + \frac{1}{2})}{n + a + \frac{1}{2}} P_n(\cos(\theta)) P_n(\cos(\phi)). \tag{12}$$

3. Properties of K_a. Let $I = (0, \pi)$, $J = [0, \pi]$.

THEOREM 1. *If $u = u(\theta)$ is piecewise continuous on J, then*

$$v = v(\phi) = \int_0^\pi u(\theta) K_a(\theta, \phi) \sin(\theta)\, d\theta \in H(I) \cap C(J).$$

We give an outline of a proof of this theorem. Let

$$K(j,k,\theta,\phi) = (a+\tfrac{1}{2})(-a)^{j+k}\sum_{n=0}^{\infty}\frac{P_n(\cos(\theta))P_n(\cos(\phi))}{(n+\frac{1}{2})^j(n+a+\frac{1}{2})^k}.$$

One can write

$$K_a(\theta,\phi) = K_0(\theta,\phi) + \sum_{j=1}^{3} K(j,0,\theta,\phi) + K(4,1,\theta,\phi) \tag{13}$$

and prove the theorem for each term on the right-hand side of (13). First, it is known [1] that $|P_n'(x)| \leqslant n^2$, $|x| \leqslant 1$. It follows that $\partial K/\partial\phi(4,1,\theta,\phi) \in C(J)$. Next, it can be shown [3] that

$$K_0(\theta,\phi) = \frac{1}{2\pi}\int_0^m \frac{ds}{K(s,\theta)K(s,\phi)}, \tag{14}$$

where $K(s,\theta) = (|\cos(s) - \cos(\theta)|)^{1/2}$ and $m = \min(\theta,\phi)$. Let $M = \max(\theta,\phi)$. A change in the integration variable and a trigonometric identity yields

$$K_0(\theta,\phi) = \frac{1}{\pi}\sec(m/2)\csc(M/2)\int_0^{\pi/2}\frac{dx}{(1 - z^2\sin^2(x))^{1/2}} \tag{15}$$

where $z = z(\theta,\phi) = \tan(m/2)\cot(M/2)$. Moreover,

$$\int_0^{\pi/2}\frac{dx}{(1-z^2\sin^2(x))^{1/2}} = \frac{\pi}{2} - 2 + \frac{1}{z}\ln\frac{(1+z)}{(1-z)} + Q(z), \quad |z| < 1, \tag{16}$$

where $Q(z) = \sum_{n=1}^{\infty} a_n z^{2n}$,

$$a_n = \frac{1}{2n+1}q\prod_{k=1}^{n}\left(1 - \left(\frac{1}{2k}\right)^2\right)\left(1 - \prod_{k=n+1}^{\infty}\left(1 - \left(\frac{1}{2k}\right)^2\right)\right)$$
$$= O(1/n^2).$$

Using basic properties of a logarithm, it can be shown that the integral

$$\frac{1}{\pi}\int_0^{\pi} u(\theta)\sin(\theta)\sec(m/2)\csc(M/2)\frac{1}{z}\ln\frac{1+z}{1-z}\,d\theta$$
$$= \frac{2}{\pi}\int_0^{\phi} u(\theta)\cos(\theta/2)\sec(\phi/2)\ln\frac{1+\tan(\theta/2)\cot(\phi/2)}{1-\tan(\theta/2)\cot(\phi/2)}\,d\theta$$
$$+ \frac{2}{\pi}\int_{\phi}^{\pi} u(\theta)\sin(\theta/2)\csc(\phi/2)\ln\frac{1+\tan(\phi/2)\cot(\theta/2)}{1-\tan(\phi/2)\cot(\theta/2)}\,d\theta$$

is Hölder continuous of any order less than one on $0 < \phi < \pi$, and that $\lim_{\phi\to 0^+} v(\phi)$ and $\lim_{\phi\to\pi^-} v(\phi)$ exist. Since $0 < a_n \leqslant 1/(2n+1)(C/n)$ for some constant C, it follows that $0 < dQ/dz \leqslant C\ln(1+z)/(1-z)$, $0 < z < 1$ and that

$$\int_0^{\pi} u(\theta)\sin(\theta)\sec(m/2)\csc(M/2)Q(z)\,d\theta \in C^1(I)\cap C(J).$$

