QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIV, NUMBER 3
OCTOBER 1986, PAGES 487-491

GENERALIZED MAGNETOHYDROSTATIC EQUILIBRIA*

By

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Abstract. This note shows the existence of a generalized class of magnetohydrostatic elliptic equilibria of a perfectly conducting fluid in systems with cylindrical topology for which the surfaces of constant pressure have an elliptic transverse cross section with the eccentricity given by an arbitrary function of the axial distance. Further, it is shown that a change in the parameters characterizing the solution leads to a generalized class of hyperbolic equilibria for which the surfaces of constant pressure have a hyperbolic transverse cross section with the eccentricity given by an arbitrary function of the axial distance. Whereas the first class of equilibria is of interest in a plasma-confinement system, the second class of equilibria is of interest in a magnetic-field-line reconnection system.

One has for a magnetohydrostatic equilibrium of a perfectly conducting fluid

$$\nabla p = \mathbf{J} \times \mathbf{B}, \tag{1}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \tag{3}$$

where p is the pressure of the fluid, $\mathbf{J}$ is the current density, and $\mathbf{B}$ is the magnetic field. Woolley [1] showed that Eqs. (1)–(3) yield a three-dimensional family of elliptic equilibria for which the surfaces of constant pressure have an elliptic transverse cross section with the eccentricity an arbitrary function of the axial distance z. These equilibria are of interest in a plasma-confinement system. The purpose of this note is first to show the existence of a more general class of elliptic equilibria than the one established by Woolley [1]. Further, it is shown that a change in the parameters characterizing the solution leads to a family of hyperbolic equilibria for which the surfaces of constant pressure have a hyperbolic transverse cross section with the eccentricity an arbitrary function of the axial distance z. These equilibria are of interest in a magnetic-field-line reconnection system

* Received April 12, 1985.

and represent a three-dimensional generalization of the solution given by Habbal and Tuan [2]. Magnetic field reconnection is the process by which magnetic field lines that are initially distinct link up, thereby lowering the potential energy of the magnetic field.

In a system of rectangular Cartesian coordinates (x, y, z), let us prescribe for the magnetic field

$$B_x = \mathscr{H}_1[yf(z)], \qquad B_y = \mathscr{H}_2[xh(z)], \qquad B_z = 0, \tag{4}$$

which identically satisfies Eq. (3). Here $f(z)$ and $h(z)$ are arbitrary functions of z. A similar prescription was made by Lin [3] and Shivamoggi and Uberoi [4] in constructing exact solutions to the equations of magnetohydrodynamics of a dissipative fluid. Note that the prescription for the magnetic field given by Woolley [1] is a special case of (4).

Using (4), Eq. (2) gives for the current density,

$$J_x = -\mathscr{H}_2' xh'; \qquad J_y = \mathscr{H}_1' yf', \qquad J_z = \mathscr{H}_2' h - \mathscr{H}_1' f, \tag{5}$$

where primes denote differentiation with respect to the argument.

Using (4), Eqs. (1) and (2) give

$$p = \omega(x, y) - \frac{1}{2}\left(\mathscr{H}_1^2 + \mathscr{H}_2^2\right). \tag{6}$$

Here $\omega(x, y)$ represents the total pressure—hydrodynamic plus magnetic. Using (4)–(6), Eq. (1) gives

$$\partial\omega/\partial x = \mathscr{H}_2 \mathscr{H}_1' f, \qquad \partial\omega/\partial y = \mathscr{H}_1 \mathscr{H}_2' h, \tag{7}$$

from which it is clear that one requires

$$\mathscr{H}_1[yf(z)]\,\mathscr{H}_2[xh(z)] = \mathscr{F}(x, y). \tag{8}$$

Further, from (7) the integrability conditions for the existence of $\omega(x, y)$ are

$$\mathscr{H}_2 \mathscr{H}_1'' f^2 = \mathscr{H}_1 \mathscr{H}_2'' h^2, \tag{9a}$$

or

$$\mathscr{H}_1'' f^2/\mathscr{H}_1 = \mathscr{H}_2'' h^2/\mathscr{H}_2 = \kappa(z), \tag{9b}$$

where $\kappa(z)$ is an arbitrary function of z.

Let us differentiate the first equation in (9b) with respect to y, so that there follows

$$f^2 \mathscr{H}_1''' = \kappa \mathscr{H}_1'. \tag{10}$$

Differentiating the first equation in (9b) with respect to z, one obtains

$$2ff'\mathscr{H}_1'' + yf^2 f'\mathscr{H}_1''' = \kappa y f'\mathscr{H}_1' + \kappa'\mathscr{H}_1. \tag{11}$$

Using Eq. (10), Eq. (11) becomes

$$2ff'\mathscr{H}_1'' = \kappa'\mathscr{H}_1, \tag{12}$$

which is consistent with the first equation in (9b). Similar deduction follows for the second equation in (9b). It is important to note from (12) that the present generalization exists only if $\kappa = \kappa(z)$, and it degenerates to Woolley's [1] solution if $\kappa = \text{const}$.

There exist two distinct classes of solutions according as $\kappa \neq 0$ or $\kappa = 0$. For $\kappa \neq 0$, one obtains from (7) and (9b)

$$\omega(x, y) = \mathscr{H}_2'\mathscr{H}_1' fh/\kappa + \text{const}. \tag{13}$$

Using (13), (6) becomes

$$p = p_0 + \mathscr{H}_2'\mathscr{H}_1' fh/\kappa - \tfrac{1}{2}\left(\mathscr{H}_1^2 + \mathscr{H}_2^2\right), \qquad \kappa \neq 0, \tag{14}$$

p_0 being an arbitrary constant.

