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A NOTE ON THE STRONGEST FIXED-FIXED COLUMN*

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Abstract. It is argued that the subsequent solutions generated in response to the 1962 Tadjbakhsh-Keller shape for the optimal fixed-fixed column are smooth approximations of that singular shape.

1. Introduction. The recent Kirmser-Hu paper [1] revives the discussion of the shape of the stongest fixed-fixed column of given weight, which was presented by Tadjbakhsh and Keller (T-K) in 1962. (See [2].) To paraphrase Kirmser and Hu, in 1977 Olhoff and Rasmussen [3] suggested that since the T-K solution contains a point of zero area (moment of inertia), that design must contain a hinge and (after some analysis) cannot possibly be the optimal design. There have followed a sequence of alternative designs which have come closer and closer in both shape and buckling load to the original T-K solution. It is argued in this note that the designs of this sequence are actually smooth approximations asymptotically approaching the T-K solution, which is singular.

As an aside, it may be noted that none of the subsequent workers have explained exactly what should be done to correct the T-K formulation. For those interested in the theory of optimization, this is not a minor matter, since the T-K paper is not only beautiful but also a significant contribution to the literature. We are, of course, suggesting that there is nothing wrong with it.

2. The Tadjbakhsh-Keller Solution. In the following, several points are listed in support of the T-K solution.

a. Numerical evidence. The alternative proposed solutions in the literature can themselves be regarded as supporting the T-K solution when viewed properly. In each case, their proponents have argued that (i) the T-K solution is incorrect and that (ii) the correct solution looks a lot like the T-K solution. In terms of their nondimensional buckling load,

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these solutions may be compared as:

Suggested buckling load	Reference[1]
52.3563	Olhoff and Rasmussen, 1977 [3]
52.3565	Masur, 1984 [4]
52.43432	Kirmser and Hu, 1983 [1]
$16\pi^2/3 = 52.6379$	Tadjbakhsh and Keller, 1962 [2]

In fact, Kirmser and Hu go on to suggest that the exact buckling load "could be quite close to that found by Tadjbakhsh and Keller".

b. Zero moment of inertia does not imply a hinge. The authors who have suggested alternatives to the T-K solution have assumed that the point of zero moment of inertia of that solution is equivalent to a hinge. This is simply not true.

For any beam problem and any variation of moment of inertia, a solution may or may not *exist* depending on the variation of the moment of inertia which is given. A hinge is another matter. In order to specify a hinge at a point, it is necessary to specify (in addition to whatever else is required for a well-posed problem) that the moment be zero at this point and to give up continuity of slope.

Appendix 1 gives the solution of a simple beam problem in which the moment of inertia varies as $I = |x^{1/2}|$. It can be seen in this case that $I = 0$ does not imply zero moment.

c. The T-K optimality criterion. Central to the T-K solution is the use of the calculus of variations technique of embedding an optimal function $\rho_0(x)$ in a single-parameter family of functions $\rho(x, \varepsilon)$ where

$$\rho(x, \varepsilon) = \rho_0(x) + \varepsilon\eta(x)$$

and

$$\rho_0(x) = \rho(x, 0).$$

Here ε is a scalar parameter and η is an arbitrary, smooth (as determined by the specific application) function. Using this technique, T-K derive the optimality criterion that the cross-sectional area is proportional to the two-thirds power of the bending moment in an optimal column.

It has been argued [3] that a source of difficulty with the T-K solution lies with multiple eigenvalues which can lead to constraint surfaces which are not smooth. The answer to this argument is that the T-K calculus of variations technique gives singular results as a limiting case of smooth functions. In support of this argument, we offer three points:

(i) Discussions of calculus of variations results with singular solutions as limits of smooth functions are available in the literature [5].

(ii) In the spirit of Masur's discrete example [4], Appendix 2 shows how this calculus of variations technique can produce the proper optimal design even though a constraint surface is singular at an optimal point.

(iii) Appendix 3 shows some numerical results in which asymptotic methods can be used to produce a singular solution as a limiting case of smooth solutions in another buckling problem.