Mehler's formulas for $P_n(x)$ yield

$$K(1,0,\theta,\phi) = \frac{2}{\pi}\int_0^{\pi/2}\frac{ds}{(1-\sin^2(m/2)\sin^2(s))^{1/2}}\int_0^{\pi/2}\frac{ds}{(1-\cos^2(M/2)\sin^2(s))^{1/2}}.$$

Using (16), one can show that $\int_0^\pi u(\theta)K(1,0,\theta,\phi)\sin(\theta)\,d\theta \in H(I)\cap C(J)$. Without going into details, we mention that

$$K(2,0,\theta,\phi) = \frac{1}{2\pi}\int_0^\pi \int_\phi^\pi \frac{(s+t-|s-t|)}{K(s,\theta)K(t,\phi)}\,dt\,ds$$

$$K(3,0,\theta,\phi) = \frac{\pi}{8}\int_M^\pi \int_0^m \frac{\left(2-(1-((s+t)/\pi))^2-(1-(s-t)/\pi)^2\right)}{K(s,M)K(t,m)}\,dt\,ds$$

and that these functions generate integrals with the stated property.

One consequence of Theorem 1 is:

COROLLARY 1. If $H_i \in H(I_i)\cap C(J_i)$, $i=1,2$, then

$$F = F(\phi) = H_2(\phi) + \lambda\int_0^\alpha H_1(\theta)K_a(\theta,\phi)\sin(\theta)\,d\theta \in H(I_2)\cap C(J_2).$$

It is next shown that K_a is bounded below by a positive constant which is dependent only on the parameter a. To show this, we write

$$K_a(\theta,\phi) = (2a+1)\Big(K_0(\theta,\phi) - \frac{a}{2}K(0,1,\theta,\phi)\Big) \tag{17}$$

where

$$K(0,1,\theta,\phi) = \sum_{n=0}^\infty \frac{P_n(\cos(\theta))P_n(\cos(\phi))}{n+a+\frac{1}{2}}.$$

Laplace's first integral representation of the Legendre polynomials can be written as

$$P_n(\cos(\theta)) = \frac{1}{\pi}\int_0^\pi \delta^n\,dt, \qquad P_n(\cos(\phi)) = \frac{1}{\pi}\int_0^\pi \zeta^n\,d\sigma,$$

where

$$\delta = \delta(\theta,t) = \cos(\theta) + \mathrm{i}\sin(\theta)\cos(t),$$
$$\zeta = \zeta(\phi,\sigma) = \cos(\phi) + \mathrm{i}\sin(\phi)\cos(\sigma).$$

Setting $r = \delta\zeta$ and using the identity

$$\sum_{n=0}^\infty \frac{r^n}{n+c} = \int_0^1 \frac{s^{c-1}\,ds}{1-rs}, \quad |r|<1,\ c>0,$$

one has

$$K(0,1,\theta,\phi) = \frac{1}{\pi^2}\int_0^1 s^{a-1/2}\,\mathrm{Re}\left(\int_0^\pi\int_0^\pi \frac{dt\,d\sigma}{(1-\delta\zeta s)}\right)ds. \tag{18}$$

It is without loss of generality to study (18) for $\phi<\theta$; moreover, it is easy to show that

$$\frac{1}{\pi}\int_0^\pi \frac{dt}{(1-\delta\zeta s)} = \frac{1}{\left((1-\zeta s e^{i\theta})(1-\zeta s e^{-i\theta})\right)^{1/2}}.$$

Further analysis on the remaining integrals in (18) yields

$$K(0,1,\theta,\phi) = \frac{1}{\pi}\int_0^1 s^{a-1/2}\int_0^\phi \frac{((|z|+x)/2)^{1/2}}{|z|K(\eta,\phi)}\,d\eta\,ds \tag{19}$$

where

$$z = x - iy,$$

$$x = x(\theta, \eta, s) = s(\cos(\eta) - \cos(\theta)) + \frac{(1-s)^2}{2}\cos(\eta), \tag{20}$$

$$y = y(\theta, \eta, s) = \frac{(1-s^2)}{2}\sin(\eta) \geqslant 0.$$

Equation (19) asserts that $K(0; 1, \theta, \phi) > 0$ for all $a > -\frac{1}{2}$ and, therefore, from (17) that

$$K_a(\theta, \phi) \geqslant (2a+1)K_0(\theta, \phi), \quad -\tfrac{1}{2} < a \leqslant 0.$$

