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MONOTONIC DECAY OF SOLUTIONS OF PARABOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS*

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1. Introduction. In recent papers [3, 4] Day considered parabolic equations in one space dimension with boundary conditions

$$u(-a,t)=\int_{-a}^{a} f(x)u(x,t)\,dx, \qquad u(a,t)=\int_{-a}^{a} g(x)u(x,t)\,dx.$$

The solution u represents the entropy in a quasi-static theory of thermoelasticity; for a derivation of the model see [1, 2]. Day proved that if

$$\int_{-a}^{a} |f(x)|\,dx < 1, \qquad \int_{-a}^{a} |g(x)|\,dx < 1, \tag{1.1}$$

then the maximum modulus of the entropy is decreasing in time, that is,

$$U(t) \equiv \max_{-a\leqslant x\leqslant a} |u(x,t)| \tag{1.2}$$

is decreasing in t. An example in [4] shows that if both inequalities in (1.1) are reversed, then, in general, $U(t)$ may increase exponentially with t.

The proof of (1.2) is based on estimating integrals of the form $\int_{-a}^{a} P(x)u^{2m}(x,t)\,dx$ with suitably chosen $P(x)$.

In this paper we extend the assertion (1.2) to general parabolic equations (in n dimensions) using a different and more general method based on the maximum principle. We also show that $U(t)\to 0$ exponentially fast as $t\to\infty$. Finally, we establish the strict monotonicity of $U(t)$ for all $t>0$; this implies, in particular, that $U(t)$ does not vanish in finite time. For completeness we also prove the existence and uniqueness of $u(x,t)$.

2. *U(t)* is decreasing. Let

$$A \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

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be a uniformly elliptic operator in a bounded domain Ω in $\mathbf{R}^n$, that is,

$$\sum a_{ij}(x)\xi_i\xi_j \geqslant \mu|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbf{R}^n, \tag{2.1}$$

where μ is a positive constant, and assume that, for some $0 < \alpha < 1$,

$$a_{ij},\ b_i,\ c \text{ belong to } C^\alpha(\bar{\Omega}), \tag{2.2}$$

$$c(x) \leqslant 0, \tag{2.3}$$

$$\text{for every } x \in \partial\Omega \text{ there exists a barrier for } A; \tag{2.4}$$

the last condition is satisfied if, for instance, $\partial\Omega$ is in C^2 or if Ω is convex.

Let $f(x, y)$ be a continuous function defined for $x \in \partial\Omega$, $y \in \bar{\Omega}$, and assume that

$$\theta(x) \equiv \int_\Omega |f(x, y)|\, dy \leqslant \rho < 1 \quad \forall x \in \partial\Omega. \tag{2.5}$$

For any $0 < T \leqslant \infty$ set

$$\Omega_T = \Omega \times \{0 < t < T\}.$$

Consider the parabolic problem:

$$\frac{\partial u}{\partial t} - Au = 0 \quad \text{in } \Omega_\infty, \tag{2.6}$$

$$u(x,0) = u_0(x) \quad \text{if } x \in \Omega, \tag{2.7}$$

$$u(x, y) = \int_\Omega f(x, y) u(y, t)\, dy \quad \text{if } x \in \partial\Omega, 0 < t < \infty, \tag{2.8}$$

where the initial data u_0 satisfies

$$u_0 \not\equiv 0, \quad u_0 \in C(\bar{\Omega}). \tag{2.9}$$

THEOREM 2.1. There exists a unique solution u of (2.6)–(2.8) such that $u \in C(\bar{\Omega}_\infty)$.