Appendix 1. This appendix describes the symmetric problem of a fixed-ended beam $(-1 \leqslant x \leqslant 1)$ under uniform load w when the moment of inertia varies as $I = |x^{1/2}|$. In this case, the differential equation

$$(EIy'')'' = w,$$

where

$$y = \text{beam deflection},$$
$$w = \text{applied lateral load},$$
$$E = \text{Young's modulus},$$
$$I = \text{moment of inertia},$$

can be integrated directly as

$$EIy'' = wx^2/2 + C_1x + C_2, \quad (-1 \leqslant x \leqslant 1),$$

or

$$Ey = \frac{2}{35}wx^{7/2} - \frac{4}{30}wx^{3/2} + \frac{8}{5 \times 21}w, \quad (0 \leqslant x \leqslant 1).$$

The moment at the center of the beam ($x = 0$) is $w/10$ while the moment of inertia is zero at this point.

Appendix 2. In an attempt to provide a simple example of difficulties which can arise in multiple eigenvalue problems, Masur [4] examines a problem of two-degree-of-freedom mass spring system which results in constraint surfaces which are not smooth. In this appendix it will be shown that the T-K calculus of variations method produces the proper optimal point in the case of a related example.

Given the two-degree-of-freedom system shown in Fig. 1, we can pose the problem of finding the spring stiffnesses k_1 and k_2 which maximize the natural frequency ω subject to $k_1 + k_2 = 1$ (fixed total weight). In this case, the solution is clearly $k_1 = k_2 = 1/2$ and the constraint surface $\omega^2 =$ constant is clearly singular at the optimal point. This result follows directly using the T-K calculus of variations technique as

$$\max \omega^2 \mid k_1 + k_2 = 1, \qquad k\delta = \omega^2 m\delta,$$

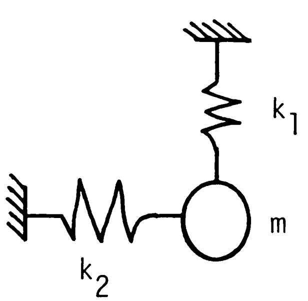

FIG. 1. Mass spring system.

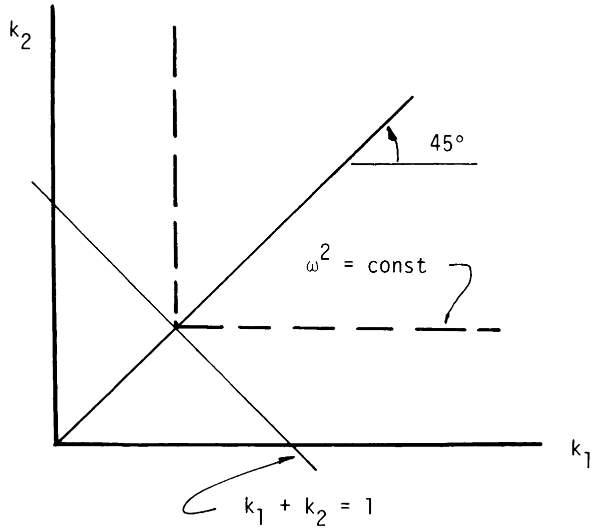

FIG. 2. Constraint surfaces.

where

$$k = \text{stiffness matrix},$$
$$m = \text{mass matrix (given)},$$
$$\delta = \text{mode shape}.$$

Let the dot represent differentiation with respect to the parameter ε in the T-K calculus of variations technique. Then

$$k\delta = \omega^2 m\delta \Rightarrow \dot{k}\delta + k\dot{\delta} = \dot{\omega}^2 m\delta + \omega^2 m\dot{\delta}.$$

It follows that

$$\delta^T \dot{k}\delta = 0 \quad and \quad \dot{k}_1 + \dot{k}_2 = 0$$

or that

$$\delta_1^2 = \delta_2^2 \Rightarrow k_1 = k_2 = 1/2.$$

Appendix 3. The purpose of this appendix is to provide numerical evidence of a singular buckling solution as a limiting case of continuous solutions. The details of the analysis leading to these results will be presented elsewhere; they are obtained by a method illustrated for a different eigenvalue problem in [6].