Using (15), one has $K_0(\theta, \phi) \geqslant \sec(m/2)\csc(M/2)(1/2) \geqslant \frac{1}{2}$, and hence that

$$K_a(\theta, \phi) \geqslant (a + \tfrac{1}{2}), \quad -\tfrac{1}{2} < a \leqslant 0. \tag{21}$$

Next, consider the case of $a > 0$. If $\eta \leqslant \phi \leqslant \pi/2$ and $\theta > \phi$, then $x = x(\theta, \eta, s)$ as given by (20) is $\geqslant 0$. If $\pi/2 < \phi \leqslant \pi$, then one may use the fact that

$$K_a(\theta, \phi) = K_a(\pi - \theta, \pi - \phi)$$

and have $0 < \theta = \pi - \theta < \Phi = \pi - \phi < \pi/2$. Thus, replacing ϕ by θ and θ by Φ in (10) gives $x = x(\phi, \eta, s) \geqslant 0$ for $\eta < \theta < \pi/2$. Therefore, it is without loss of generality to assume that $x \geqslant 0$ in (19). Let $x_1 = s(\cos\eta - \cos\theta) \leqslant x(\theta, \eta, s)$. Using (17) and (19), one has

$$K_a(\theta, \phi) = (2a+1)\left(K_0(\theta, \phi) - aK_0(\theta, \phi)\int_0^1 s^{a-1}\,ds \right.$$
$$\left. + \frac{a}{2\pi}\int_0^1 s^{a-1/2}\int_0^\phi \frac{g(\theta, \eta, s)}{K(\eta, \phi)}\,d\eta\,ds \right)$$
$$= (a + \tfrac{1}{2})\frac{1}{\pi}\int_0^1 \int_0^\phi \frac{s^{a-1/2}g(\theta, \eta, s)}{K(\eta, \phi)}\,d\eta\,ds$$

where

$$g(\theta, \eta, s) = x_1^{-1/2} - \frac{((|z| + x)/2)^{1/2}}{|z|} \geqslant x_1^{-1/2} - |z|^{-1/2}$$
$$= \frac{\frac{1}{2}(1-s)^2\cos(\eta)\left(2s(\cos(\eta) - \cos(\theta)) + \frac{1}{2}(1-s)^2\cos(\eta)\right) + \frac{1}{4}(1-s^2)^2\sin^2(\eta)}{\sqrt{|z|}\,(|z| + x_1)\left(\sqrt{|z|} + \sqrt{|x_1|}\right)\sqrt{s}\,K(\eta, \theta)}$$
$$\geqslant \tfrac{1}{4}(1-s)^4\frac{\cos^2(\eta)}{\sqrt{s}\,K(\eta, \theta)7.33}.$$

Hence for $a > 0$

$$K_a(\theta, \phi) \geqslant \frac{a(a + \frac{1}{2})}{4(7.33)}\left(\int_0^1 s^{a-1}(1-s)^4\,ds\right)\left(\frac{1}{\pi}\int_0^\phi \frac{\cos^2(\eta)\,d\eta}{K(\eta, \phi)K(\eta, \phi)}\right).$$

Since $\int_0^1 s^{a-1}(1-s)^4\,ds = 4!/(a\prod_{k=1}^4 (a+k))$ and

$$\frac{1}{\pi}\int_0^\phi \frac{\cos^2(\eta)\,d\eta}{K(\eta, \theta)K(\eta, \phi)} \geqslant 1/32,$$

it follows that

$$K_a(\theta,\phi) \geqslant \frac{3}{32(7.33)} \frac{1}{(a+2)(a+3)(a+4)}, \quad a > 0.$$

Putting these statements together, we have

THEOREM 2.

$$K_a(\theta,\phi) \geqslant m(a) = \begin{cases} a+\frac{1}{2}, & -\frac{1}{2} < a \leqslant 0 \\ 0.012/\prod_{k=2}^{4}(a+k), & a > 0. \end{cases}$$

Theorem 2 and the orthogonality of the Legendre polynomials over the interval $0 \leqslant \theta \leqslant \pi$ yield

$$\begin{aligned} \int_\alpha^\pi |K_a(\theta,\phi)|\sin(\theta)\,d\theta &= \int_\alpha^\pi K_a(\theta,\phi)\sin(\theta)\,d\theta \\ &= \int_0^\pi K_a(\theta,\phi)\sin(\theta)\,d\theta - \int_0^\alpha K_a(\theta,\phi)\sin(\theta)\,d\theta \\ &\leqslant 1 - m(a)\int_0^\alpha \sin(\theta)\,d\theta. \end{aligned}$$

This is restated as

COROLLARY 2. $\int_\alpha^\pi |K_a(\theta,\phi)|\sin(\theta)\,d\theta \leqslant 1 - 2m(a)\sin^2(\alpha/2) < 1$.