Proof. It suffices to prove existence and uniqueness in Ω_T, for any $T < \infty$. Define a sequence $u_m(x, t)$ inductively as follows: (i) $u_0(x, t) \equiv u_0(x)$; (ii) given u_m, let

$$\tilde{u}_{m+1}(x, t) = \int_\Omega f(x, y) u_m(y, t)\, dy \quad (x \in \partial\Omega)$$

and let u_{m+1} be the solution of

$$\left(\frac{\partial}{\partial t} - A\right) u_{m+1} = 0 \quad \text{in } \Omega_T,$$

$$u_{m+1} = \tilde{u}_{m+1} \quad \text{on } \partial\Omega \times (0, T),$$

$$u_{m+1}(x,0) = u_0(x) \quad \text{if } x \in \Omega.$$

Conditions (2.1)–(2.4) ensure that u_{m+1} exists and is continuous in $\bar{\Omega}_T$ (provided the same is true of u_m). Further, since

$$\left(\frac{\partial}{\partial t} - A\right)(u_{m+1} - u_m) = 0 \quad \text{in } \Omega_T,$$

$$(u_{m+1} - u_m)(x,0) = 0 \quad \text{if } x \in \Omega,$$

the maximum principle [5, Chapter 2] gives

$$\sup_{\Omega_T} |u_{m+1}(x,t) - u_m(x,t)| = \sup_{\substack{x\in\partial\Omega \\ 0\leqslant t\leqslant T}} \left| \int_\Omega f(x,y)(u_m(y,t) - u_{m-1}(y,t))\,dy \right|$$
$$\leqslant \rho \sup_{\Omega_T} |u_m - u_{m-1}|, \tag{2.10}$$

or

$$\sup_{\Omega_T} |u_{m+1} - u_m| \leqslant C\rho^m \quad \left(C = \sup_{\Omega_T} |u_1 - u_0|\right).$$

It follows that $\{u_m\}$ is a uniform Cauchy sequence and, by standard theory [5], $u \equiv \lim u_m$ is the solution of (2.6)–(2.8).

Finally, if v is another solution then [cf. (2.10)]

$$\sup_{\Omega_T} |v - y| \leqslant \rho \sup_{\Omega_T} |v - u|,$$

giving $v \equiv u$.

Set

$$U(t) = \max_{x\in\bar\Omega} |u(x,t)|. \tag{2.11}$$

THEOREM 2.2. The function $U(t)$ is monotone decreasing in t.

Proof. If the assertion is not true then there exists a $t_0 > 0$ and sequences of positive numbers $\varepsilon_m \downarrow 0$, $\delta_m \downarrow 0$ such that

$$\delta_m + \max_{x\in\bar\Omega} |u(x,t_0)| < \max_{x\in\bar\Omega} |u(x,t_0+\varepsilon_m)|. \tag{2.12}$$

Without loss of generality we may assume that

$$\max_{x\in\bar\Omega} |u(x,t_0+\varepsilon_m)| = \max_{x\in\bar\Omega} u(x,t_0+\varepsilon_m) \geqslant -\min_{x\in\bar\Omega} u(x,t_0+\varepsilon_m). \tag{2.13}$$

Choose $x_m \in \Omega$ such that

$$u(x_m, t_0+\varepsilon_m) > \max_{x\in\bar\Omega} u(x,t_0+\varepsilon_m) - \delta_m.$$

Then, by (2.12)

$$u(x_m, t_0+\varepsilon_m) > \max_{x\in\bar\Omega} |u(x,t_0)|. \tag{2.14}$$

Applying the strong maximum principle to u in $\Omega \times (t_0, t_0+\varepsilon_m)$ and using (2.14) we conclude that there exist $y_m \in \partial\Omega$ and $\tilde\varepsilon_m \in (0,\varepsilon_m)$ such that

$$u(x_m, t_0+\varepsilon_m) < u(y_m, t_0+\tilde\varepsilon_m). \tag{2.15}$$

But, by (2.8) and (2.5),

$$u(y_m, t_0+\tilde\varepsilon_m) \leqslant \int_\Omega |f(y_m,x)| \left[\max_{x\in\bar\Omega} |u(x,t_0+\tilde\varepsilon_m)|\right] dx$$
$$\leqslant \rho\left\{\max_{x\in\bar\Omega} |u(x,t_0)| + \eta_m\right\},$$

