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INTERACTIONS IN A STRETCHED STRING*

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1. Introduction. In this note we shall discuss the interaction of constant stretch traveling waves in an infinitely long elastic string.

The motion of such a string is described by a complex-valued function

$$Z(x,t) = \chi(x,t) + i\mathscr{Y}(x,t), \tag{1}$$

where χ and $\mathscr{Y}$ represent the horizontal and vertical positions of a mass point x at time t. The equilibrium or rest configuration of the string is taken to be

$$Z(x,t) \equiv x + i0. \tag{2}$$

In the absence of body forces the equations of motion for the string are given by

$$\rho_0 \frac{\partial^2 Z(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{T(x,t)}{\gamma(x,t)}\frac{\partial Z(x,t)}{\partial x}\right), \tag{3}$$

where ρ_0 is the constant mass density of points in the reference state $Z \equiv x + i0$, $T(x,t)$ is the tension at the displaced point $(\chi,\mathscr{Y})(x,t)$ and is labeled by its material coordinate x and time t, and $\gamma(x,t)$ is the stretch associated with the displaced point $(\chi,\mathscr{Y})(x,t)$ and is given by

$$\gamma(x,t) \overset{\text{def}}{=} \sqrt{\left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \mathscr{Y}}{\partial x}\right)^2}\,(x,t). \tag{4}$$

We shall assume that the string is elastic, that is, that

$$T(x,t) = \hat{\tau}(\gamma(x,t)), \tag{5}$$

where $\hat{\tau}(\cdot)$ is a positive-valued, monotone increasing function of the stretch γ. For any constant stretch $\gamma_0 > 0$, equations (3) and (5) support solutions

$$Z(x,t) = \hat{\mathscr{Z}}(x \mp c_0 t) \pm \gamma_0 c_0 t, \tag{6}$$

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where

$$0 < c_0 = \sqrt{\frac{\hat{\tau}(\gamma_0)}{\rho_0\gamma_0}} \tag{7}$$

and

$$\hat{\mathscr{Z}}(\xi) = \hat{\chi}(\xi) + i\hat{\mathscr{Y}}(\xi) \tag{8}$$

is any smooth function satisfying

$$0 < \frac{d\hat{\chi}}{d\xi}(\xi), \qquad \left|\frac{d\hat{\mathscr{Z}}}{d\xi}\right|(\xi) = \sqrt{\left(\frac{d\hat{\chi}}{d\xi}\right)^2 + \left(\frac{d\hat{\mathscr{Y}}}{d\xi}\right)^2}(\xi) \equiv \gamma_0, \qquad \text{and} \lim_{|\xi|\to\infty} \hat{\mathscr{Y}}(\xi) = 0. \tag{9}$$

Such solutions represent traveling waves moving to the right (respectively, the left) in a strained medium which is at rest ahead of the wave. The interaction problem is generated by superposing two such traveling waves. Specifically, if we let $\hat{\mathscr{Y}}_l(\cdot)$ and $\hat{\mathscr{Y}}_r(\cdot)$ be two smooth functions satisfying

$$\left.\begin{array}{l} \text{support of } \hat{\mathscr{Y}}_l(\cdot) = (a,0), \quad -\infty < a < 0, \\ \text{support of } \hat{\mathscr{Y}}_r(\cdot) = (0,b), \quad 0 < b < \infty, \\ 0 < \left|\dfrac{d\hat{\mathscr{Y}}_l}{d\xi}\right|(\xi) < \gamma_0, \quad \text{and} \quad 0 < \left|\dfrac{d\hat{\mathscr{Y}}_r}{d\xi}\right|(\xi) < \gamma_0, \end{array}\right\} \tag{10}$$

and define

$$\hat{\mathscr{Y}}_*(\xi) = \begin{cases} \hat{\mathscr{Y}}_l(\xi), & \xi < 0, \\ \hat{\mathscr{Y}}_r(\xi), & \xi > 0, \end{cases} \tag{11}$$

$$\hat{\chi}_*(\xi) = \gamma_0\xi + \int_{-\infty}^{\xi}\left(\sqrt{\gamma_0^2 - \left(\frac{d\hat{\mathscr{Y}}_*}{dr}\right)^2}(r) - \gamma_0\right) dr, \tag{12}$$

and

$$\hat{\mathscr{Z}}_*(\xi) = \hat{\chi}_*(\xi) + i\hat{\mathscr{Y}}_*(\xi), \tag{13}$$

then it is easily checked that the incident wave function

$$Z_{\text{inc}}(x,t) \stackrel{\text{def}}{=} \begin{cases} \hat{\mathscr{Z}}_*(x - c_0t) + \gamma_0c_0t, & x < c_0t, \\ \hat{\mathscr{Z}}_*(0) + \gamma_0x, & c_0t < x < -c_0t, \\ \hat{\mathscr{Z}}_*(x + c_0t) - \gamma_0c_0t, & -c_0t < x, \end{cases} \tag{14}$$

with

$$c_0 = \sqrt{\frac{\hat{\tau}(\gamma_0)}{\rho_0\gamma_0}} \tag{15}$$

is a solution to (3) and (5) for all $t \leqslant 0$ and represents two traveling waves advancing on one another which collide at $x = 0$ and at time $t = 0$.

The problem we shall study is the continuation of Z_{inc} to the upper half plane $t \geqslant 0$. The solutions we obtain are approximate and are based on the assumption that the *shear wave speed* at γ_0, namely the constant c_0 defined in (15), is much smaller than the *longitudinal wave speed* at γ_0, namely the constant

$$c_{\text{long}}(\gamma_0) = \sqrt{\frac{1}{\rho_0}\frac{d\hat{\tau}(\gamma_0)}{d\gamma}}\,.$$

A similar hypothesis was invoked by Carrier [1] and Dickey [2, 3] in their analysis of the vibrations of a finite string.

Since our primary interest is in the behavior of the vertical displacement field $\mathscr{Y}$ and not the detailed structure of the longitudinal field, we shall *assume that* $\hat{\tau}(\cdot)$ *behaves linearly near* γ_0. This assumption guarantees that longitudinal shock waves are not generated spontaneously. It also guarantees that the incident wave field Z_{inc} defined in (14) represents the solution to the continuation problem in the region $|x| \geqslant c_{\text{long}}(\gamma_0)t$ with $t \geqslant 0$.

The organization of the remainder of this note is as follows. We shall conclude this section with a derivation of the approximate equation for the vertical component of the motion, $\hat{\mathscr{Y}}(x,t)$, which is valid in the region $|x| \leqslant c_{\text{long}}t$, $t \geqslant 0$, and with a statement of our principal results for the approximating equation (WE). These results consist of a priori and decay estimates for solutions of the approximating equation (WE). Section 2 is devoted to proving these estimates.

Derivation of Approximate Equation

There is no loss in generality to take the density ρ_0, stretch γ_0, and the tension $\hat{\tau}(\gamma_0)$ all equal to unity.[2] This, of course, yields $c_0 = 1$. With this normalization and our previous hypothesis that $\hat{\tau}$ behaves linearly near the stretch $\gamma = 1$, we have

$$\tau(\gamma) = 1 + \frac{(\gamma - 1)}{\varepsilon^2}, \tag{16}$$

and the *longitudinal wave speed* c_{long} is given by

$$c_{\text{long}} = 1/\varepsilon. \tag{17}$$

Moreover, the hypothesis that $1 = c_0 \ll c_{\text{long}}$ reduces to the assumption that $0 < \varepsilon \ll 1$. We shall restrict our attention to small-amplitude motions of the form

$$Z(x,t) = \big(x + \varepsilon^2 u(x,t;\,\varepsilon)\big) + i\varepsilon w(x,t;\,\varepsilon), \tag{18}$$

where u and w have an asymptotic development in ε, and shall content ourselves with determining the zeroth-order terms in these expansions. In the sequel we shall adopt the notation

$$Z_0(x,t) = \big(x + \varepsilon^2 u_0(x,t)\big) + i\varepsilon w_0(x,t), \tag{19}$$

and by this we mean that

$$\text{Re}(Z - Z_0) = O(\varepsilon^4) \quad \text{and} \quad \text{Im}(Z - Z_0) = O(\varepsilon^3). \tag{20}$$

We start with an expansion of the functions $\hat{\chi}_*$ and $\hat{\mathscr{Y}}_0$ defined in (10)–(13). The basic *ansatz* (19) implies that

$$\hat{\mathscr{Y}}_*(\xi) = \varepsilon w_*(\xi), \tag{21}$$

[2] If this hypothesis is not met, then with a simple renormalization it is always achievable.

where

$$w_*(\xi) = \begin{cases} w_l(\xi), & \xi < 0, \\ w_r(\xi), & \xi > 0, \end{cases} \tag{22}$$

and $w_l(\cdot)$ and $w_r(\cdot)$ are smooth functions satisfying

$$\left.\begin{array}{ll} \text{support of } w_l(\xi) = (a,0), & -\infty < a < 0, \\ \text{support of } w_r(\xi) = (0,b), & 0 < b < \infty. \end{array}\right\} \tag{23}$$