The idea is that the buckling problem of a fixed beam with a center hinge can be approached as a limiting case of a uniform beam with a variable notch at the center by using the method of matched asymptotic expansions. For the case illustrated, the beam is fixed at $x = 0, 1$ and the dimensionless area varies according to

$$A(x) = \begin{cases} \alpha + (1-\alpha)|x - 0.5|/\delta & |x - 0.5| < \delta \\ 1 & |x - 0.5| > \delta. \end{cases}$$

The small notch width parameter δ is taken equal to 0.02 for these results.

Figure 3 shows the behavior of the mode shape as the notch is made increasingly severe, i.e., as the center area ratio α decreases. In every case, the slope of the beam is continuous (zero) at $x = 0.5$. Figure 4 illustrates the variation with α of the dimensionless buckling load λ ($\lambda = 4\pi^2$ for $\alpha = 1$; $\lambda = \pi^2$ for a uniform beam hinged at $x = 0.5$).

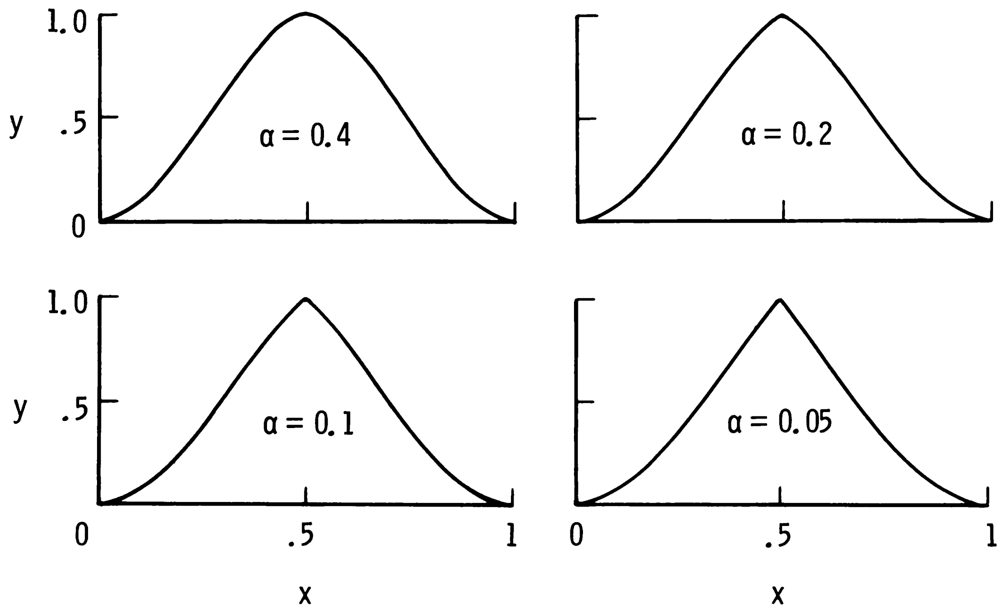

FIG. 3. Variation of mode shape for decreasing area ratio α.

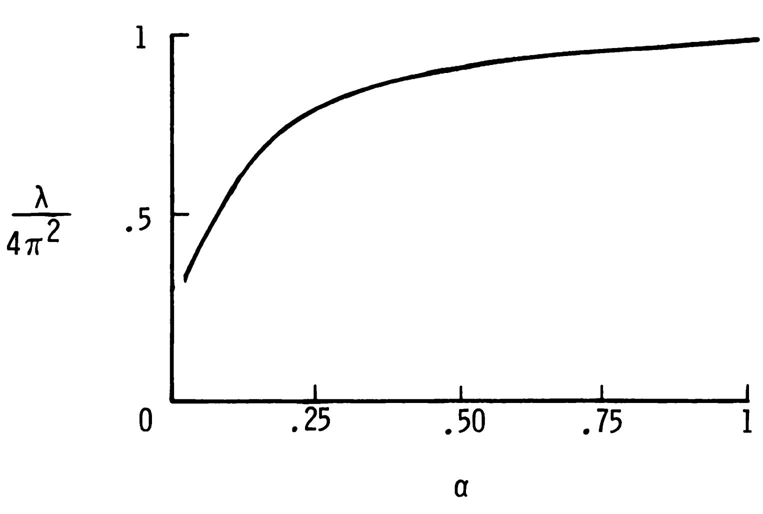

FIG. 4. Variation of buckling load with area ratio α; $\delta = 0.02$.

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