where $\eta_m \to 0$ by the continuity of u. Substituting this into (2.15) and recalling (2.14), we get

$$u(x_m, t_0 + \varepsilon_m) \leqslant \rho u(x_m, t_0 + \varepsilon_m) + \eta_m$$

or $(1 - \rho)u(x_m, t_0 + \varepsilon_m) \leqslant \eta_m$. Taking $m \to \infty$ we deduce, upon using (2.14), that

$$u(x, t_0) \equiv 0. \tag{2.16}$$

By the uniqueness part of Theorem 2.1 it follows that $u(x, t) = 0$ if $t > t_0$, a contradiction to (2.12).

THEOREM 2.3. There exist positive constants C, γ such that

$$U(t) \leqslant Ce^{-\gamma t} \quad \text{for all } t > 0. \tag{2.17}$$

Proof. Let $|u_0| \leqslant M$. The proof of existence shows that $|u_m| \leqslant M$ and therefore also $|u| \leqslant M$ in Ω_∞. Hence, if $y \in \partial\Omega$, $t > 0$,

$$|u(y, t)| \leqslant \int_\Omega |f(y, x)u(x, t)|\, dx \leqslant \rho M.$$

By standard results on the asymptotic behavior of the solution of parabolic equations [5, Chapter 6] it follows that

$$\limsup_{x \in \Omega,\, t \to \infty} |u(x, t)| \leqslant \rho M.$$

Hence, if $\rho_* = (\rho + 1)/2$ then

$$|u(x, t)| \leqslant \rho_* M \quad \text{if } x \in \Omega,\ t > t_1$$

for some large enough t_1. Similarly, we obtain by induction

$$|u(x, t)| \leqslant (\rho_*)^m M \quad \text{if } t > mt_1,$$

so that

$$U(t) \leqslant (\rho_*)^m M \quad \text{if } mt_1 \leqslant t \leqslant (m + 1)t_1.$$

It follows that

$$U(t) \leqslant \frac{M}{\rho_*}(\rho_0)^{m+1} \leqslant \frac{M}{\rho_*}(\rho_*)^{t/t_1}$$

from which (2.17) follows.

3. *U(t)* is strictly decreasing.

LEMMA 3.1. There exists a $0 < T_* \leqslant \infty$ such that $U(t)$ is strictly decreasing if $0 < t < T_*$ and $U(t) \equiv 0$ if $t > T_*$.

Proof. Suppose $U(t)$ is not strictly decreasing for all $t > 0$. Then there exist $0 < t_0 < t_1 < \infty$ such that

$$U(t) = U(t_0) = U(t_1) \quad \text{for all } t_0 < t < t_1.$$

In order to prove the lemma it suffices to show that for any such t_0, t_1 we have $U(t_1) = 0$. We shall assume that

$$U(t_1) > 0 \tag{3.1}$$

and derive a contradiction.

By the maximum principle we then have, for any $x \in \Omega$,

$$
\begin{aligned}
|u(x,t_1)| &\leqslant \max\left\{ \max_{y\in\partial\Omega,\ t_1\leqslant s\leqslant t} |u(y,s)|,\ \max_{\bar{x}\in\bar{\Omega}} |u(\bar{x},t_0)| \right\} \\
&= \max_{t_0\leqslant t\leqslant t_1} U(t) = U(t_1) = \max_{\bar{x}\in\bar{\Omega}} u(\bar{x},t_1) \qquad (3.2)\\
&= \max_{y\in\partial\Omega} u(y,t_1) = u(y_1,t_1)
\end{aligned}
$$

for some $y_1 \in \partial\Omega$, where for definiteness we have assumed that

$$U(t_1) = \max_{\bar{x}\in\bar{\Omega}} u(\bar{x},t_1) \geqslant -\min_{\bar{x}\in\bar{\Omega}} u(\bar{x},t_1).$$

Hence

$$
\begin{aligned}
u(y_1,t_1) &= \int_\Omega f(y_1,x)u(x,t_1)\,dx \leqslant \rho \max_{x\in\bar{\Omega}} |u(x,t_1)| \\
&\leqslant \rho u(y_1,t_1). \qquad (3.3)
\end{aligned}
$$

Since $\rho < 1$ it follows that $u(y_1,t_1) = 0$ and, by (3.2), $U(t_1) = 0$, which contradicts (3.1).