To within order ε^4 the function $\hat{\chi}_*$ is given by

$$\hat{\chi}_{*,0}(\xi) = \xi - \frac{\varepsilon^2}{2}\int_{-\infty}^{\xi} \left(\frac{dw_*}{dr}\right)^2(r)\,dr, \tag{24}$$

and thus

$$\hat{\mathscr{Z}}_{*,0}(\xi) = \left(\xi - \frac{\varepsilon^2}{2}\int_{-\infty}^{\xi} \left(\frac{dw_*}{dr}\right)^2(r)\,dr\right) + i\varepsilon w_*(\xi). \tag{25}$$

Equations (14) and (25) in turn yield the following expansion for the incident wave field:

$$(Z_{\text{inc}})_0(x,t) = \begin{cases} x - \dfrac{\varepsilon^2}{2}\displaystyle\int_{-\infty}^{x-t} \left(\frac{dw_*}{dr}\right)^2(r)\,dr + i\varepsilon w_*(x-t), & x < t, \\ x - \dfrac{\varepsilon^2}{2}\displaystyle\int_{-\infty}^{0} \left(\frac{dw_*}{dr}\right)^2(r)\,dr, & t < x < -t, \\ x - \dfrac{\varepsilon^2}{2}\displaystyle\int_{-\infty}^{x+t} \left(\frac{dw_*}{dr}\right)^2(r)\,dr + i\varepsilon w_*(x+t), & -t < x, \end{cases} \tag{26}$$

and this is a valid asymptotic representation of the incident field in the lower half space $t < 0$ and in the region $|x| \geqslant t/\varepsilon$ when $t \geqslant 0$.

It should also be noted that if we write the incident wave field of (26) as

$$(Z_{\text{inc}})_0 = (x + \varepsilon^2 u_{\text{inc},0}) + i\varepsilon w_{\text{inc},0}, \tag{27}$$

then the pair $(u_{\text{inc},0}, w_{\text{inc},0})$ satisfies

$$\left.\begin{array}{l} \dfrac{\partial u_{\text{inc},0}}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w_{\text{inc},0}}{\partial x}\right)^2 = 0, \\ \dfrac{\partial^2 w_{\text{inc},0}}{\partial t^2} - \dfrac{\partial^2 w_{\text{inc},0}}{\partial x^2} = 0, \end{array}\right\} \tag{28}$$

in both $t \leqslant 0$ and in the region $|x| \geqslant t/\varepsilon$, $t \geqslant 0$, and the initial conditions

$$w_{\text{inc},0}(x,0) = w_*(x) \overset{\text{def}}{=} \begin{cases} w_l(x), & x < 0, \\ w_r(x), & x > 0, \end{cases} \tag{29}$$

and

$$\frac{\partial w_{\text{inc},0}(x,0)}{\partial t} = \begin{cases} -\dfrac{dw_l(x)}{dx}, & x < 0, \\ \dfrac{dw_r(x)}{dx}, & x > 0, \end{cases} \tag{30}$$

where again $w_l(\cdot)$ and $w_r(\cdot)$ satisfy (23).

Our remaining task is to obtain evolution equations for the functions u_0 and w_0 defined in (19) in the region $|x| \leqslant t/\varepsilon$ when $t \geqslant 0$. The ansatz (19) when combined with (3), (5), (16), and the identity

$$\gamma = 1 + \varepsilon^2\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) + O(\varepsilon^4) \tag{31}$$

yields the following system of partial differential equations for u_0 and w_0:

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) = 0,\\ &\frac{\partial^2 w_0}{\partial t^2} - \frac{\partial}{\partial x}\left(\left(1 + \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)\frac{\partial w_0}{\partial x}\right) = 0 \end{aligned}\right\} \quad -\frac{t}{\varepsilon} < x < \frac{t}{\varepsilon} \text{ and } t \geqslant 0. \tag{32}$$

These equations are supplemented with the following compatibility conditions across the curves $x = \mp t/\varepsilon$, $t \geqslant 0$:

$$u_0(\mp t/\varepsilon, t) = u_{\text{inc},0}(\mp t/\varepsilon, t), \quad t \geqslant 0, \tag{33}$$

$$w_0(\mp t/\varepsilon, t) = w_{\text{inc},0}(\mp t/\varepsilon, t), \quad t \geqslant 0, \tag{34}$$

and

$$\mp\frac{1}{\varepsilon}\frac{\partial w_0}{\partial t}(\mp t/\varepsilon, t) + \left(1 + \left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)(\mp t/\varepsilon, t)\right)\frac{\partial w_0}{\partial x}(\mp t/\varepsilon, t)$$
$$= \mp\frac{1}{\varepsilon}\frac{\partial w_{\text{inc},0}}{\partial t}(\mp t/\varepsilon, t) + \frac{\partial w_{\text{inc},0}}{\partial x}(\mp t/\varepsilon, t). \tag{35}$$

Equations (33) and (34) reflect the fact that the incident and outgoing waves must agree on the curves $x = \mp t/\varepsilon$, $t \geqslant 0$, while (35) follows from the Rankine–Hugoniot conditions for the original system (3) and (5) when $\hat{\tau}$ is given by (16) and guarantees that to within terms of $O(\varepsilon^2)$ vertical momentum is conserved across the curves $x = \mp t/\varepsilon$, $t \geqslant 0$.

The longitudinal momentum equation $(32)_1$ implies that

$$\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 = a(t), \quad -\frac{t}{\varepsilon} < x < \frac{t}{\varepsilon}, \tag{36}$$

and this, when combined with (33), yields

$$a(t) = \frac{\varepsilon}{4t}\left[\int_{-t/\varepsilon}^{t/\varepsilon}\left(\frac{\partial w_0}{\partial r}\right)^2(r,t)\,dr - \int_{-(1+1/\varepsilon)t}^{(1+1/\varepsilon)t}\left(\frac{d\hat{w}_*}{dr}\right)^2(r)\,dr\right]. \tag{37}$$

Combining this last result with (32)–(35), we arrive at the following problem for the vertical displacement w_0:

$$\frac{\partial^2 w_0}{\partial t^2} - c^2(t)\frac{\partial^2 w_0}{\partial x^2} = 0, \quad -\frac{t}{\varepsilon} < x < \frac{t}{\varepsilon}, \; t \geqslant 0, \tag{WE}$$

$$c^2(t) = 1 + \frac{\varepsilon}{4t}\left[\int_{-t/\varepsilon}^{t/\varepsilon}\left(\frac{\partial w_0}{\partial r}\right)^2(r,t)\,dr - \int_{-(1+1/\varepsilon)t}^{(1+1/\varepsilon)t}\left(\frac{dw_*}{dr}\right)^2(r)\,dr\right], \tag{C}$$

$$\left.\begin{aligned} w_0(\mp t/\varepsilon, t) &= w_*(\mp(1 + 1/\varepsilon)t),\\ \mp\frac{1}{\varepsilon}\frac{\partial w_0}{\partial t}(\mp t/\varepsilon, t) + c^2(t)\frac{\partial w_0}{\partial x}(\mp t/\varepsilon, t) &= \frac{(1+\varepsilon)}{\varepsilon}\frac{dw_*}{d\xi}(\mp(1 + 1/\varepsilon)t), \end{aligned}\right\} \tag{BC}$$

where again

$$w_*(\xi) = \begin{cases} w_l(\xi), & \xi < 0, \\ w_r(\xi), & \xi > 0. \end{cases} \tag{38}$$

It should be noted that if one of the two functions $w_l(\cdot)$ or $w_r(\cdot)$ is identically zero, then the resulting solution to (WE), (C), and (BC) is simply the input traveling wave and $c^2(t) \equiv 1$. Thus the system (WE), (C), and (BC) is consistent with the original one, namely (3), (5), and (16).