COROLLARY 3. The operator $A(\mu)(\phi) = F(\phi) + \lambda\int_{I_2} K_a(\theta,\phi)\sin(\theta)\mu(\theta)\,d\theta$ is a contraction of the space $C(J_2)$ into itself under the maximum norms $\|\mu\| = \max_{\phi\in I_2}|\mu(\phi)|$ and $\|K_a\| = \max_{\phi\in I_2}\int_{I_2}|K_a(\theta,\phi)|\sin(\theta)\,d\theta$.

COROLLARY 4. Equation (10) has a unique solution $\mu \in C(J_2)$; furthermore, by Theorem 1 $\mu \in H(I_2) \cap C(J_2)$. Introducing the sequence

$$\mu_0(\phi) = F(\phi), \qquad \mu_{k+1}(\phi) = F(\phi) + \lambda\int_{I_2} K_a(\theta,\phi)\sin(\theta)\mu_k(\theta)\,d\theta, \tag{22}$$

one can state

COROLLARY 5. $\lim_{k\to\infty}\mu_k(\phi) = \mu(\phi)$ where the limit is uniform on J_2. Other direct consequences of the above results are

COROLLARY 6.

$$\mu \leqslant H\csc^2(\alpha/2)/m(a) \quad \text{and} \quad u \leqslant H\csc^2(\alpha/2)/m(a). \tag{23}$$

COROLLARY 7. If $H(\phi) \geqslant 0$, $\phi \in I_1^0 \cup I_2^0$, then $\mu(\phi) \geqslant 0$ and $u(\phi) \geqslant 0$.

Proof. From Theorem 2 $K_a(\theta,\phi) > 0$, since $H(\phi) \geqslant 0$, it follows that $F(\phi) \geqslant 0$, and that all members of the sequence $\mu_k(\phi)$ defined by (22) are $\geqslant 0$. Hence by Corollary 5 $\mu(\phi) \geqslant 0$.

4. The solution w. Now that a suitable function μ has been shown to exist, again consider

$$w = w(\rho,\phi) = \sum_{n=0}^{\infty} A_n P_n(\cos(\phi))(\rho/R)^n, \quad 0 \leqslant \rho < R$$

where A_n is defined by (6). Clearly $w \in C^2$ and $\nabla^2 w = 0$ for $0 \leqslant \rho < R$. Recall that $I = (0, \pi)$, $J = [0, \pi]$, and set

$$W(\phi) = \sum_{n=0}^{\infty} A_n P_n(\cos(\phi)) = \frac{1}{h_1} \int_0^{\pi} K_a(\theta, \phi) \sin(\theta) u(\theta)\, d\theta, \quad \phi \in J. \tag{24}$$

By Theorem 1 $W(\phi) \in H(I) \cap C(J)$; moreover, by Abel's limit theorem [5] $\lim_{\rho \to R} w(\rho, \phi) = W(\phi)$, $\phi \in I$. We define $w(R, \phi) = W(\phi)$. Differentiating (3) with respect to ρ one has

$$w_\rho(\rho, \phi) = \sum_{n=0}^{\infty} A_n \frac{n}{R} P_n(\cos(\phi))(\rho/R)^{n-1}, \quad \rho < R.$$

Let

$$W_1(\phi) = \sum_{n=0}^{\infty} A_n \frac{n}{R} P_n(\cos(\phi))$$

$$= \sum_{n=0}^{\infty} A_n \frac{(n + h_1 R)}{R} P_n(\cos(\phi)) - h_1 \sum_{n=0}^{\infty} A_n P_n(\cos(\phi)). \tag{25}$$