In the sequel we impose the following additional conditions:

$$a_{ij} \in C^{1+\alpha}(\bar{\Omega}), \qquad (3.4)$$

$$\partial\Omega \quad \text{is in } C^{2+\alpha}, \qquad (3.5)$$

and there exists an extension of f into a function $f(x,y)$ defined for all $x \in \bar{\Omega}$, $y \in \bar{\Omega}$ such that

$$f(x,y) \in C^{2+\alpha}(\bar{\Omega}\times\bar{\Omega}). \qquad (3.6)$$

We can clearly redefine f, if necessary, such that also

$$\theta(x) \equiv \int_\Omega |f(x,y)|\,dy \leqslant \rho_0 < 1 \quad \text{for all } x \in \bar{\Omega}. \qquad (3.7)$$

LEMMA 3.2. For any $x \in \Omega$ the function $u(x,t)$ is analytic in t, for $0 < t < \infty$.

Proof. Introduce the function

$$V(x,t) = u(x,t) - \int_\Omega f(x,y)u(y,t)\,dy. \qquad (3.8)$$

We can solve u in terms of U, at least formally, by

$$u(x,t) = V(x,t) + \sum_{m=0}^{\infty} \int_\Omega f^{(m)}(x,y)V(y,t)\,dy, \qquad (3.9)$$

where $f^{(0)}(x,y) = f(x,y)$ and

$$f^{(m)}(x,y) = \int_\Omega f^{(m-1)}(x,z)f(z,y)\,dz = \int_\Omega f(x,z)f^{(m-1)}(z,y)\,dz. \qquad (3.10)$$

In view of (3.7)

$$|f^{(m)}(x,y)| \leqslant \rho_0 \sup_{z\in\Omega} |f^{(m-1)}(z,y)|,$$

from which we deduce that

$$\sup_{x,y} |f^{(m)}(x,y)| \leqslant C(\rho_0)^{m-1}. \qquad (3.11)$$

It follows that the series $\sum f^{(m)}(x, y)$ is uniformly convergent; in fact, it even converges in the $C^{2+\alpha}(\bar{\Omega} \times \bar{\Omega})$ sense. We can thus write

$$u(x,t) = V(x,t) + \int_\Omega F(x,y)V(y,t)\,dy \tag{3.12}$$

with F in $C^{2+\alpha}$.

The function V satisfies:

$$\begin{aligned} V_t - AV &= -\int_\Omega f(x,y)u_t(y,t)\,dy + \int_\Omega (Af(x,y))u(y,t)\,dy \\ &= -\int_\Omega f(x,y)\left[\sum a_{ij}(y)\frac{\partial^2 u}{\partial y_i \partial y_j} + \sum b_i(y)\frac{\partial u}{\partial y} + c(y)u\right]dy \\ &\quad + \int_\Omega \left[\sum a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial f}{\partial x_i} + c(x)f\right]u(y,t)\,dy. \end{aligned}$$