Our principal results consist of a priori estimates for the solution of (WE), (C), and (BC). The most basic of these is a uniform bound for the energy:

$$\mathscr{E}(t) \overset{\text{def}}{=} \int_{-t/\varepsilon}^{t/\varepsilon} \left[\left(\frac{\partial w_0}{\partial t} \right)^2 + \left(\frac{\partial w_0}{\partial x} \right)^2 \right](x,t)\, dx \leqslant \mathscr{E}_0, \tag{39}$$

where $\mathscr{E}_0$ is a constant depending only on the data $w_l(\cdot)$ and $w_r(\cdot)$. This estimate implies that as t tends to infinity the sound speed $c(t)$ satisfies

$$|c(t) - 1| \leqslant \varepsilon \mathscr{E}_1 / t, \tag{40}$$

and $\mathscr{E}_1$ is a constant depending on $w_l(\cdot)$ and $w_r(\cdot)$. If the data $w_l(\cdot)$ and $w_r(\cdot)$ are smooth [recall that (23) guarantees they each have compact support], then not only is (40) valid but

$$|\mathring{c}(t)| \leqslant \varepsilon \mathscr{E}_2 / t^2, \quad t \to \infty, \tag{41}$$

and $\mathscr{E}_2$ depends on $w_l(\cdot)$ and $w_r(\cdot)$. To obtain more detailed information it is convenient to introduce the new timelike or phase variable φ defined as an appropriately normalized solution of

$$\frac{d\varphi}{dt} = c(t) \tag{42}$$

and to regard w_0 as a function of x and φ, that is, to let

$$\delta(x, \varphi) = w_0(x, T(\varphi)), \tag{43}$$

where $\varphi \to T(\varphi)$ is the inverse of $t \to \varphi(t)$. Our principal results for δ involve the existence of functions $\delta_r(\cdot)$ and $\delta_l(\cdot)$ such that the following limiting relations obtain:

$$\lim_{\varphi \to \infty} (\delta(x + \varphi, \varphi), \delta(x - \varphi, \varphi)) = (\delta_r(x), \delta_l(x)), \tag{44}$$

$$\lim_{\varphi \to \infty} \int_{-T(\varphi)/\varepsilon}^{T(\varphi)/\varepsilon} \left(\frac{\partial \delta}{\partial \varphi} \right)^2 (x, \varphi)\, dx = \int_{-\infty}^{\infty} \left(\left(\frac{d\delta_l}{dx} \right)^2 + \left(\frac{d\delta_r}{dx} \right)^2 \right)(x)\, dx, \tag{45}$$

and

$$\lim_{\phi \to \infty} \int_{-T(\varphi)/\varepsilon}^{T(\varphi)/\varepsilon} \left(\frac{\partial \delta}{\partial x} \right)^2 (x, \varphi)\, dx = \int_{-\infty}^{\infty} \left(\left(\frac{d\delta_l}{dx} \right)^2 + \left(\frac{d\delta_r}{dx} \right)^2 \right)(x)\, dx. \tag{46}$$

Equations (45) and (46) assert that the energy equipartions as φ tends to infinity.

2. A priori estimates. In view of our remarks in the introduction it suffices to focus on the function w_0 defined by

$$\frac{\partial^2 w_0}{\partial t^2} - c^2(t)\frac{\partial^2 w_0}{\partial x^2} = 0, \quad -\frac{t}{\varepsilon} < x < \frac{t}{\varepsilon} \text{ and } t \geqslant 0, \tag{1}$$

$$c^2(t) = 1 + \frac{\varepsilon}{4t}\left[\int_{-t/\varepsilon}^{t/\varepsilon}\left(\frac{\partial w_0}{\partial x}\right)^2(x,t)\,dx - \int_{-(1+\varepsilon)t/\varepsilon}^{(1+\varepsilon)t/\varepsilon}\left(\frac{dw_*}{dx}\right)^2(x)\,dx\right], \tag{2}$$

$$\left.\begin{aligned} w_0(\mp t/\varepsilon, t) &= w_*\left(\mp\frac{(1+\varepsilon)t}{\varepsilon}\right),\\ \mp\frac{1}{\varepsilon}\frac{\partial w_0}{\partial t}(\mp t/\varepsilon, t) + c^2(t)\frac{\partial w_0}{\partial x}(\mp t/\varepsilon, t) &= \frac{(1+\varepsilon)}{\varepsilon}\frac{dw_*}{d\xi}\left(\mp\frac{(1+\varepsilon)t}{\varepsilon}\right),\end{aligned}\right\} \tag{3}$$

where again w_* is given by (1.22) and (1.23) and satisfies $w_*(0) = 0$.

Our prime concern is the long-time behavior of w_0. Some important facts about the short-time behavior are summarized below. The first of these are the identities

$$\left.\begin{aligned} \frac{d}{dt}\int_{-t/\varepsilon}^{t/\varepsilon}\frac{\partial w_0}{\partial t}(x,t)\,dx &= \frac{(1+\varepsilon)}{\varepsilon}\left(\frac{dw_*}{d\xi}\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)t\right) - \frac{dw_*}{d\xi}\left(-\left(\frac{1+\varepsilon}{\varepsilon}\right)t\right)\right),\\ \int_{-t/\varepsilon}^{t/\varepsilon}\frac{\partial w_0}{\partial t}(x,t)\,dx &= w_0\left(\left(\frac{1+\varepsilon}{\varepsilon}\right)t\right) + w_*\left(-\left(\frac{1+\varepsilon}{\varepsilon}\right)t\right).\end{aligned}\right\} \tag{4}$$

The identities (1.22) and (1.23) together with (3) and (4) then guarantee that for times $t \geqslant t(a,b,\varepsilon) = \max[|a|\varepsilon/(1+\varepsilon), b\varepsilon/(1+\varepsilon)] < 1$,

$$w_0(\mp t/\varepsilon, t) = \frac{\partial w_0}{\partial t}(\mp t/\varepsilon, t) = \frac{\partial w_0}{\partial x}(\mp t/\varepsilon, t) = \int_{-t/\varepsilon}^{t/\varepsilon}\frac{\partial w_0}{\partial t}(x,t)\,dx = 0, \tag{5}$$

and that w_0 has compact support in the region

$$|x| \leqslant \frac{t(a,b,\varepsilon)}{\varepsilon} + \int_{t(a,b,\varepsilon)}^{t} c(s)\,ds, \quad t \geqslant t(a,b,\varepsilon).$$

On the strip $0 \leqslant t \leqslant 1$, w_0 is endowed with the same regularity properties as the data w_* and there exists an order one constant K_1 such that

$$\begin{aligned} |w_0|_N &= \sup_{\substack{-\infty < x < \infty\\ 0 \leqslant t \leqslant 1}} \sum_{m+n \leqslant N} |\partial_x^n \partial_t^m w_0(x,t)|\\ &\leqslant K_1 \sup_{-\infty < x < \infty} \sum_{n=0}^{N}\left|\frac{d^n w_*(x)}{dx^n}\right| \stackrel{\text{def}}{=} K_1 |w_*|_N,\end{aligned} \tag{6}$$

and

$$\max\left[\sup_{0 \leqslant t \leqslant 1}|c^2(t) - 1|, \sup_{0 \leqslant t \leqslant 1}\left|\frac{dc(t)}{dt}\right|\right] \leqslant K_1 \varepsilon \max[|a|, b]|w_0|_0^2. \tag{7}$$

For times $t \geqslant 1$, w_0 is continued as the solution of

$$\frac{\partial^2 w_0}{\partial t^2} - c^2(t)\frac{\partial^2 w_0}{\partial x^2} = 0, \quad -\infty < x < \infty \text{ and } t \geqslant 1, \tag{WE}$$

$$c^2(t) = 1 + \frac{\varepsilon}{4t}\left[\int_{-\infty}^{\infty}\left(\left(\frac{\partial w_0}{\partial x}\right)^2(x,t) - \left(\frac{dw_*}{dx}\right)^2(x)\right)dx\right], \tag{C}$$

$$\lim_{t\to 1^+}\left(w_0(x,t), \frac{\partial w_0}{\partial t}(x,t)\right) = (\varphi(x), \Psi(x)), \quad -\infty < x < \infty, \tag{IC$_1$}$$

where of course $\lim_{t\to 1^-}(w_0(x,t), (\partial w_0/\partial t)(x,t)) \overset{\text{def}}{=} (\varphi(x), \Psi(x))$, (φ, Ψ) has compact support in $|x| < t(a,b,\varepsilon)/\varepsilon + \int_{t(a,b,\varepsilon)}^{1} c(s)\,dx = O(1)$ and $\int_{-\infty}^{\infty}\Psi(x)\,dx = 0$. To obtain our desired estimates we shall have to constrain the size of (φ, Ψ). The basic inequality (6) guarantees that any such constraint can be realized by constraining the original data

$$w_*(x) = \begin{cases} w_l(x), & x < 0, \\ w_r(x), & x > 0. \end{cases}$$

Although some of our results are obtainable from equations (WE), (C), and (IC)$_1$ directly, we find it convenient to operate with the Fourier transform of the solution. We let

$$\hat{w}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} w_0(x,t)\,dx \tag{8}$$

and note that

$$\bar{\hat{w}}(k,t) = \hat{w}(-k,t) \quad \text{and} \quad w_0(x,t) = \frac{1}{2\pi}\lim_{K\to\infty}\int_{-K}^{K} e^{ikx} w(k,t)\,dk. \tag{9}$$

The evolution equation for $\hat{w}$ is obtained by multiplying (WE) by e^{-ikx} and integrating the resulting expression from $x = -\infty$ to $x = +\infty$. The result is

$$\frac{d^2\hat{w}(k,t)}{dt^2} + k^2c^2(t)\hat{w}(k,t) = 0, \quad t \geqslant 1. \tag{WEFT}$$

Parseval's identity applied to (C) also yields

$$c^2(t) = 1 + \frac{\varepsilon}{8\pi t}\int_{-\infty}^{\infty} k^2(\hat{w}(k,t)\hat{w}(-k,t) - \hat{w}_*(k)\hat{w}_*(-k))\,dk. \tag{CFT}$$

Our first estimate bounds

$$\mathscr{E}_0(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty}\left(\frac{d\hat{w}(k,t)}{dt}\,\frac{d\hat{w}(-k,t)}{dt} + k^2\hat{w}(k,t)\hat{w}(-k,t)\right)dk. \tag{10}$$

THEOREM 1. The following identity holds on $t \geqslant 1$:

$$\mathscr{E}_0(t) + \frac{4\pi t}{\varepsilon}(c^2(t) - 1)^2 + \frac{4\pi}{\varepsilon}\int_1^t (c^2(s) - 1)^2\,ds = \mathscr{E}_0(1) + \frac{4\pi}{\varepsilon}(c^2(1) - 1)^2, \tag{11}$$

where $c^2(t)$ is given by (CFT), $\mathscr{E}_0(t)$ by (10).