Corresponding to $\kappa = 0$, (9) gives

$$\mathscr{H}_1 = Gyf(z), \qquad \mathscr{H}_2 = Hxh(z), \tag{15}$$

where G and H are arbitrary constants. Using (15), (8) gives

$$\sqrt{f(z)h(z)} = \text{const}. \tag{16}$$

First let us choose this constant to be imaginary, say iC, where C is real. This choice leads to a solution that is relevant to confining-field equilibria because in this solution the pressure p decreases monotonically as one moves away from the axis in a transverse plane $z = \text{const}$.

Using (7), (15), and (16), (6) becomes

$$p = p_0 - \frac{1}{2}\left[\left(\frac{C^2H}{Gf^2} + 1\right)\mathscr{H}_1^2 + \left(\frac{C^2G}{Hh^2} + 1\right)\mathscr{H}_2^2\right]. \tag{17}$$

For the case $G = H = 1$, (17) becomes

$$p = p_0 - \tfrac{1}{2}C^2\left[1 + \rho^2(z)\right]\left[x^2/\rho^2(z) + y^2\right], \tag{18}$$

where

$$\rho(z) \equiv f(z)/c.$$

Note that (18) is identical to the case obtained by Woolley [1] for $\kappa = 0$.

The intersection of the surfaces of constant pressure with a transverse plane $z = \text{const}$ gives a family of nested ellipses

$$x^2/a^2 + y^2/b^2 = 1 \tag{19}$$

within the bounding ellipse corresponding to $p = 0$ (assuming that $p = 0$ describes the boundary of the plasma). Here

$$a^2(z) = \frac{2(p_0 - p)\rho^2(z)}{C^2\left[1 + \rho^2(z)\right]}, \qquad b^2(z) = \frac{2(p_0 - p)}{C^2\left[1 + \rho^2(z)\right]}. \tag{20}$$

The eccentricity σ of the loci $p = \text{const}$ in a given plane $z = \text{const}$ is given by

$$\sigma(z) = \frac{b^2 - a^2}{b^2 + a^2} = \frac{1 - \rho^2(z)}{1 + \rho^2(z)}. \tag{21}$$

On the other hand, if we choose the constant in (16) to be real, say A, (6) becomes

$$p = p_0 - \tfrac{1}{2}\left[\left(1 - \frac{A^2H}{Gf^2}\right)\mathscr{H}_1^2 + \left(1 - \frac{A^2G}{Hh^2}\right)\mathscr{H}_2^2\right]. \tag{22}$$

Choosing $H = kG$, where k is a positive constant, (22) becomes

$$p = p_0 + \frac{1}{2}kG^2A^2[1 - \xi^2(z)]\left[y^2 - \frac{x^2}{\xi^2(z)}\right], \tag{23}$$

where

$$\xi(z) \equiv f(z)/A\sqrt{k}\,.$$

The intersection of the surfaces of constant pressure with a transverse plane z = const gives a family of hyperbolae

$$y^2/b^2 - x^2/a^2 = 1, \tag{24}$$

where

$$a^2(z) = \frac{2(p_0 - p)\xi^2(z)}{kG^2A^2[1 - \xi^2(z)]s},$$

$$b^2(z) = \frac{2(p_0 - p)}{kG^2A^2[1 - \xi^2(z)]}.$$

The eccentricity ν of the loci p = const in a given plane z = const is given by

$$\nu(z) = \frac{b^2 + a^2}{b^2 - a^2} = \frac{1 + \xi^2(z)}{1 - \xi^2(z)}. \tag{25}$$

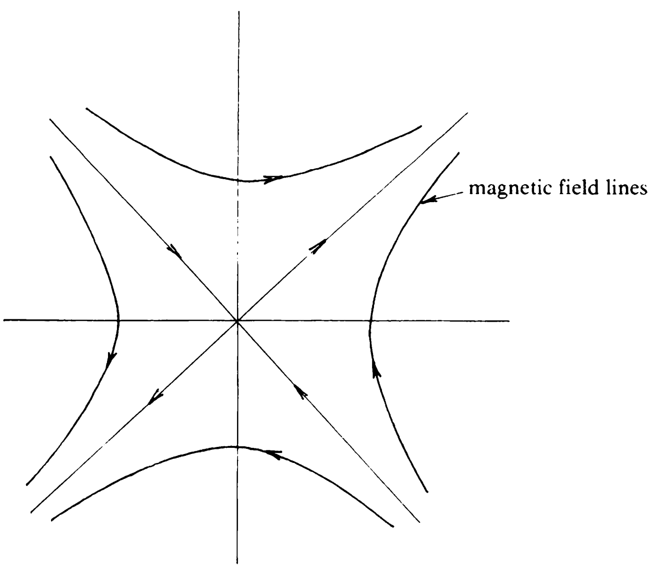

FIG. 1

This solution is of relevance to the problem of magnetic-field-line reconnection, and represents a three-dimensional generalization of the solution obtained by Habbal and Tuan [2]. For the case considered by Habbal and Tuan, $f(z) = h(z) = 1$, and then (4) gives

$$B_x = Gy, \qquad HB_y = kGx. \tag{26}$$

(26) represents a magnetic field line configuration near a X-type magnetic neutral point that is relevant for the magnetic field reconnection (see Figure 1). For this case, the isobars follow the field lines closely. Further, we obtain

$$J_x = J_y = 0, \qquad J_z = G(k - 1), \tag{27}$$

so that it is necessary to have $k \neq 1$ in order to allow the magnetic field reconnection to take place.

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