The first term in (25) is of the form of the right-hand side of (9) with $f = u$ and the second term is $h_1 W(\phi)$. Hence,

$$W_1(\phi) = u(\phi) - h_1 W(\phi), \quad \phi \in I.$$

Another application of Abel's limit theorem gives $\lim_{\rho \to R} w_\rho(\rho, \phi) = W_1(\phi)$, $\phi \in I$, and defining $w_\rho(R, \phi) = \lim_{\rho \to R^-} w_\rho(\rho, \phi)$ yields

$$w_\rho(R, \phi) = u(\phi) - h_1 w(R, \phi), \quad \phi \in I. \tag{26}$$

For $\phi \in I^0$, $u(\phi) = H_1(\phi)$ and (26) become

$$w_\rho(R, \phi) = H_1(\phi) - h_1 w(R, \phi), \quad \phi \in I_1^0. \tag{27}$$

Next, consider (10) written as

$$\mu(\phi) - H_2(\phi) = \lambda \int_0^{\pi} u(\theta) K_a(\theta, \phi) \sin(\theta)\, d\theta = \lambda h_1 W(\phi) = (h_1 - h_2) w(R, \phi). \tag{28}$$

Equation (26) with $\phi \in I_2^0$ and (28) give

$$w_\rho(R, \phi) = H_2(\phi) - h_2 w(R, \phi), \quad \phi \in I_2^0. \tag{29}$$

Note that (26) also applies when $\phi = \alpha$, because

$$u(\alpha) = \frac{H_1(\alpha^-) + \mu(\alpha^+)}{2} = \frac{H_1(\alpha^-) + H_2(\alpha^+) + (h_1 - h_2) w(R, \alpha)}{2}. \tag{30}$$

Using (27), (29), and (30), one has

$$w_\rho(R, \alpha) = \frac{w_\rho(R, \alpha^-) + w_\rho(R, \alpha^+)}{2},$$

$$w_\rho(R, \alpha^+) - w_\rho(R, \alpha^-) = H_2(\alpha^+) - H_1(\alpha^-) + (h_1 - h_2) w(R, \alpha). \tag{31}$$

It should further be noted that the right-hand sides of (27) and (29) have limits as $\phi \to 0^+$ and $\phi \to \pi^-$, respectively. Thus, w defined by (3) and (6) is a solution of the boundary value problem. The uniqueness of a solution of the problem is well known, and can be shown through a Green integral identity.

Other consequences of the above theorems and corollaries and the maximum principle are

COROLLARY 8. $|w(\rho,\phi)| \leqslant \max_{\phi\in J}|w(R,\phi)| \leqslant \|H\|\csc^2(\zeta/2)/h_1 m(a)$.

COROLLARY 9. *If* $H(\phi) \geqslant 0$, *then* $w(R,\phi) \geqslant 0$ *and* $w(\rho,\phi) \geqslant 0$. *Corollary* 9 *and Eq.* (31) *yield conditions under which* $w_\rho(R,\cdot)$ *has a jump discontinuity at* $\phi = \alpha$.

5. Numerical results. In consideration of Eq. (6) written as

$$A_n = \frac{R(n+\frac{1}{2})}{(n+h_1R)}\left(\int_0^\alpha H_1(\theta)P_n(\cos(\theta))\sin(\theta)\,d\theta + \int_\alpha^\pi \mu(\theta)P_n(\cos(\theta))\sin(\theta)\,d\theta\right)$$

and the sequence defined by (22), we let

$$x_{n,k} = \frac{R(n+\frac{1}{2})}{(n+h_1R)}\left(\int_0^\alpha H_1(\theta)P_n(\cos(\theta))\sin(\theta)\,d\theta + \int_\alpha^\pi \mu_k(\theta)P_n(\cos(\theta))\sin(\theta)\,d\theta\right), \quad k \geqslant 0. \tag{32}$$

Equations (32) and (22) lead to the iterative system of equations

$$x_{n,k} = \frac{R(n+\frac{1}{2})}{(n+h_1R)}\left(\int_0^\pi H(\theta)P_n(\cos(\theta))\sin(\theta)\,d\theta + (h_1-h_2)\sum_{j=0}^{\infty} a_{j,n}x_{j,k-1}\right), \quad k \geqslant 0 \tag{33}$$

where

$$x_{j,-1} = \frac{R(j+\frac{1}{2})}{(j+h_1R)}\int_0^\pi H_1(\theta)P_j(\cos(\theta))\sin(\theta)\,d\theta,$$

$$a_{j,n} = \int_{-1}^X P_j(x)P_n(x)\,dx = a_{n,j}, \quad X = \cos(\alpha).$$

The elements of the matrix $[a_{j,n}]$ can be found through the recurrence relationship

$$a_{0,0} = 1 + X, \qquad a_{0,n} = \frac{P_{n+1}(X) - P_{n-1}(X)}{(2n+1)},$$

$$a_{1,n} = \frac{(n+1)a_{0,n+1} + na_{0,n-1}}{2n+1}, \quad n \geqslant 1,$$

$$a_{j,n} = \frac{(2j-1)((n+1)a_{j-1,n+1} + na_{j-1,n-1})}{(2n+1)} - \frac{(j-1)a_{j-2,n}}{j}, \quad j \geqslant 2,\ n \geqslant 1.$$