Proof. If we multiply (WEFT) by $d\hat{w}(-k,t)/dt$, the conjugate equation by $d\hat{w}(k,t)/dt$, and add and integrate the resulting expression with respect to k we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left[\frac{d\hat{w}(k,t)}{dt}\,\frac{d\hat{w}(-k,t)}{dt} + k^2\hat{w}(k,t)\hat{w}(-k,t)\right]dk =$$

$$-(c^2(t) - 1)\int_{-\infty}^{\infty} k^2\left[\frac{d\hat{w}}{dt}(-k,t)\hat{w}(k,t) + \frac{d\hat{w}}{dt}(k,t)\hat{w}(-k,t)\right]dk. \tag{12}$$

But equation (CFT) implies that

$$\frac{8\pi}{\varepsilon}\frac{d}{dt}\big(t(c^2(t)-1)\big)=\int_{-\infty}^{\infty}k^2\left[\frac{d\hat{w}}{dt}(-k,t)\hat{w}(k,t)+\frac{d\hat{w}}{dt}(k,t)\hat{w}(-k,t)\right]dk, \tag{13}$$

and this, when combined with (12), yields

$$\frac{d}{dt}\left\{\int_{-\infty}^{\infty}\left[\frac{d\hat{w}(k,t)}{dt}\,\frac{d\hat{w}(-k,t)}{dt}+k^2\hat{w}(k,t)\hat{w}(-k,t)\right]dk+\frac{4\pi t}{\varepsilon}(c^2-1)^2\right\}$$

$$=-\frac{4\pi}{\varepsilon}(c^2-1)^2. \tag{14}$$

The theorem now follows from (14). □

A direct consequence of Theorem 1, (CFT), (6), and (7) is

COROLLARY 1. The following relations obtain:

$$\mathscr{E}_0(t)\leqslant 12\pi\max[|a|,b]|w_0|_1^2+4\pi K_1^2\varepsilon(\max[|a|,b])^2|w_0|_2^4, \tag{15}$$

$$\frac{\varepsilon}{8\pi t}\int_{-\infty}^{\infty}k^2\hat{w}_*(k)\hat{w}_*(-k)\,dk\leqslant c^2(t)-1\leqslant\frac{\varepsilon\mathscr{E}_0(t)}{4\pi t}, \tag{16}$$

and

$$\lim_{t\to\infty}\mathscr{E}_0(t)=\mathscr{E}_0(1)+\frac{4\pi}{\varepsilon}(c^2(1)-1)^2-\frac{4\pi}{\varepsilon}\int_1^{\infty}(c^2(s)-1)^2\,ds. \tag{17}$$

To proceed it is convenient to introduce the following change of independent and dependent variables. We introduce the phase φ by

$$\varphi(t)=\int_1^t c(s)\,ds, \tag{18}$$

let $t=T(\varphi)$ denote the inverse $t\to\varphi(t)$, and define

$$\mathbb{C}(\varphi)=c(T(\varphi)) \tag{19}$$

and

$$\hat{\delta}(k,\varphi)=\hat{w}(k,T(\varphi)). \tag{20}$$

The fact that $\hat{w}(k,t)$ satisfies (WEFT) on $t\geqslant 1$ guarantees that $\hat{\delta}(k,\varphi)$ satisfies

$$\frac{d^2\hat{\delta}}{d\varphi^2}(k,\varphi)+k^2\hat{\delta}(k,\varphi)=-\frac{1}{\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}\frac{d\hat{\delta}}{d\varphi}(k,\varphi) \tag{21}$$

on $\varphi\geqslant 0$. Moreover, $\mathbb{C}^2(\varphi)$ is given by

$$\mathbb{C}^2(\varphi)=1+\frac{\varepsilon}{8\pi T(\varphi)}\int_{-\infty}^{\infty}k^2[\hat{\delta}(k,\varphi)\hat{\delta}(-k,\varphi)-\hat{w}_*(k)\hat{w}_*(-k)]\,dk. \tag{22}$$

The analysis of (21) and (22) is facilitated via the introduction of functions $\hat{A}(k,\varphi)$ and $\hat{B}(k,\varphi)$ defined by

$$\hat{A}(k,\varphi)=\mathbb{C}^{1/2}(\varphi)e^{ik\varphi}\left(\frac{d\hat{\delta}}{d\varphi}(k,\varphi)-ik\hat{\delta}(k,\varphi)\right) \tag{23}$$

and

$$\hat{B}(k,\varphi)=\mathbb{C}^{1/2}(\varphi)e^{-ik\varphi}\left(\frac{d\hat{\delta}}{d\varphi}(k,\varphi)+ik\hat{\delta}(k,\varphi)\right). \tag{24}$$

The fact that $\hat{\delta}(k,\varphi)$ satisfies (21) implies that

$$\frac{d\hat{A}}{d\varphi}(k,\varphi) = -\frac{1}{2\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}e^{2ik\varphi}\hat{B}(k,\varphi) \tag{25}$$

and

$$\frac{d\hat{B}}{d\varphi}(k,\varphi) = -\frac{1}{2\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}e^{-2ik\varphi}\hat{A}(k,\varphi), \tag{26}$$

while (22)–(26) imply that

$$\begin{aligned}\mathbb{C}^2(\varphi) - 1 = {} & \frac{\varepsilon}{32\pi\mathbb{C}(\varphi)T(\varphi)}\int_{-\infty}^{\infty}\left[\hat{A}(k,\varphi)\hat{A}(-k,\varphi) + \hat{B}(k,\varphi)\hat{B}(-k,\varphi)\right]dk \\ & - \frac{\varepsilon}{16\pi\mathbb{C}(\varphi)T(\varphi)}\int_{-\infty}^{\infty}\left[e^{-2ik\varphi}\hat{A}(k,\varphi)\hat{B}(-k,\varphi) + e^{2ik\varphi}\hat{A}(-k,\varphi)\hat{B}(k,\varphi)\right]dk \\ & - \frac{\varepsilon}{8\pi T(\varphi)}\int_{-\infty}^{\infty}k^2\hat{w}_*(k)\hat{w}_*(-k)\,dk,\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\frac{d}{d\varphi}\left(T(\varphi)(\mathbb{C}^2(\varphi) - 1)\right) \\ &\quad = \frac{i\varepsilon}{16\pi\mathbb{C}(\varphi)}\int_{-\infty}^{\infty}k\left[\hat{A}(k,\varphi)\hat{B}(-k,\varphi)e^{-2ik\varphi} - \hat{A}(-k,\varphi)\hat{B}(k,\varphi)e^{2ik\varphi}\right]dk.\end{aligned} \tag{28}$$