Writing

$$\begin{aligned} -\int_\Omega f(x,y)a_{ij}(y)\frac{\partial^2 u(y)}{\partial y_i \partial y_j}\,dy = &-\int_{\partial\Omega} f(x,y)a_{ij}(y)\nu_i \frac{\partial u(y,t)}{\partial y_j}\,dS_y \\ &+ \int_\Omega \frac{\partial}{\partial y_i}(f(x,y)a_{ij}(y))\frac{\partial u(y,t)}{\partial y_j}\,dy \end{aligned}$$

we see that

$$V_t - AV = G \quad \text{in } \Omega_\infty, \tag{3.13}$$

where

$$\begin{aligned} G = \sum \int_{\partial\Omega} \tilde{\alpha}_j(x,y)\frac{\partial u(y,t)}{\partial y_j}\,dS_y &+ \sum \int_\Omega \tilde{\beta}_j(x,y)\frac{\partial u(y,t)}{\partial y_j}\,dy \\ &+ \int_\Omega \tilde{\gamma}(x,y)u(y,t)\,dy \end{aligned}$$

and $\tilde{\alpha}_j$, $\tilde{\beta}_j$, $\tilde{\gamma}$ are in C^α. Substituting u and $\partial u/\partial y_j$ from (3.12) we find that

$$\begin{aligned} G = G(V) = \sum \int_{\partial\Omega} \alpha_j(x,y)\frac{\partial V(y,t)}{\partial y_j}\,dS_y &+ \sum \int_\Omega \beta_j(x,y)\frac{\partial V(y,t)}{\partial y_j}\,dy \\ &+ \int_\Omega \gamma(x,y)V(y,t)\,dy, \end{aligned} \tag{3.14}$$

where α_j, β_j, γ are in C^α.

Note that

$$V(x,t) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0, \tag{3.15}$$

and

$$V(x,0) = V_0(x) \quad \text{if } x \in \Omega,\ V_0 \in C(\bar{\Omega}). \tag{3.16}$$

The parabolic problem (3.13), (3.15), (3.16) is not a standard one since G is a nonlocal (linear) operator of V. Nevertheless, one can still apply Schauder-type estimates in order to derive inductively that, for any $0 < T < \infty$, $1 \leqslant m < \infty$,

$$|D_t^m V(y,t)| \leqslant \frac{K_0 K^{m-1}}{t^{m-1+n/2}} \frac{m!}{m^2} \quad \text{in } \Omega_T, \tag{3.17}$$

with similar bounds on $D_t^{m-1} D_y^2 V(y,t)$ and on the corresponding Hölder coefficients. Indeed, in [6] we considered the parabolic problem

$$v_{yy} - v_t = v_y(1,t) v_y \quad \text{in } \{0 < y < 1, 0 < t < T\},$$

$$v(0,t) = v(1,t) = 0 \quad \text{if } 0 < t < T,$$

and established estimates of the type (3.17). The proof can be extended to parabolic equations with variable coefficients in n-dimensional domains [and with $v_y(1,t)v_y$ replaced by $G(V)$]. In fact, the basic estimates from which one deduces (3.17) (for v) are the interior-boundary Schauder estimates stated in Theorem 1.1 of [6]. This theorem can be extended to general parabolic operators in general domains by a standard argument used to derive the Schauder estimates for variable coefficients and general domains, once they have been proven for constant coefficients in rectangular domains; see [5, Chapter 4]. Since the calculations are routine, we omit the details.

The assertion (3.17) now follows from the interior-boundary Schauder estimates in a way similar to that for v in [6].

From (3.17) and (3.12) we see that $u(x,t)$ is analytic in t, for each $x \in \Omega$.

THEOREM 3.3. If (3.4)–(3.6) hold, then $U(t)$ is strictly decreasing in t for all $t > 0$.

Proof. If the assertion is not true then, by Lemma 3.1, $u(x,t) \equiv 0$ for all $t > T_*$. By Lemma 3.2 it then follows that $u(x,t) \equiv 0$ in Ω_∞, a contradiction to $u_0 \not\equiv 0$.

REMARK. Theorems 2.1–2.3 and Lemma 3.1 extend to the case where the coefficients of A also depend on t. Lemma 3.2 and Theorem 3.3 also extend to this case provided the coefficients are analytic in t.

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