Equations (33) have been programmed with the infinite series replaced by a finite number of terms N. This has been done for $h_1 = 1$, $R = 1$, $N = 26$, and $k = 15$. Two selections of the boundary function $H(\phi)$ and various choices of the parameters α and h_2 were made. In addition, the harmonic functions

$$w = w_{k,N}(\rho, \phi) = \sum_{n=0}^{N} x_{n,k} P_n(\cos(\phi))(\rho/R)^n$$

were computed for $\rho = 0, 0.5, 1$, $\phi = j\pi/4$, $j = 0, 1, 2, 3, 4$, and $\alpha = m\pi/4$, $m = 1, 2, 3$. The results are listed in the table that follows. A proof that the sequence $x_{n,k}$, generated by replacing the infinite series in (33) by the finite sum $\Sigma_{j=0}^{N}$, is convergent to A_n is given in the appendix at the end of the paper.

CASE I $H_1(\phi) = 1$, $H_2(\phi) = 1$, $h_2 = .5$

j	$m = 1$ $w(\frac{1}{2}, \phi)$	$m = 2$ $w(\frac{1}{2}, \phi)$	$m = 3$ $w(\frac{1}{2}, \phi)$	$m = 1$ $w(1, \phi)$	$m = 2$ $w(1, \phi)$	$m = 3$ $w(1, \phi)$
0	1.6447	1.2505	1.0517	1.4374	1.1511	1.0295
1	1.7065	1.2821	1.0579	1.6209	1.1809	1.0349
2	1.7954	1.3865	1.0823	1.8339	1.3797	1.0557
3	1.8366	1.4954	1.1409	1.8757	1.6027	1.1926
4	1.8478	1.5306	1.1847	1.8869	1.6418	1.3360

CASE II $H_1(\phi) = 1$, $H_2(\phi) = 1$, $h_2 = 0$

j	$m = 1$ $w(\frac{1}{2}, \phi)$	$m = 2$ $w(\frac{1}{2}, \phi)$	$m = 3$ $w(\frac{1}{2}, \phi)$	$m = 1$ $w(1, \phi)$	$m = 2$ $w(1, \phi)$	$m = 3$ $w(1, \phi)$
0	6.7809	2.1196	1.1441	4.8957	1.6754	1.0825
1	7.3484	2.2608	1.1614	6.4657	1.8062	1.0973
2	8.2290	2.7321	1.2295	8.5744	2.6624	1.1550
3	8.6837	3.2519	1.3937	9.1397	3.7706	1.5309
4	8.8309	3.4325	1.5184	9.3024	4.0098	1.9618

CASE III $H_1(\phi) = 1/2$, $H_2(\phi) = 1/3$, $h_2 = 0$

j	$m = 1$ $w(\frac{1}{2}, \phi)$	$m = 2$ $w(\frac{1}{2}, \phi)$	$m = 3$ $w(\frac{1}{2}, \phi)$	$m = 1$ $w(1, \phi)$	$m = 2$ $w(1, \phi)$	$m = 3$ $w(1, \phi)$
0	2.6101	.71571	.31006	1.8405	.53093	.28439
1	2.8417	.77446	.31730	2.4817	.58533	.29054
2	3.2009	.97046	.34561	3.3419	.94149	.31460
3	3.3903	1.1867	.41403	3.5720	1.4024	.47121
4	3.4461	1.2617	.46598	3.6382	1.5018	.65075

	CASE I	CASE II	CASE III
m	$w(0,\phi)$	$w(0,\phi)$	$w(0,\phi)$
1	1.7795	8.0904	3.1443
2	1.3880	2.7480	.97708
3	1.0938	1.2618	.35908

Case III is one of the boundary value problems considered by Kelman in [4] with his $h = h_1 = 1$.