If we now let

$$E_0(k,\varphi) \stackrel{\text{def}}{=} \hat{A}(k,\varphi)\hat{A}(-k,\varphi) + \hat{B}(k,\varphi)\hat{B}(-k,\varphi), \tag{29}$$

$$E_1(k,\varphi) \stackrel{\text{def}}{=} \hat{A}(k,\varphi)\hat{B}(-k,\varphi), \tag{30}$$

and

$$q(\varphi) \stackrel{\text{def}}{=} \frac{1}{\mathbb{C}(\varphi)}\frac{d\mathbb{C}(\varphi)}{d\varphi}, \tag{31}$$

then it follows directly from (25)–(28) and the identity

$$-\int_{-\infty}^{\infty}kE_1(-k,\varphi)e^{2ik\varphi}\,dk = \int_{-\infty}^{\infty}kE_1(k,\varphi)e^{-2ik\varphi}\,dk \tag{32}$$

that these functions satisfy

$$\frac{dE_0(k,\varphi)}{d\varphi} = -\frac{q(\varphi)}{2}\left(e^{-2ik\varphi}E_1(k,\varphi) + e^{2ik\varphi}E_1(-k,\varphi)\right), \tag{33}$$

$$\frac{dE_1}{d\varphi}(k,\varphi) = -\frac{q(\varphi)}{2}\left(e^{2ik\varphi}E_0(k,\varphi)\right), \tag{34}$$

and

$$q(\varphi) = \frac{i\varepsilon}{16\pi\mathbb{C}^3(\varphi)T(\varphi)}\int_{-\infty}^{\infty}ke^{-2ik\varphi}E_1(k,\varphi)\,dk - \frac{(\mathbb{C}^2(\varphi) - 1)}{2\mathbb{C}^3(\varphi)T(\varphi)}, \tag{35}$$

and these in turn imply that for integers $p = 0$ and 1 the functions

$$\left.\begin{aligned} H_{0,p}(\varphi, s) &\overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-2ik\varphi} k^p E_0(k,s)\,dk, \quad -\infty < \varphi < \infty \text{ and } s \geqslant 0, \\ H_{1,p}(\varphi) &\overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-2ik\varphi} k^p E_1(k,\varphi)\,dk, \quad \varphi \geqslant 0, \end{aligned}\right\} \tag{36}$$

satisfy

$$H_{0,0}(-\varphi, s) = H_{0,0}(\varphi, s) \quad \text{and} \quad H_{0,1}(-\varphi, s) = -H_{0,1}(\varphi, s), \tag{37}$$

$$H_{1,p}(\varphi) = \int_{-\infty}^{\infty} e^{-2ik\varphi} k^p E_1(k,0)\,dk - \frac{1}{2}\int_0^{\varphi} q(\eta) H_{0,p}(\varphi - \eta, \eta)\,d\eta, \tag{38}$$

and

$$\begin{aligned} H_{0,p}(\varphi, s) = \int_{-\infty}^{\infty} & e^{-2ik\varphi} k^p E_0(k,0)\,dk \\ & - \frac{1}{2}\int_0^s q(\eta) \int_{-\infty}^{\infty} k^p \big(e^{-2ik(\varphi+\eta)} E_1(k,0) + e^{-2ik(\varphi-\eta)} E_1(-k,0)\big)\,dk\,d\eta \\ & + \frac{1}{4}\int_0^s q(\eta) \int_0^{\eta} q(r) \big(H_{0,p}(\varphi + \eta - r, r) + H_{0,p}(\varphi + r - \eta, r)\big)\,dr\,d\eta, \end{aligned} \tag{39}$$

while

$$q(\varphi) = \frac{i\varepsilon H_{1,1}(\varphi)}{16\pi \mathbb{C}^3(\varphi) T(\varphi)} - \frac{\big(\mathbb{C}^2(\varphi) - 1\big)}{2\mathbb{C}^3(\varphi) T(\varphi)}. \tag{40}$$

It should be noted that the system of equations $(38)_{p=1}$, $(39)_{p=1}$, and (40), together with

$$\left.\begin{aligned} \frac{d\mathbb{C}}{d\varphi} &= q(\varphi)\mathbb{C}(\varphi) \\ \mathbb{C}(0) &= c(t=1) \end{aligned}\right\} \tag{41}$$

and

$$T(\varphi) = 1 + \int_0^{\varphi} \frac{ds}{\mathbb{C}(s)} \tag{42}$$

represent a closed system for $H_{1,1}(\cdot)$, $H_{0,1}(\cdot,\cdot)$, $q(\cdot)$, $\mathbb{C}(\cdot)$, and $T(\cdot)$. Moreover, the properties of $E_0(k,0)$ and $E_1(k,0)$ guarantee that $H_{1,1}(\cdot)$ and $H_{0,1}(\cdot,\cdot)$ are pure imaginary and thus that $q(\cdot)$, $\mathbb{C}(\cdot)$, and $T(\cdot)$ are real valued. Our principal results are estimates for this system. The first is summarized in

LEMMA 1. (a) Suppose $k \to k^p E_n(\pm k, 0)$ is smooth and satisfies

$$\lim_{|k|\to\infty} \frac{d^m}{dk^m}\big(k^p E_n(\pm k, 0)\big) = 0 \tag{43}$$

and

$$\int_{-\infty}^{\infty} \left| \frac{d^m}{dk^m}\big(k^p E_n(\pm k, 0)\big) \right| dk = Q_{n,p,m} < \infty \tag{44}$$

for indices $n = 0,1$; $p = 0,1$; and $m = 0,1,2$. Then,

$$\sup_{\varphi \geqslant 0}(1+\varphi^2)\left|\int_{-\infty}^{\infty} e^{-2ik\varphi}k^pE_n(\pm - k,0)\,dk\right| \leqslant Q_{n,p,0} + \frac{1}{4}Q_{n,p,2}. \tag{45}$$

(b) Suppose in addition to the hypotheses of part (a) the function $\varphi \to q(\varphi)$ satisfies

$$\sup_{\varphi \geqslant 0}(1+\varphi^2)|q(\varphi)| \overset{\text{def}}{=} q_2 < \infty. \tag{46}$$

Then

$$\sup_{\substack{\varphi \geqslant 0\\ s\geqslant 0}}(1+\varphi^2)\left|\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2ik(\varphi\pm\eta)}k^pE_n(\pm k,0)\,dk\,d\eta\right| \leqslant \frac{9\pi}{2}q_2\left(Q_{n,p,0} + \frac{Q_{n,p,2}}{4}\right). \tag{47}$$

Proof. We first observe that

$$\begin{aligned}(1+\varphi^2)\int_{-\infty}^{\infty} e^{-2ik\varphi}k^pE_n(\pm k,0)\,dk &= \int_{-\infty}^{\infty} k^pE_n(\pm k,0)\left[\left(1-\frac{1}{4}\frac{d^2}{dk^2}\right)e^{-2ik\varphi}\right]dk\\ &= \int_{-\infty}^{\infty} e^{-2ik\varphi}\left[\left(1-\frac{1}{4}\frac{d^2}{dk^2}\right)(k^pE_n(\pm k,0))\right]dk.\end{aligned} \tag{48}$$

That (45) is true now follows from the integrability hypotheses (44) and (48). The inequality (47) follows from similar reasoning. Specifically, we have the identities

$$\begin{aligned}&\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2ik(\varphi\pm\eta)}k^pE_n(\pm k,0)\,dk\,d\eta\\ &\quad= \int_0^s \frac{q(\eta)}{1+(\varphi\pm\eta)^2}\int_{-\infty}^{\infty} k^pE_n(\pm k,0)\left[\left(1-\frac{1}{4}\frac{d^2}{dk^2}\right)e^{-2ik(\varphi\pm\eta)}\right]dk\,d\eta\\ &\quad= \int_0^s \frac{q(\eta)}{1+(\varphi\pm\eta)^2}\int_{-\infty}^{\infty} e^{-2ik(\varphi\pm\eta)}\left[\left(1-\frac{1}{4}\frac{d^2}{dk^2}\right)(k^pE_n(\pm k,0))\right]dk\,d\eta,\end{aligned}$$

and these, when combined with (44), yield

$$\left|\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2ik(\varphi\pm\eta)}k^pE_n(\pm k,0)\,dk\,d\eta\right| \leqslant \left|\int_0^s \frac{q(\eta)\,d\eta}{1+(\varphi\pm\eta)^2}\right|(Q_{n,p,0} + Q_{n,p,2}/4). \tag{49}$$

The hypothesis (46) on $q(\cdot)$ then implies that for any $s \geqslant 0$ and $\varphi \geqslant 0$

$$\left|\int_0^s \frac{q(\eta)\,d\eta}{1+(\varphi\pm\eta)^2}\right| \leqslant q_2\int_0^s \frac{d\eta}{(1+\eta^2)(1+(\varphi-\eta)^2)}, \tag{50}$$

and the result now follows from (49), (50), and the fact that

$$\sup_{s\geqslant 0}\int_0^s \frac{d\eta}{(1+\eta^2)(1+(\varphi-\eta)^2)} \leqslant \frac{9\pi}{2(1+\varphi^2)}. \tag{51}$$