Appendix: A proof of the convergence of the numerical procedure. Let

$$u_k^N(\theta) = \sum_{n=0}^{N} x_{n,k} \frac{\left(n + a + \frac{1}{2}\right)}{R} P_n(\cos(\theta)).$$

It follows that

$$x_{n,k} = \frac{R\left(n + \frac{1}{2}\right)}{\left(n + a + \frac{1}{2}\right)} \int_0^{\pi} \left(u_k^N(\theta) P_n(\cos(\theta)) \sin(\theta)\, d\theta\right).$$

Using Eq. (6) and the above equation, one has

$$x_{n,k} - A_n = \frac{R\left(n + \frac{1}{2}\right)}{\left(N + a + \frac{1}{2}\right)} \int_0^{\pi} \left(u_k^N(\theta) - u(\theta)\right) P_n(\cos(\theta)) \sin(\theta)\, d\theta.$$

It will be shown that

$$\lim_{\substack{k\to\infty \\ N\to\infty}} \max_{0 \leqslant n \leqslant N} \left| \int_0^{\pi} \left(u_k^N(\theta) - u(\theta)\right) P_n(\cos(\theta)) \sin(\theta)\, d\theta \right| = 0, \quad n \geqslant 0. \tag{34}$$

Let

$$K_a^N(\psi,\phi) = \sum_{n=0}^{N} \frac{\left(a + \frac{1}{2}\right)\left(n + \frac{1}{2}\right)}{\left(n + a + \frac{1}{2}\right)} P_n(\cos(\psi)) P_n(\cos(\phi)),$$

$$k_a^N(\theta,\phi) = \sum_{n=0}^{N} \left(n + \tfrac{1}{2}\right)\left(\int_{\alpha}^{\pi} K_a^N(\psi,\phi) P_n(\cos(\psi)) \sin(\psi)\, d\psi\right) P_n(\cos(\theta)),$$

$$H_N(\theta) = \sum_{n=0}^{N} \left(n + \tfrac{1}{2}\right)\left(\int_0^{\pi} H(\phi) P_n(\cos(\phi)) \sin(\phi)\, d\phi\right) P_n(\cos(\theta)), \quad N > 2.$$

It can be shown that

$$u_k^N(\theta) = H_N(\theta) + \lambda \int_0^{\pi} k_a^N(\theta,\phi) u_{k-1}^N(\phi) \sin(\phi)\, d\phi. \tag{35}$$

It is also known that

$$u(\theta) = \begin{cases} H_1(\theta), & 0 \leqslant \theta < \alpha, \\ H_2(\theta) + \lambda \int_0^{\pi} K_a(\theta,\phi) u(\phi) \sin(\phi)\, d\phi, & \alpha < \theta \leqslant \pi. \end{cases} \tag{36}$$

Let

$$C_{0,n}^{N}(\theta) = P_n(\cos(\theta)),$$
$$C_{j,n}^{N}(\theta) = \int_{\alpha}^{\pi} K_a^N(\theta,\phi) C_{j-1,n}^{N}(\phi)\sin(\phi)\,d\phi, \quad j \geqslant 1,$$
$$F_{j,n}^{N} = \lambda \int_0^{\pi} u(\phi)\sin(\phi)\int_{\alpha}^{\pi}\Big(\big(K_a^N(\theta,\phi) - K_a(\theta,\phi)\big)C_{j,n}^{N}(\theta)\sin(\theta)\,d\theta\Big)\,d\phi.$$

Let $\Omega = 1 - 2m(a)\sin^2(\alpha/2) > 0$. It is easy to show that

$$\left|C_{j,n}^{N}(\theta)\right| \leqslant \Omega^{j}$$

and

$$\int_0^{\pi}\big(u_k^N(\theta) - u(\theta)\big)C_{j,n}^{N}(\theta)\sin(\theta)\,d\theta = \int_0^{\pi}\big(u_{k-1}^N(\theta) - u(\theta)\big)C_{j+1,n}^{N}(\theta)\sin(\theta)\,d\theta + F_{j,n}^{N}. \tag{37}$$

Iterating on equation (37) leads to

$$\int_0^{\pi}\big(u_k^N(\theta) - u(\theta)\big)P_n(\cos(\theta))\sin(\theta)\,d\theta$$
$$= \lambda^k \int_0^{\pi}\big(u_0^N(\theta) - u(\theta)\big)C_{k,n}^{N}(\theta)\sin(\theta)\,d\theta + \sum_{j=0}^{k-1}\lambda^j F_{j,n}^{N}. \tag{38}$$

Equation (38) implies equation (34).

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