LEMMA 2. Suppose the functions $k \to k^p E_n(\pm k, 0)$ and $\varphi \to q(\varphi)$ satisfy the hypotheses of Lemma 1. Then

$$\sup_{\substack{\varphi \geq 0 \\ s \geq 0}} (1+\varphi^2)\left|H_{0,p}(\varphi, s)\right| \leqslant \frac{\left[Q_{0,p,0} + \frac{1}{4}Q_{0,p,2} + (9\pi q_2/2)\left(Q_{1,p,0} + Q_{1,p,2}/4\right)\right]}{1 - 9\pi^2 q_2^2/8} \tag{52}$$

and

$$\sup_{\varphi \geq 0} (1+\varphi^2)\left|H_{1,p}(\varphi)\right| \leqslant \left[Q_{1,p,0} + \frac{1}{4}Q_{1,p,2}\right] + \pi q_2 \frac{\left[Q_{0,p,0} + \frac{1}{4}Q_{0,p,2} + (9\pi q_2/2)\left(Q_{1,p,0} + Q_{1,p,2}/4\right)\right]}{1 - 9\pi^2 q_2^2/8} \tag{53}$$

provided

$$1 - \frac{9\pi^2 q_2^2}{8} > 0. \tag{54}$$

Proof. Our starting point for (52) is the identity (39). An easy consequence of (37), (39), and the inequalities (45) and (47) is that $\bar{h}_{0,p} \overset{\text{def}}{=} \sup_{\varphi \geq 0;\, s \geq 0}(1+\varphi^2)|H_{0,p}(\varphi, s)|$ must satisfy

$$\bar{h}_{0,p} \leqslant \left[Q_{0,p,0} + Q_{0,p,2}/4\right] + \frac{9\pi q_2}{2}\left[Q_{1,p,0} + Q_{1,p,2}/4\right] + \left(q_2^2 \bar{h}_{0,p}/4\right) \times \tag{55}$$

$$\left[\sup_{\substack{\varphi \geq 0 \\ s \geq 0}} (1+\varphi^2)\int_0^s \frac{1}{1+\eta^2}\int_0^\eta \frac{1}{1+r^2}\left(\frac{1}{1+(\varphi+\eta-r)^2} + \frac{1}{1+(\varphi+r-\eta)^2}\right) dr\, d\eta\right].$$

The fact that

$$(1+\varphi^2)\int_0^s \frac{1}{1+\eta^2}\int_0^\eta \frac{dr\, d\eta}{(1+r^2)\left(1+(\varphi+\eta-r)^2\right)} \leqslant \frac{\pi^2}{4}, \quad \varphi \geq 0 \text{ and } s \geq 0,$$

and

$$(1+\varphi^2)\int_0^s \frac{1}{1+\eta^2}\int_0^\eta \frac{dr\, d\eta}{(1+r^2)\left(1+(\varphi+r-\eta)^2\right)} \leqslant \begin{cases} 2\pi^2, & 0 \leqslant s \leqslant \varphi, \\ \dfrac{17\pi^2}{4}, & 0 \leqslant \varphi \leqslant s, \end{cases}$$

when combined with (55) yields

$$\bar{h}_{0,p} \leqslant \left[Q_{0,p,0} + Q_{0,p,2}/4\right] + \frac{9\pi q_2}{2}\left[Q_{1,p,0} + Q_{1,p,2}/4\right] + \frac{9\pi^2 q_2^2}{8}\bar{h}_{0,p}, \tag{56}$$

and this yields (52) provided $1 - 9\pi^2 q_2^2/8 > 0$. The inequality (53) follows directly from (38), (45), (52), and the fact that

$$\left|\int_0^\varphi q(\eta) H_{0,p}(\varphi-\eta, \eta)\, d\eta\right| \leqslant q_2 \bar{h}_{0,p}\int_0^\varphi \frac{d\eta}{(1+\eta^2)\left(1+(\varphi-\eta)^2\right)}$$

$$\leqslant \frac{2\pi q_2 \bar{h}_{0,p}}{(1+\varphi^2)},$$

where again $\bar{h}_{0,p} = \sup_{\varphi \geq 0; s \geq 0}(1+\varphi^2)|H_{0,p}(\varphi, s)|$.

We now turn our attention to the system

$$\frac{d\mathbb{C}}{d\varphi} = q(\varphi)\mathbb{C}(\varphi), \qquad \mathbb{C}(0) = c(t = 1), \tag{57}$$

$$\frac{dT}{d\varphi} = \frac{1}{\mathbb{C}(\varphi)}, \qquad T(0) = 1, \tag{58}$$

$$q(\varphi) = \frac{i\varepsilon h(\varphi)}{16\pi\mathbb{C}^3(\varphi)T(\varphi)} - \frac{\left(\mathbb{C}^2(\varphi) - 1\right)}{2\mathbb{C}^3(\varphi)T(\varphi)}, \tag{59}$$

where $\varphi \to h(\varphi)$ is a smooth, imaginary-valued function satisfying

$$\sup_{\varphi \geqslant 0} (1 + \varphi^2)|h(\varphi)| \stackrel{\text{def}}{=} h_1 < \infty. \tag{60}$$

Our basic results for (57)–(59) are summarized in

LEMMA 3. If $\delta \stackrel{\text{def}}{=} |\mathbb{C}^2(0) - 1| < 2/3$ and $0 < \varepsilon h_1 < 8\delta\sqrt{1 - 3\delta/2}$, then the solutions of (57)–(60) satisfy the following estimates:

$$\sqrt{1 - \frac{3\delta}{2}} < \mathbb{C}(\varphi) < \sqrt{1 + \frac{3\delta}{2}}, \tag{61}$$

$$T(\varphi) \geqslant 1 + \frac{\varphi}{\sqrt{1 + 3\delta/2}}, \tag{62}$$

$$|\mathbb{C}^2(\varphi) - 1| < \frac{3\delta}{2\left(1 + \varphi/\sqrt{1 + 3\delta/2}\right)}, \tag{63}$$

$$q(\varphi) \leqslant \frac{\varepsilon h_1}{16\pi(1 - 3\delta/2)^{3/2}\left(1 + \varphi/\sqrt{1 + d\delta/2}\right)(1 + \varphi^2)} + \frac{3\delta}{4(1 - 3\delta/2)^{3/2}\left(1 + \varphi/\sqrt{1 + 3\delta/2}\right)^2}. \tag{64}$$

Proof. We start with the observation that (57)–(59) is equivalent to

$$\frac{d}{d\varphi}\left(T(\mathbb{C}^2(\varphi) - 1)\right) = \frac{i\varepsilon h(\varphi)}{8\pi\mathbb{C}(\varphi)} \tag{65}$$

and

$$\frac{dT}{d\varphi} = \frac{1}{\mathbb{C}(\varphi)}, \quad T(0) = 1. \tag{66}$$

Integrating (65) yields

$$T(\mathbb{C}^2(\varphi) - 1) = \left(\mathbb{C}^2(0) - 1\right) + \frac{i\varepsilon}{8\pi}\int_0^{\varphi} \frac{h(s)}{\mathbb{C}(s)}\, ds, \tag{67}$$

and this combined with $T(\varphi) \geqslant 1$ yields

$$|\mathbb{C}^2(\varphi) - 1| \leqslant \delta + \frac{\varepsilon h_1}{8\pi c_{\min}} \cdot \frac{\pi}{2}. \tag{68}$$

Moreover, the right-hand side of (68) is bounded from above by $3\delta/2$ provided $\varepsilon h_1 < 8\delta\sqrt{1 - 3\delta/2}$. To obtain the last inequality we have used the fact that $|\mathbb{C}^2(\varphi) - 1| \leqslant 3\delta/2$ iff $\sqrt{1 - 3\delta/2} < \mathbb{C}(\varphi) < \sqrt{1 + 3\delta/2}$. The upper bound (61) now combines with (66) to yield (62), and (62), (67), and $\varepsilon h_1 < 8\delta\sqrt{1 - 3\delta/2}$ combine to yield (63). The inequality (64) follows from (59), (61)–(63), and the hypothesis $\sup(1 + \varphi^2)|h(\varphi)| = h_1 < \infty$.

The results of Lemmas 1–3 together with

$$\left(\mathbb{C}^2(0) - 1\right) = \frac{\varepsilon}{32\pi\mathbb{C}(0)}\left(Q_{0,0,0} - 2H_{1,0}(0)\right) - \frac{\varepsilon}{8\pi}\int_{-\infty}^{\infty} \hat{w}_*(k)\hat{w}_*(-k)\,dk \tag{69}$$

now easily combine to give

THEOREM 2. If

$$\delta_2^2 = \max\left(\max_{\substack{n=0,1\\ p=0,1\\ m=0,1,2}} Q_{n,p,m}, \int_{-\infty}^{\infty} k^2\hat{w}_*(k)\hat{w}_*(-k)\,dk\right) \tag{70}$$

is sufficiently small[3], then $q(\varphi) = (1/\mathbb{C}(\varphi))\,d\mathbb{C}/d\varphi$ satisfies

$$q_2 \overset{\text{def}}{=} \sup_{0 \leqslant \varphi}\left(1 + \varphi^2\right)|q(\varphi)| \leqslant K_2\varepsilon\delta_2^2 \tag{71}$$

for some order one constant K_2.

We now turn to a discussion of the system

$$\frac{d\hat{A}}{d\varphi}(k,\varphi) = -\frac{q(\varphi)}{2}e^{2ik\varphi}\hat{B}(k,\varphi), \tag{25}$$

$$\frac{d\hat{B}}{d\varphi}(k,\varphi) = -\frac{q(\varphi)}{2}e^{-2ik\varphi}\hat{A}(k,\varphi), \tag{26}$$

$$\begin{aligned}\hat{A}(k,0) &= \mathbb{C}^{1/2}(0)\left(\frac{d\hat{\delta}}{d\varphi}(k,0) - ik\hat{\delta}(k,0)\right)\\ &= \frac{1}{\mathbb{C}^{1/2}(0)}\frac{d\hat{w}_0}{dt}(k,1^-) - ik\mathbb{C}^{1/2}(0)\hat{w}_0(k,1^-),\end{aligned} \tag{72}$$

and

$$\begin{aligned}\hat{B}(k,0) &= \mathbb{C}^{1/2}(0)\left(\frac{d\hat{\delta}}{d\varphi}(k,0) + ik\hat{\delta}(k,0)\right)\\ &= \frac{1}{\mathbb{C}^{1/2}(0)}\frac{d\hat{w}_0}{dt}(k,1^-) + ik\mathbb{C}^{1/2}(0)\hat{w}_0(k,1^-),\end{aligned} \tag{73}$$

where $H_{0,p}(\cdot,\cdot)$, $H_{1,p}(\cdot)$, $\mathbb{C}(\cdot)$, $q(\cdot)$, and $T(\cdot)$ are determined by solving the closed system (37)–(42). In the sequel we shall assume that δ_2 is small enough that the system (37)–(42) has a solution satisfying the estimates of Lemmas 1–3 and Theorem 2. We note

[3] This condition may be achieved by taking $|w_*|_4$ small enough.

that this constraint can be achieved if $|w_*|_4$ is small enough. It is not difficult to show that, as defined, $\hat{A}$ and $\hat{B}$ satisfy the following consistency conditions:

$$H_{0,p}(\varphi, s) = \int_{-\infty}^{\infty} e^{-2ik\varphi} k^p (\hat{A}(k,s)\hat{A}(-k,s) + \hat{B}(k,s)\hat{B}(-k,s))\, dk, \quad p = 0,1, \tag{74}$$

$$H_{1,p}(\varphi) = \int_{-\infty}^{\infty} e^{-2ik\varphi} k^p \hat{A}(k,\varphi)\hat{B}(-k,\varphi)\, dk, \quad p = 0,1, \tag{75}$$

and

$$\mathbb{C}^2(\varphi) - 1 = \frac{\varepsilon}{32\pi C(\varphi)T(\varphi)}(H_{0,0}(0,\varphi) - 2H_{1,0}(\varphi)) - \frac{\varepsilon}{8\pi T(\varphi)}\int_{-\infty}^{\infty} k^2 \hat{w}_*(k)\hat{w}_*(-k)\, dk. \tag{76}$$

To obtain additional information about $\hat{A}$ and $\hat{B}$ we note that

$$\begin{pmatrix}\hat{A}\\ \hat{B}\end{pmatrix}(k,\varphi) = \begin{pmatrix}\alpha(k,\varphi), & \beta(-k,\varphi)\\ \beta(k,\varphi), & \alpha(-k,\varphi)\end{pmatrix}\begin{pmatrix}\hat{A}\\ \hat{B}\end{pmatrix}(k,0), \tag{77}$$

where $\alpha(k,\varphi)$ and $\beta(k,\varphi)$ satisfy

$$\left.\begin{aligned}\alpha(k,\varphi) &= 1 - \frac{1}{2}\int_0^{\varphi} e^{2iks} q(s)\beta(k,s)\, ds,\\ \beta(k,\varphi) &= -\frac{1}{2}\int_0^{\varphi} e^{-2iks} q(s)\alpha(k,s)\, ds,\end{aligned}\right\} \tag{78}$$

and

$$\alpha(k,\varphi)\alpha(-k,\varphi) - \beta(k,\varphi)\beta(-k,\varphi) = 1. \tag{79}$$

Moreover, if $1 - q_2^2\pi^2/16 > 0$, α and β satisfy

$$\sup_{\substack{0\leqslant\varphi\\ -\infty<k<\infty}} |\alpha(k,\varphi)| \leqslant \frac{1}{1 - q_2^2\pi^2/16}, \tag{80}$$

$$\sup_{\substack{0\leqslant\varphi\\ -\infty<k<\infty}} |\beta(k,\varphi)| \leqslant \frac{q_2\pi}{4(1 - q_2^2\pi^2/16)}, \tag{81}$$

$$\sup_{-\infty<k<\infty} |\alpha(k,\varphi) - \alpha(k,\infty)| \leqslant \frac{q_2(\pi/2 - \arctan\varphi)}{2(1 - q_2^2\pi^2/16)}, \tag{82}$$

and

$$\sup_{-\infty<k<\infty} |\beta(k,\varphi) - \beta(k,\infty)| \leqslant \frac{q_2(\pi/2 - \arctan\varphi)}{2(1 - q_2^2\pi^2/16)}. \tag{83}$$

Equations (77) and (80)–(83) then imply that for all $\varphi \geqslant 0$

$$\sup(|\hat{A}(k,\varphi)|, |\hat{B}(k,\varphi)|) \leqslant \frac{(1 + q_2^2\pi^2/16)^{1/2}}{(1 - q_2^2\pi^2/16)}\sqrt{|\hat{A}(k,0)|^2 + |\hat{B}(k,0)|^2}, \tag{84}$$

and

$$\sup(|\hat{A}(k,\varphi)-\hat{A}(k,\infty)|,|\hat{B}(k,\varphi)-\hat{B}(k,\infty)|)$$
$$\leqslant \frac{q_2}{2^{1/2}}\frac{(\pi/2-\arctan\varphi)}{(1-q_2^2\pi^2/4)}\sqrt{|\hat{A}(k,0)|^2+|\hat{B}(k,0)|^2}. \quad (85)$$

In what follows we shall assume that

$$\int_{-\infty}^{\infty} k^{2p}\left(|\hat{A}(k,0)|^2+|\hat{B}(k,0)|^2\right)dk$$
$$=\int_{-\infty}^{\infty} k^{2p}(\hat{A}(k,0)\hat{A}(-k,0)+\hat{B}(k,0)\hat{B}(-k,0))\,dk \overset{\text{def}}{=} 2\pi M_p^2<\infty \quad (86)$$

for indices $p=0,1$, and 2. The assumption (86), together with (84) and (85), implies that the functions

$$\mathscr{A}(x,\varphi)\overset{\text{def}}{=}\frac{1}{2\pi\mathbb{C}^{1/2}(\varphi)}\int_{-\infty}^{\infty} e^{ik(x-\varphi)}\hat{A}(k,\varphi)\,dk \quad (87)$$

and

$$\mathscr{B}(x,\varphi)\overset{\text{def}}{=}\frac{1}{2\pi\mathbb{C}^{1/2}(\varphi)}\int_{-\infty}^{\infty} e^{ik(x+\varphi)}\hat{B}(k,\varphi)\,dk \quad (88)$$

are well defined in $\varphi\geqslant 0$ and satisfy the estimates

$$\sum_{n=0}^{2}\int_{-\infty}^{\infty}\left(\frac{\partial^n}{\partial x^n}\mathscr{A}(x,\varphi)\right)^2dx\leqslant\frac{(1+q_2^2\pi^2/16)}{(1-q_2^2\pi^2/16)^2\mathbb{C}(\varphi)}\sum_{n=0}^{2}M_p^2 \quad (89)$$

and

$$\sum_{n=0}^{2}\int_{-\infty}^{\infty}\left(\frac{\partial^n}{\partial x^n}\mathscr{B}(x,\varphi)\right)^2dx\leqslant\frac{(1+q_2^2\pi^2/16)}{(1-q_2^2\pi^2/16)^2\mathbb{C}(\varphi)}\sum_{n=2}^{\infty}M_p^2. \quad (90)$$

Equations (85) and (86) also imply that the functions

$$\mathscr{A}_\infty(x)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{A}(k,\infty)\,dk \quad (91)$$

and

$$\mathscr{B}_\infty(x)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{B}(k,\infty)\,dk \quad (92)$$

satisfy

$$\lim_{\varphi\to\infty}\mathscr{A}(x+\varphi,\varphi)=\mathscr{A}_\infty(x) \quad (93)$$

and

$$\lim_{\varphi\to\infty}\mathscr{B}(x-\varphi,\varphi)=\mathscr{B}_\infty(x) \quad (94)$$

in the following strong sense:

$$\sum_{n=0}^{2}\int_{-\infty}^{\infty}\left(\frac{\partial^n}{\partial x^n}(\mathscr{A}(x+\varphi,\varphi)-\mathscr{A}_\infty(x))\right)^2dx\leqslant\frac{q_2^2(\pi^2/2-\arctan\varphi)^2}{2(1-q_2^2\pi^2/16)^2}\sum_{n=0}^{2}M_p^2 \quad (95)$$

and

$$\sum_{n=0}^{2}\int_{-\infty}^{\infty}\left(\frac{\partial^n}{\partial x^n}(\mathscr{B}(x-\varphi,\varphi)-\mathscr{B}_\infty(x))\right)^2 dx \leqslant \frac{q_2^2(\pi/2-\arctan\varphi)^2}{2(1-q_2^2\pi^2/16)^2}\sum_{n=0}^{2}M_p^2. \quad (96)$$

Definitions (87) and (88), together with (25) and (26), also imply that

$$\frac{\partial\mathscr{A}}{\partial\varphi}+\frac{\partial\mathscr{A}}{\partial x}=-\frac{1}{2\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}(\mathscr{B}+\mathscr{A}), \quad (97)$$

$$\frac{\partial\mathscr{B}}{\partial\varphi}-\frac{\partial\mathscr{B}}{\partial x}=-\frac{1}{2\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}(\mathscr{B}+\mathscr{A}), \quad (98)$$

$$\mathscr{A}(x,0)=\frac{1}{\mathbb{C}(0)}\frac{\partial w_0}{\partial t}(x,1^-)-\frac{\partial w_0}{\partial x}(x,1^-), \quad (99)$$

$$\mathscr{B}(x,0)=\frac{1}{\mathbb{C}(0)}\frac{\partial w_0}{\partial t}(x,1^-)+\frac{\partial w_0}{\partial x}(x,1^-), \quad (100)$$

and these equations, together with the fact that $(\partial w_0/\partial t)(x,1^-)$ and $(\partial w_0/\partial x)(x,1^-)$ have compact support in $|x|\leqslant l_1 \stackrel{\text{def}}{=} t(a,b,\varepsilon)/\varepsilon+\int_{t(a,b,\varepsilon)}^{1}c(s)\,ds$, guarantee that the functions $\mathscr{A}$ and $\mathscr{B}$ are supported in $|x|\leqslant l_1+\varphi$, $\varphi\geqslant 0$, and that the functions $\mathscr{A}_\infty(\cdot)$ and $\mathscr{B}_\infty(\cdot)$ defined in (93) and (94) satisfy

$$\mathscr{A}_\infty(x)\equiv 0,\quad x>l_1 \qquad \text{and} \qquad \mathscr{B}_\infty(x)\equiv 0,\quad x<-l_1. \quad (101)$$

Equations (97) and (98) also imply that

$$\left.\begin{aligned}\frac{\partial}{\partial\varphi}(\mathscr{B}-\mathscr{A})-\frac{\partial}{\partial x}(\mathscr{B}+\mathscr{A})&=0,\\ \frac{\partial}{\partial\varphi}(\mathbb{C}(\varphi)(\mathscr{B}+\mathscr{A}))-\mathbb{C}(\varphi)\frac{\partial}{\partial x}(\mathscr{B}-\mathscr{A})&=0,\end{aligned}\right\} \quad (102)$$

and (102), when combined with

$$\int_{-\infty}^{\infty}\frac{\partial w_0}{\partial x}(x,1^-)\,dx=\int_{-\infty}^{\infty}\frac{\partial w_0}{\partial t}(x,1^-)\,dx=0,^4 \quad (103)$$

implies that for all $\varphi\geqslant 0$

$$\int_{-\infty}^{\infty}\mathscr{A}(x,\varphi)\,dx=\int_{-l_1-\varphi}^{l_1+\varphi}\mathscr{A}(x,\varphi)\,dx=\int_{-\infty}^{\infty}\mathscr{B}(x,\varphi)\,dx=\int_{-l_1-\varphi}^{l_1+\varphi}\mathscr{B}(x,\varphi)\,dx=0. \quad (104)$$

Equations (102) and (104) also guarantee that the potential

$$\delta(x,\varphi)\stackrel{\text{def}}{=}\frac{1}{2}\int_{-\infty}^{x}(\mathscr{B}-\mathscr{A})(x,\varphi)=\frac{1}{2}\int_{-l_1-\varphi}^{x}\mathscr{B}(\xi,\varphi)\,d\xi+\frac{1}{2}\int_{x}^{l_1+\varphi}\mathscr{A}(\xi,\varphi)\,d\xi \quad (105)$$

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[4] See Eq. (5) of this section.

satisfies

$$\frac{\partial^2\delta}{\partial\varphi^2} - \frac{\partial^2\delta}{\partial x^2} = -\frac{1}{\mathbb{C}(\varphi)}\frac{d\mathbb{C}}{d\varphi}\frac{\partial\delta}{\partial\varphi}, \quad \varphi \geqslant 0, \tag{106}$$

$$\delta(x,0) = w_0(x,1^-) \quad \text{and} \quad \frac{\partial\delta}{\partial\varphi}(x,0) = \frac{1}{\mathbb{C}(0)}\frac{\partial w_0}{\partial t}(x,1^-), \tag{107}$$

$$\mathbb{C}^2(\varphi)\int_{-\infty}^{\infty}\left(\frac{\partial\delta}{\partial\varphi}\right)^2(x,\varphi)\,dx = \frac{\mathbb{C}(\varphi)}{8\pi}\left(H_{0,0}(0,\varphi) + 2H_{1,0}(\varphi)\right), \tag{108}$$

$$\int_{-\infty}^{\infty}\left(\frac{\partial\delta}{\partial x}\right)^2(x,\varphi)\,dx = \frac{1}{8\pi\mathbb{C}(\varphi)}\left(H_{0,0}(0,\varphi) - 2H_{1,0}(\varphi)\right), \tag{109}$$

$$\mathbb{C}^2(\varphi) - 1 = \frac{\varepsilon}{4T(\varphi)}\int_{-\infty}^{\infty}\left(\left(\frac{\partial\delta}{\partial x}\right)^2(x,\varphi) - \left(\frac{dw_*}{dx}\right)^2(x)\right)dx, \tag{110}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\mathbb{C}^2(\varphi)\left(\frac{\partial\delta}{\partial\varphi}\right)^2 + \left(\frac{\partial\delta}{\partial x}\right)^2\right)(x,\varphi)\,dx + \frac{2T(\varphi)}{\varepsilon}\left(\mathbb{C}^2(\varphi) - 1\right)^2 \\ + \frac{2}{\varepsilon}\int_0^{\varphi}\frac{\left(\mathbb{C}^2(s) - 1\right)}{\mathbb{C}(s)}\,ds \\ = \int_{-\infty}^{\infty}\left(\left(\frac{\partial w_0}{\partial t}\right)^2 + \left(\frac{\partial w_0}{\partial x}\right)^2\right)(x,1^-)\,dx \\ + \frac{\varepsilon}{8}\left(\int_{-\infty}^{\infty}\left(\left(\frac{\partial w_0}{\partial x}\right)^2(x,1^-) - \left(\frac{dw_*}{dx}\right)^2(x)\right)dx\right)^2.\end{aligned} \tag{111}$$

We are now set up to prove the asymptotic results claimed in the introduction. The identities

$$\int_{-l_1-\varphi}^{x}\mathscr{B}(\xi,\varphi)\,d\xi = \int_{-l_1}^{x+\varphi}\mathscr{B}(\xi-\varphi,\varphi)\,d\xi, \tag{112}$$

$$\int_{x}^{l_1+\varphi}\mathscr{A}(\xi,\varphi)\,d\xi = \int_{x-\varphi}^{l_1}\mathscr{A}(\xi+\varphi,\varphi)\,d\xi, \tag{113}$$

together with (104), (105), and (93)–(96), establish that

$$\lim_{\varphi\to\infty}\delta(x+\varphi,\varphi) = \delta_r(x) \overset{\text{def}}{=} \frac{1}{2}\int_x^{l_1}\mathscr{A}_\infty(\xi)\,d\xi \tag{114}$$

and

$$\lim_{\varphi\to\infty}\delta(x-\varphi,\varphi) = \delta_l(x) \overset{\text{def}}{=} \frac{1}{2}\int_{-l_1}^{x}\mathscr{B}_\infty(\xi)\,d\xi. \tag{115}$$

That

$$\lim_{\varphi\to\infty}\int_{-\infty}^{\infty}\left(\frac{\partial\delta}{\partial\varphi}\right)^2(x,\varphi)\,dx = \int_{-\infty}^{\infty}\left(\left(\frac{d\delta_l}{dx}\right)^2 + \left(\frac{d\delta_r}{dx}\right)^2\right)(x)\,dx \tag{116}$$

and

$$\lim_{\varphi \to \infty} \int_{-\infty}^{\infty} \left(\frac{\partial \delta}{\partial x} \right)^2 (x, \varphi)\, dx = \int_{-\infty}^{\infty} \left(\left(\frac{d\delta_l}{dx} \right)^2 + \left(\frac{d\delta_r}{dx} \right)^2 \right)(x)\, dx \tag{117}$$

follows from (108) and (109), the identities $\lim_{\varphi \to \infty} H_{1,0}(\varphi) = 0$, $\lim_{\varphi \to \infty} C(\varphi) = 1$, and (73). The latter equation implies

$$\lim_{\varphi \to \infty} H_{0,0}(0, \varphi) = \int_{-\infty}^{\infty} (\hat{A}(k, \infty)\hat{A}(-k, \infty) + \hat{B}(k, \infty)\hat{B}(-k, \infty))\, dk, \tag{118}$$

and this, together with (91)–(94), Parseval's identity, (114), and (115), yields the desired result.

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